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A unified rational expansion method to construct a series of explicit exact solutions to nonlinear evolution equations

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Abstract

A unified basic frame of rational expansion methods is presented, which leads to closed-form solutions of nonlinear evolution equations (NLEEs). The new unified algorithms are given to find exact rational formal polynomial solutions of NLEEs in terms of Jacobi elliptic functions, solutions of the Riccati equation and solutions of the generalized Riccati equation. They can be implemented in symbolic computation system *Maple*. As applications of the methods, we choose some physical significance NLEEs to illustrate the methods. As a consequence, we not only can successfully obtain the solutions found by most existing Jacobi elliptic function methods and tanh methods, but also find other new and more general solutions at the same time.

Keywords: Symbolic computation; Exact solutions; Nonlinear evolution equations; Jacobi elliptic function rational expansion method; Riccati equation rational expansion method

1. Introduction

An exciting and extremely active area of research investigation during the past four years has been the study of solitons and the related issue of the construction of solutions to a wide class of NLEEs. There has been a great amount of activities aiming to find methods for exact solution of NLEEs, such as Bäcklund transformation, Darboux transformation, variable separation approach, various tanh methods, Painleve method, generalized hyperbolic-function method, homogeneous balance method, similarity reduction method and so on [1–14]. The tanh method [5–7] is considered to be one of the most straightforward and effective algorithm to obtain solitary wave solutions for a large of NLEEs. In line with the development of computerized symbolic computation, much work has been concentrated on the various extensions and applications of the tanh method [8–16].

Recently, we present various rational function expansions methods [17] and various subequation expansion methods [18] to construct new and more general solutions of NLEEs, the solutions obtained contain not only

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the results obtained by using the various tanh methods [5–16] but also other types of solutions. The main purpose of this paper is to summary and extend our methods [17,18] to unified basic frame so that it can be used to obtained more types and general formal solutions. For illustration, we apply the generalized method to some NLEEs and successfully construct new and more general solutions including rational form Jacobi elliptic function solutions, solitary wave solutions, triangular periodic solutions. The related further discuss and comparison with other tanh method will be given in Section 5.

The paper is organized as follows: in Section 2, we give the main steps of our algorithms for computing exact solutions of nonlinear polynomial NLEEs. In Section 3, we consider the Jacobi elliptic function rational expansion method. In Section 4, we show the Riccati equation rational expansion method. In Section 5, we further extend the Riccati equation rational expansion method to a generalized form. Conclusions will be presented finally.

2. Summary of the rational expansion method

In the following we would like to outline the main steps of our method:

Step 1. Given a system of polynomial NLEEs with constant coefficients, with some physical fields $u_i(x,y,t)$ in three variables x, y, t,

$$\Delta(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \ldots) = 0, \tag{2.1}$$

use the wave transformation

$$u_i(x,y,t) = U_i(\xi), \quad \xi = k(x + ly + \lambda t), \tag{2.2}$$

where k, l and λ are constants to be determined later. Then the nonlinear partial differential system (2.1) is reduced to a nonlinear ordinary differential equation (ODE)

$$\Theta(U_i,\ U_i',\ U_i'', \ldots) = 0. \tag{2.3}$$

Step 2. We introduce a new ansätz in terms of finite rational formal expansion in the following forms:

$$U_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum_{r_1+r_2=j} a^{ij}_{r_1 r_2} F^{r_1}(\xi) G^{r_2}(\xi)}{(\mu_1 F(\xi) + \mu_2 G(\xi) + 1)^j}, \tag{2.4}$$

where a_{i0}, $a^{ij}_{r_1 r_2}$, μ_1 and μ_2 $(i = 1,2,\ldots)$ are constants to be determined later and the new variables $F = F(\xi)$ and $G = G(\xi)$ satisfy

$$\frac{\mathrm{d}F}{\mathrm{d}\xi} = K_1(F,G), \qquad \frac{\mathrm{d}G}{\mathrm{d}\xi} = K_2(F,G), \tag{2.5}$$

where K_1 and K_2 are polynomial of F and G.

Step 3. Determine the m_i of the rational formal polynomial solutions (2.4) by respectively balancing the highest nonlinear terms and the highest-order partial derivative terms in the given system equations (see Ref. [9–13] for details), and then give the formal solutions.

Step 4. Substitute (2.4) into (2.3) along with (2.5) and then set all coefficients of $F^p(\xi)G^q(\xi)$, $(p = 1,2,\ldots;$ $q = 0,1)$ of the resulting system's numerator to be zero to get an over-determined system of nonlinear algebraic equations with respect to k, a_{i0}, $a^{ij}_{r_1 r_2}$, μ_1 and μ_2 $(i = 1,2,\ldots)$.

Step 5. By solving the over-determined system of nonlinear algebraic equations by use of symbolic computation system *Maple*, we end up with the explicit expressions for k, a_{i0}, $a^{ij}_{r_1 r_2}$, μ_1 and μ_2 $(i = 1,2,\ldots)$.

Step 6. According to system (2.2) and (2.4), the general solutions of system (2.5) and the conclusions in Step 5, we can obtain rational formal exact solutions of system (2.1).

Remark 1. The method proposed here is more general than the various exist method for finding exact solutions of NLEEs. We can easily draw this conclusion when F and G take particular variables in the following sections. The appeal and success of the method lies in the fact that writing the exact solutions of

NLEEs as polynomials of F and G whose derivatives are in closed-form, the equations can changed into a nonlinear system of algebraic equations. The system can be solved with help of symbolic computation.

3. Jacobi elliptic function rational expansion method

In this section we would like to apply our method to obtain rational formal Jacobi elliptic function solutions of NLEEs, i.e., restricting F and G in Jacobi elliptic functions.

Here $\operatorname{sn}\xi$, $\operatorname{cn}\xi$, $\operatorname{dn}\xi$, $\operatorname{ns}\xi$, $\operatorname{cs}\xi$, $\operatorname{sc}\xi$, $\operatorname{nc}\xi$, $\operatorname{sd}\xi$ and $\operatorname{nd}\xi$ are the Jacobian elliptic sine function, the Jacobian elliptic cosine function and the Jacobian elliptic function of the third kind and other Jacobian functions which is denoted by Glaishers symbols and are generated by these three kinds of functions, namely [19],

$$\operatorname{ns}\xi=\frac{1}{\operatorname{sn}\xi},\quad \operatorname{nc}\xi=\frac{1}{\operatorname{cn}\xi},\quad \operatorname{nd}\xi=\frac{1}{\operatorname{dn}\xi},\quad \operatorname{sd}\xi=\frac{\operatorname{sn}\xi}{\operatorname{dn}\xi}, \tag{3.1.1}$$

$$\operatorname{sc}\xi=\frac{\operatorname{sn}\xi}{\operatorname{cn}\xi},\quad \operatorname{cs}\xi=\frac{\operatorname{cn}\xi}{\operatorname{sn}\xi},\quad \operatorname{ds}\xi=\frac{\operatorname{dn}\xi}{\operatorname{sn}\xi}, \tag{3.1.2}$$

which are double periodic and posses the following properties

1. Properties of triangular function

$$\operatorname{cn}^2\xi+\operatorname{sn}^2\xi=\operatorname{dn}^2\xi+m^2\operatorname{sn}^2\xi=1, \tag{3.2.1}$$

$$\operatorname{ns}^2\xi=1+\operatorname{cs}^2\xi,\quad \operatorname{ns}^2\xi=m^2+\operatorname{ds}^2\xi, \tag{3.2.2}$$

$$\operatorname{sc}^2\xi+1=\operatorname{nc}^2\xi,\quad m^2\operatorname{sd}^2\xi+1=\operatorname{nd}^2\xi. \tag{3.2.3}$$

2. Derivatives of the Jacobi elliptic functions

$$\operatorname{sn}'\xi=\operatorname{cn}\xi\operatorname{dn}\xi,\quad \operatorname{cn}'\xi=-\operatorname{sn}\xi\operatorname{dn}\xi,\quad \operatorname{dn}'\xi=-m^2\operatorname{sn}\xi\operatorname{cn}\xi, \tag{3.3.1}$$

$$\operatorname{ns}'\xi=-\operatorname{ds}\xi\operatorname{cs}\xi,\quad \operatorname{ds}'\xi=-\operatorname{cs}\xi\operatorname{ns}\xi,\quad \operatorname{cs}'\xi=-\operatorname{ns}\xi\operatorname{ds}\xi, \tag{3.3.2}$$

$$\operatorname{sc}'\xi=\operatorname{nc}\xi\operatorname{dc}\xi,\quad \operatorname{nc}'\xi=\operatorname{sc}\xi\operatorname{dc}\xi,\quad \operatorname{cd}'\xi=\operatorname{cd}\xi\operatorname{nd}\xi,\quad \operatorname{nd}'\xi=m^2\operatorname{sd}\xi\operatorname{cd}\xi, \tag{3.3.3}$$

   where m is a modulus.
3. Properties of limit

$$\lim_{m\to1}\operatorname{sn}\xi=\tanh\xi,\quad \lim_{m\to1}\operatorname{cn}\xi=\operatorname{sech}\xi,\quad \lim_{m\to1}\operatorname{dn}\xi=\operatorname{sech}\xi, \tag{3.4.1}$$

$$\lim_{m\to1}\operatorname{ns}\xi=\coth\xi,\quad \lim_{m\to1}\operatorname{cs}\xi=\operatorname{csch}\xi,\quad \lim_{m\to1}\operatorname{ds}\xi=\operatorname{csch}\xi, \tag{3.4.2}$$

$$\lim_{m\to0}\operatorname{sn}\xi=\sin\xi,\quad \lim_{m\to0}\operatorname{cn}\xi=\cos\xi,\quad \lim_{m\to0}\operatorname{dn}\xi=1, \tag{3.4.3}$$

$$\lim_{m\to0}\operatorname{ns}\xi=\csc\xi,\quad \lim_{m\to0}\operatorname{cs}\xi=\cot\xi,\quad \lim_{m\to0}\operatorname{ds}\xi=\csc\xi. \tag{3.4.4}$$

The Jacobi–Glaisher functions for elliptic function can be found in Ref. [19]. It can easily be seen that the Jacobi elliptic functions satisfy system (2.5).

The six main steps of the Jacobi elliptic function (here just consider the condition $F=\operatorname{sn}\xi$ and $G=\operatorname{cn}\xi$) rational expansion method are illustrated with the $(2+1)$-dimensional dispersive long wave equation (DLWE), i.e.,

$$\begin{aligned} &u_{yt}+v_{xx}+(uu_x)_y=0,\\ &v_t+u_x+(uv)_x+u_{xxy}=0. \end{aligned} \tag{3.5}$$

According to the *Step 1* in Section 2, we make the following travelling wave transformation:

$$u(x,y,t) = U(\xi), \quad v(x,y,t) = V(\xi), \quad \xi = k(x + ly + \lambda t), \tag{3.6}$$

where k, l and λ are constants to be determined later, and thus system (3.5) becomes

$$\begin{aligned} &\lambda l U'' + V'' + lU'^2 + lUU'' = 0, \\ &\lambda V' + U' + (UV)' + k^2 l U''' = 0. \end{aligned} \tag{3.7}$$

According to *Step 2* in Section 2, we expand the solution of system (3.7) in the form

$$\begin{aligned} U(\xi) &= a_0 + \sum_{j=1}^{m_u} \frac{a_j \mathrm{sn}^j(\xi) + b_j \mathrm{sn}^{j-1}(\xi)\mathrm{cn}(\xi)}{(\mu_1 \mathrm{sn}(\xi) + \mu_2 \mathrm{cn}(\xi) + 1)^j}, \\ V(\xi) &= A_0 + \sum_{j=1}^{m_v} \frac{A_j \mathrm{sn}^j(\xi) + B_j \mathrm{sn}^{j-1}(\xi)\mathrm{cn}(\xi)}{(\mu_1 \mathrm{sn}(\xi) + \mu_2 \mathrm{cn}(\xi) + 1)^j}, \end{aligned} \tag{3.8}$$

According to *Step 3* in Section 2, by balancing $U'''(\xi)$ and $(V(\xi)U(\xi))'$ in system (3.7) and by balancing $V''(\xi)$ and $U(\xi)U''(\xi)$ in system (3.7), we can obtain that $m_u = 1$ and $m_v = 2$. So we have

$$\begin{aligned} U(\xi) &= a_0 + \frac{a_1 \mathrm{sn}(\xi) + b_1 \mathrm{cn}(\xi)}{\mu_1 \mathrm{sn}(\xi) + \mu_2 \mathrm{cn}(\xi) + 1}, \\ V(\xi) &= A_0 + \frac{A_1 \mathrm{sn}(\xi) + B_1 \mathrm{cn}(\xi)}{\mu_1 \mathrm{sn}(\xi) + \mu_2 \mathrm{cn}(\xi) + 1} + \frac{A_2 \mathrm{sn}^2(\xi) + B_2 \mathrm{sn}(\xi)\mathrm{cn}(\xi)}{(\mu_1 \mathrm{sn}(\xi) + \mu_2 \mathrm{cn}(\xi) + 1)^2}, \end{aligned} \tag{3.9}$$

where a_0, a_1, b_1, A_0, A_1, A_2, B_1, B_2, μ_1 and μ_2 are constants to be determined later.

According to *Step 4* in Section 2, with the aid of *Maple*, substituting (3.9) into (3.7), yields a set of algebraic equations for $\mathrm{sn}^i(\xi)\mathrm{cn}^j(\xi)$ $(i = 0, 1, \ldots; j = 0, 1)$. Setting the coefficients of these terms $\mathrm{sn}^i(\xi)\mathrm{cn}^j(\xi)$ of the resulting system's numerator to be zero yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, A_0, A_1, B_1, A_2, B_2, k, l and λ.

According to *Step 5* in Section 2, by use of the *Maple* soft package "Charsets" by Dongming Wang, which is based on the Wu-elimination method [20], solving the over-determined algebraic equations, we get explicit expressions for a_0, a_1, b_1, A_0, A_1, B_1, A_2, B_2, μ_1, μ_2, k, l and λ.

According to *Step 6* in Section 2, we get following solutions of $(2+1)$-dimension DLWE:

Family 1

$$\begin{aligned} u_1 = {} & \frac{\pm(2k\mu_2^3 m^2 - 2k\mu_2^3 - 2k\mu_2 m^2 + k\mu_2) - \sqrt{-\mu_2^4 m^2 + \mu_2^4 + 2m^2\mu_2^2 - \mu_2^2 - m^2}\,\lambda}{\sqrt{-\mu_2^4 m^2 + \mu_2^4 + 2m^2\mu_2^2 - \mu_2^2 - m^2}} \\ & \pm 2\frac{\sqrt{-\mu_2^4 m^2 + \mu_2^4 + 2m^2\mu_2^2 - \mu_2^2 - m^2}\,k\,\mathrm{cn}(k(x + ly + \lambda t))}{1 + \mu_2 \mathrm{cn}(k(x + ly + \lambda t))}, \end{aligned} \tag{3.10.1}$$

$$\begin{aligned} v_1 = {} & \frac{\mu_2^4 - \mu_2^2 - \mu_2^4 m^2 - m^2 - 2lk^2\mu_2^2 m^2 + lk^2 m^2 + 2m^2\mu_2^2 + \mu_2^4 lk^2 m^2 - \mu_2^4 lk^2}{\mu_2^4 m^2 - \mu_2^4 - 2m^2\mu_2^2 + \mu_2^2 + m^2} \\ & - 2\frac{lk^2\mu_2 \mathrm{cn}(k(x + ly + \lambda t))}{1 + \mu_2 \mathrm{cn}(k(x + ly + \lambda t))} + \frac{(2lk^2\mu_2^2 m^2 - 2lk^2\mu_2^2 - 2lk^2 m^2)\mathrm{sn}^2(k(x + ly + \lambda t))}{(1 + \mu_2 \mathrm{cn}(k(x + ly + \lambda t)))^2}, \end{aligned} \tag{3.10.2}$$

where μ_2, k, l and λ are arbitrary constants.

Family 2

$$u_2 = \frac{\pm(k\mu_2^3 - \mu_2^3 m^2 k - k\mu_2 + k\mu_2 m^2) - \lambda\sqrt{\mu_2^4 - \mu_2^4 m^2 - \mu_2^2 + 2m^2\mu_2^2 - m^2}}{\sqrt{\mu_2^4 - \mu_2^4 m^2 - \mu_2^2 + 2m^2\mu_2^2 - m^2}} \pm \frac{\sqrt{-m^2\mu_2^2 + \mu_2^2 + m^2}k\,\mathrm{sn}(k(x+ly+\lambda t))}{1+\mu_2\mathrm{cn}(k(x+ly+\lambda t))} \pm \frac{\sqrt{\mu_2^4 - \mu_2^4 m^2 - \mu_2^2 + 2m^2\mu_2^2 - m^2}k\,\mathrm{cn}(k(x+ly+\lambda t))}{1+\mu_2\mathrm{cn}(k(x+ly+\lambda t))}, \tag{3.11.1}$$

$$v_2 = -\frac{-\mu_2^2 + m^2\mu_2^2 - m^2 + lk^2m^2}{m^2\mu_2^2 - \mu_2^2 - m^2} - \frac{lk^2\mu_2\mathrm{cn}(k(x+ly+\lambda t))}{1+\mu_2\mathrm{cn}(k(x+ly+\lambda t))} \pm \frac{l\sqrt{-m^2\mu_2^2 + \mu_2^2 + m^2}k^2\mu_2(-\mu_2^2 + m^2\mu_2^2 + 1 - m^2)\mathrm{sn}(k(x+ly+\lambda t))}{\sqrt{\mu_2^4 - \mu_2^4 m^2 - \mu_2^2 + 2m^2\mu_2^2 - m^2}(1+\mu_2\mathrm{cn}(k(x+ly+\lambda t)))} + \frac{(-lk^2\mu_2^2 + lk^2\mu_2^2m^2 - lk^2m^2)\mathrm{sn}^2(k(x+ly+\lambda t))}{(1+\mu_2\mathrm{cn}(k(x+ly+\lambda t)))^2} \pm l\sqrt{-m^2\mu_2^2 + \mu_2^2 + m^2} \times \frac{k^2\sqrt{\mu_2^4 - \mu_2^4 m^2 - \mu_2^2 + 2m^2\mu_2^2 - m^2}\mathrm{sn}(k(x+ly+\lambda t))\mathrm{cn}(k(x+ly+\lambda t))}{(1+\mu_2\mathrm{cn}(k(x+ly+\lambda t)))^2}, \tag{3.11.2}$$

where μ_2, k, l and λ are arbitrary constants.

Family 3

$$u_3 = \frac{\pm(k\mu_1^3 - k\mu_1) - \lambda\sqrt{\mu_1^4 - \mu_1^2 - \mu_1^2 m^2 + m^2}}{\sqrt{\mu_1^4 - \mu_1^2 - \mu_1^2 m^2 + m^2}} \pm \frac{\sqrt{\mu_1^2 - m^2}k\,\mathrm{cn}(k(x+ly+\lambda t))}{\mu_1\mathrm{sn}(k(x+ly+\lambda t)) + 1} \pm \frac{\sqrt{\mu_1^4 - \mu_1^2 - \mu_1^2 m^2 + m^2}k\,\mathrm{sn}(k(x+ly+\lambda t))}{\mu_1\mathrm{sn}(k(x+ly+\lambda t)) + 1}, \tag{3.12.1}$$

$$v_3 = \frac{lk^2\mu_1^4 - 2lk^2\mu_1^2m^2 + \mu_1^2 - m^2 + lm^2k^2}{-\mu_1^2 + m^2} + \frac{(2lk^2\mu_1^3 - lk^2\mu_1 - lk^2\mu_1m^2)\mathrm{sn}(k(x+ly+\lambda t))}{\mu_1\mathrm{sn}(k(x+ly+\lambda t)) + 1} \pm \frac{l\sqrt{-m^2 + \mu_1^2}k^2\mu_1(\mu_1^2 - 1)\mathrm{cn}(k(x+ly+\lambda t))}{\sqrt{\mu_1^4 - \mu_1^2 - \mu_1^2 m^2 + m^2}(\mu_1\mathrm{sn}(k(x+ly+\lambda t)) + 1)} + \frac{(-lk^2\mu_1^4 + lk^2\mu_1^2 + lk^2\mu_1^2m^2 - lm^2k^2)\mathrm{sn}^2(k(x+ly+\lambda t))}{(\mu_1\mathrm{sn}(k(x+ly+\lambda t)) + 1)^2} \pm \frac{l\sqrt{-m^2 + \mu_1^2}k^2\sqrt{\mu_1^4 - \mu_1^2 - \mu_1^2 m^2 + m^2}\mathrm{sn}(k(x+ly+\lambda t))\mathrm{cn}(k(x+ly+\lambda t))}{(\mu_1\mathrm{sn}(k(x+ly+\lambda t)) + 1)^2}, \tag{3.12.2}$$

where μ_1, k, l and λ are arbitrary constants.

Family 4

$$u_4 = \frac{\pm(m^2\mu_1k + k\mu_1 - 2k\mu_1^3) - \lambda\sqrt{\mu_1^4 - \mu_1^2m^2 - \mu_1^2 + m^2}}{\sqrt{\mu_1^4 - \mu_1^2m^2 - \mu_1^2 + m^2}} - 2\frac{\sqrt{\mu_1^4 - \mu_1^2m^2 - \mu_1^2 + m^2}k\,\mathrm{sn}(k(x+ly+\lambda t))}{\mu_1\mathrm{sn}(k(x+ly+\lambda t)) + 1}, \tag{3.13.1}$$

$$v_4 = -\frac{3lk^2\mu_1^4 - \mu_1^4 - 6lk^2\mu_1^2m^2 - m^2 + lm^2k^2 + lk^2m^4 - 2lk^2\mu_1^6 + 3lk^2\mu_1^4m^2 + \mu_1^2m^2 + \mu_1^2}{-\mu_1^4 + \mu_1^2m^2 + \mu_1^2 - m^2} + \frac{(-2lk^2\mu_1m^2 - 2lk^2\mu_1 + 4lk^2\mu_1^3)\mathrm{sn}(k(x+ly+\lambda t))}{\mu_1\mathrm{sn}(k(x+ly+\lambda t)) + 1} + \frac{(-2lk^2\mu_1^4 + 2lk^2\mu_1^2m^2 + 2lk^2\mu_1^2 - 2lm^2k^2)(\mathrm{sn}(k(x+ly+\lambda t)))^2}{(\mu_1\mathrm{sn}(k(x+ly+\lambda t)) + 1)^2}, \tag{3.13.2}$$

where μ_1, k, l and λ are arbitrary constants.

Family 5

$$u_5 = -\frac{1}{2}a_1 - \lambda + \frac{a_1 \mathrm{sn}\left(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}\right)}{\pm\mathrm{sn}\left(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}\right) \pm \mathrm{cn}\left(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}\right) + 1}, \tag{3.14.1}$$

$$\begin{aligned} v_5 = {} & \frac{2A_1 + la_1^2 m^2 + 4 - 4m^2 - 2A_1 m^2}{4(m^2-1)} - \frac{\frac{1}{2}la_1^2 \mathrm{sn}^2\left(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}\right)}{\left(\pm\mathrm{sn}\left(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}\right) \pm \mathrm{cn}\left(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}\right) + 1\right)^2} \\ & + \frac{A_1 \mathrm{sn}\left(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}\right)}{\pm\mathrm{sn}\left(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}\right) \pm \mathrm{cn}\left(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}\right) + 1} + \frac{\left(-\frac{1}{2}la_1^2 + A_1\right)\mathrm{cn}\left(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}\right)}{\pm\mathrm{sn}\left(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}\right) \pm \mathrm{cn}\left(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}\right) + 1} \\ & + \frac{\left(\frac{1}{2}la_1^2 - A_1\right)\mathrm{sn}\left(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}\right)\mathrm{cn}\left(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}\right)}{\left(\pm\mathrm{sn}(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}) \pm \mathrm{cn}\left(\pm\frac{a_1(x+ly+\lambda t)}{2\sqrt{1-m^2}}\right) + 1\right)^2}, \end{aligned} \tag{3.14.2}$$

where a_1, A_1, l and λ are arbitrary constants.

Remark 2. The solutions (3.10.1) and (3.10.2) reproduce the solutions (16) in [10], when $\mu_2 = 0$, $\lambda = -\lambda^*$ and $k = \sqrt{\frac{2C_1+2+l\lambda^2}{2l(1+m^2)}}$. The solutions (3.12.1) and (3.12.2) reproduce the solutions (17) in [10], when $\mu_1 = 0$, $\lambda = -\lambda^*$ and $k = \sqrt{\frac{2C_1+2+l\lambda^2}{l(2-m^2)}}$. The solutions (3.13.1) and (3.13.2) reproduce the solutions (15) in [10], when $\mu_1 = 0$, $\lambda = -\lambda^*$ and $k = \sqrt{\frac{2C_1+2+l\lambda^2}{2l(1-2m^2)}}$. The other solutions obtained here, to our knowledge, are all new families of rational formal doubly periodic solution of the $(2+1)$-dimension DLWE.

Remark 3. If we replace the Jacobi elliptic functions $\mathrm{sn}(\xi)$, $\mathrm{cn}(\xi)$ in the ansätzes (3.8) with other pairs of Jacobi elliptic functions such as $\mathrm{sn}(\xi)$ and $\mathrm{dn}(\xi)$; $\mathrm{ns}(\xi)$ and $\mathrm{cs}(\xi)$; $\mathrm{ns}(\xi)$ and $\mathrm{ds}(\xi)$; $\mathrm{sc}(\xi)$ and $\mathrm{nc}(\xi)$; $\mathrm{dc}(\xi)$ and $\mathrm{nc}(\xi)$; $\mathrm{sd}(\xi)$ and $\mathrm{nd}(\xi)$; $\mathrm{cd}(\xi)$ and $\mathrm{nd}(\xi)$ (It is necessary to point out that they are satisfying system (2.5)), therefore other new double periodic wave solutions can be obtained for some system. For simplicity, we omit them here.

4. Riccati equation rational expansion method

In this section we would like apply our method to obtain rational form soliton solutions of NLEEs, i.e., restricting F and G in solutions of the Riccati equation.

The six main steps of the Riccati equation rational expansion method are illustrated with the Whitham–Broer–Kaup (WBK) system, i.e.,

$$\begin{aligned} & u_t + uu_x + v_x + \beta u_{xx} = 0, \\ & v_t + (uv)_x + \alpha u_{xxx} - \beta v_{xx} = 0, \end{aligned} \tag{4.1}$$

where $\alpha, \beta \neq 0$ are all constants.

According to the *Step 1* in Section 2, we make the following travelling wave transformation:

$$u(x,t) = U(\xi), \qquad v(x,t) = V(\xi), \quad \xi = k(x + \lambda t), \tag{4.2}$$

where k and λ are constants to be determined later, and thus system (4.1) becomes

$$\begin{aligned} & \lambda U' + UU' + V' + k\beta U'' = 0, \\ & \lambda V' + (UV)' + \alpha k^2 U''' - \beta k V'' = 0. \end{aligned} \tag{4.3}$$

According to *Step 2* in Section 2, We suppose that WBK system has the following formal travelling wave solution:

$$
\begin{aligned}
U(\xi) &= a_0 + \sum_{i=1}^{m_u} \frac{a_i\phi^i(\xi) + b_i\phi^{i-1}(\xi)\sqrt{R+\phi^2(\xi)}}{\left(\mu_1\phi(\xi) + \mu_2\sqrt{R+\phi^2(\xi)} + 1\right)^i}, \\
V(\xi) &= A_0 + \sum_{i=1}^{m_v} \frac{A_i\phi^i(\xi) + B_i\phi^{i-1}(\xi)\sqrt{R+\phi^2(\xi)}}{\left(\mu_1\phi(\xi) + \mu_2\sqrt{R+\phi^2(\xi)} + 1\right)^i}
\end{aligned}
\tag{4.4}
$$

and the new variable $\phi = \phi(\xi)$ satisfying the Riccati equation, i.e.,

$$
\phi' - (R+\phi^2) = \frac{d\phi}{d\xi} - (R+\phi^2) = 0, \tag{4.5}
$$

where R, a_0, a_i, b_i, A_0, A_i and B_i $(i = 1, 2, \ldots, m_i)$ are constants to be determined later.

For the WBK system, According to *Step 3* in Section 2, by balancing the highest nonlinear terms and the highest-order partial derivative terms in (4.3), gives $m_u = 1$ and $m_v = 2$. So we have

$$
\begin{aligned}
U(\xi) &= a_0 + \frac{a_1\phi(\xi) + b_1\sqrt{R+\phi^2(\xi)}}{\mu_1\phi(\xi) + \mu_2\sqrt{R+\phi^2(\xi)} + 1}, \\
V(\xi) &= A_0 + \frac{A_1\phi(\xi) + B_1\sqrt{R+\phi^2(\xi)}}{\mu_1\phi(\xi) + \mu_2\sqrt{R+\phi^2(\xi)} + 1} + \frac{A_2\phi^2(\xi) + B_2\phi(\xi)\sqrt{R+\phi^2(\xi)}}{\left(\mu_1\phi(\xi) + \mu_2\sqrt{R+\phi^2(\xi)} + 1\right)^2}.
\end{aligned}
\tag{4.6}
$$

According to *Step 4* in Section 2, with the aid of *Maple*, substituting (4.6) along with (4.5) into (4.3), yields a set of algebraic equations for $\phi^i(\xi)(\sqrt{R+\phi^2(\xi)})^j$, $(i = 0, 1, \ldots; j = 0, 1)$. Setting the coefficients of these terms $\phi^i(\xi)\left(\sqrt{R+\phi^2(\xi)}\right)^j$ of the resulting system's numerator to zero yields a set of over-determined algebraic equations.

According to *Step 5* in Section 2, by use of the *Maple* soft package "Charsets" by Dongming Wang, we get the explicit expressions for a_0, a_1, b_1, A_0, A_1, B_1 A_2, B_2, μ_1, μ_2, k and λ.

It is well known that the general solutions of Eq. (4.5) are

(1) when $R < 0$,

$$
\phi(\xi) = -\sqrt{-R}\tanh(\sqrt{-R}\xi), \qquad \phi(\xi) = -\sqrt{-R}\coth(\sqrt{-R}\xi), \tag{4.7}
$$

(2) when $R = 0$,

$$
\phi(\xi) = -\frac{1}{\xi}, \tag{4.8}
$$

(3) when $R > 0$,

$$
\phi(\xi) = \sqrt{R}\tan(\sqrt{R}\xi), \qquad \phi(\xi) = -\sqrt{R}\cot(\sqrt{R}\xi). \tag{4.9}
$$

Thus according to system (4.2), (4.7)–(4.9) and the conclusions in *Step 5*, we can obtain following rational formal travelling-wave solutions of WBK system.

Note: Since tan- and cot-type solution appear in pairs with tanh- and coth-type solutions, respectively, we omit them in this paper, i.e., just considering the condition $R < 0$. In addition, some rational solutions are also omitted.

Family 1

$$u_{11} = -\lambda + \frac{\pm\sqrt{\alpha+\beta^2}k\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm \sqrt{-\alpha\mu_2^2R - \beta^2\mu_2^2R + \alpha + \beta^2}k\sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi)}{1 \pm \mu_2\sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi)}, \tag{4.10.1}$$

$$\begin{aligned} v_{11} = {} & \frac{k^2R\left(-\alpha\sqrt{\alpha+\beta^2} - \sqrt{\alpha+\beta^2}\beta^2 \pm \beta\alpha \pm \beta^3\right)}{\sqrt{\alpha+\beta^2}} + \frac{\left(\alpha+\beta^2+\beta\sqrt{\alpha+\beta^2}\right)k^2R\tanh^2(\sqrt{-R}\xi)}{(1 \pm \mu_2\sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi))^2} \\ & \pm \frac{\left(k^2R\mu_2\alpha + k^2R\mu_2\beta^2 \pm \beta k^2\sqrt{\alpha+\beta^2}\mu_2R\right)\sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi)}{1 \pm \mu_2\sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi)} \\ & \pm \frac{\left(\sqrt{\alpha+\beta^2}k^2 \pm \beta k^2\right)\sqrt{-\alpha\mu_2^2R - \beta^2\mu_2^2R + \alpha + \beta^2}\tanh(\sqrt{-R}\xi)R\mathrm{isech}(\sqrt{-R}\xi)}{(1 \pm \mu_2\sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi))^2}, \end{aligned} \tag{4.10.2}$$

$$u_{12} = -\lambda + \frac{\pm\sqrt{\alpha+\beta^2}k\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \sqrt{-\alpha\mu_2^2R - \beta^2\mu_2^2R + \alpha + \beta^2}k\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi)}{1 \pm \mu_2\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi)}, \tag{4.11.1}$$

$$\begin{aligned} v_{12} = {} & \frac{k^2R\left(-\alpha\sqrt{\alpha+\beta^2} - \sqrt{\alpha+\beta^2}\beta^2 \pm \beta\alpha \pm \beta^3\right)}{\sqrt{\alpha+\beta^2}} + \frac{\left(\alpha+\beta^2+\beta\sqrt{\alpha+\beta^2}\right)k^2R\coth^2(\sqrt{-R}\xi)}{(1 \pm \mu_2\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi))^2} \\ & \pm \frac{\left(k^2R\mu_2\alpha + k^2R\mu_2\beta^2 \pm \beta k^2\sqrt{\alpha+\beta^2}\mu_2R\right)\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi)}{1 \pm \mu_2\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi)} \\ & \pm \frac{\left(\sqrt{\alpha+\beta^2}k^2 \pm \beta k^2\right)\sqrt{-\alpha\mu_2^2R - \beta^2\mu_2^2R + \alpha + \beta^2}\coth(\sqrt{-R}\xi)R\mathrm{csch}(\sqrt{-R}\xi)}{(1 \pm \mu_2\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi))^2}, \end{aligned} \tag{4.11.2}$$

where $\xi = k(x+\lambda t)$, $R < 0$, μ_2, k and λ are arbitrary constants.

Family 2

$$u_{21} = a_0 \pm 2\frac{\sqrt{-\alpha\mu_2^2R - \beta^2\mu_2^2R + \alpha + \beta^2}k\sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi)}{1 \pm \mu_2\sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi)}, \tag{4.12.1}$$

$$\begin{aligned} v_{21} = {} & A_0 \pm \frac{(2k^2R\mu_2\alpha + 2k^2R\mu_2\beta^2)\sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi)}{1 \pm \mu_2\sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi)} + \frac{2k^2(\alpha+\beta^2)R\tanh^2(\sqrt{-R}\xi)}{(1 \pm \mu_2\sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi))^2} \\ & \pm \frac{2\beta k^2\sqrt{-\alpha\mu_2^2R - \beta^2\mu_2^2R + \alpha + \beta^2}\tanh(\sqrt{-R}\xi)R\mathrm{isech}(\sqrt{-R}\xi)}{(1 \pm \mu_2\sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi))^2}, \end{aligned} \tag{4.12.2}$$

$$u_{22} = a_0 \pm 2\frac{\sqrt{-\alpha\mu_2^2R - \beta^2\mu_2^2R + \alpha + \beta^2}k\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi)}{1 \pm \mu_2\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi)}, \tag{4.13.1}$$

$$\begin{aligned} v_{22} = {} & A_0 \pm \frac{(2k^2R\mu_2\alpha + 2k^2R\mu_2\beta^2)\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi)}{1 \pm \mu_2\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi)} + \frac{2k^2(\alpha+\beta^2)R\coth^2(\sqrt{-R}\xi)}{(1 \pm \mu_2\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi))^2} \\ & \pm \frac{2\beta k^2\sqrt{-\alpha\mu_2^2R - \beta^2\mu_2^2R + \alpha + \beta^2}\coth(\sqrt{-R}\xi)R\mathrm{csch}(\sqrt{-R}\xi)}{(1 \pm \mu_2\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi))^2}, \end{aligned} \tag{4.13.2}$$

where $\xi = k(x+\lambda t)$, $a_0 = -\frac{\lambda\sqrt{-\alpha\mu_2^2R-\beta^2\mu_2^2R+\alpha+\beta^2}\pm k\alpha\mu_2R\pm k\beta^2\mu_2R}{\sqrt{-\alpha\mu_2^2R-\beta^2\mu_2^2R+\alpha+\beta^2}}$, $A_0 = -\frac{(2\alpha\mu_2^2R-\alpha+2\beta^2\mu_2^2R-\beta^2)k^2R}{\mu_2^2R-1}$, $R<0$, μ_2, k and λ are arbitrary constants.

Family 3

$$u_{31} = -\lambda \pm 2\frac{\sqrt{\beta^2R\mu_1^2+\alpha+\beta^2+\alpha R\mu_1^2}k\sqrt{-R}i\mathrm{sech}(\sqrt{-R}\xi)}{-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi)+1}, \tag{4.14.1}$$

$$\begin{aligned} v_{31} = A_0 &+ \frac{-4(\beta^2R\mu_1^2+\alpha+\beta^2+\alpha R\mu_1^2)k^2\mu_1R\sqrt{-R}\tanh(\sqrt{-R}\xi)}{-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi)+1} \\ &\pm \frac{2\beta k^2\sqrt{\beta^2R\mu_1^2+\alpha+\beta^2+\alpha R\mu_1^2}R\sqrt{-R}i\mu_1\mathrm{sech}(\sqrt{-R}\xi)}{-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi)+1} - \frac{A_2R\tanh^2(\sqrt{-R}\xi)}{(-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi)+1)^2} \\ &\pm \frac{\left(2\beta k^2\sqrt{\beta^2R\mu_1^2+\alpha+\beta^2+\alpha R\mu_1^2}(R\mu_1^2+1)\right)\tanh(\sqrt{-R}\xi)Ri\mathrm{sech}(\sqrt{-R}\xi)}{(-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi)+1)^2}, \end{aligned} \tag{4.14.2}$$

$$u_{32} = -\lambda \pm 2\frac{\sqrt{\beta^2R\mu_1^2+\alpha+\beta^2+\alpha R\mu_1^2}k\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi)}{-\mu_1\sqrt{-R}\coth(\sqrt{-R}\xi)+1}, \tag{4.15.1}$$

$$\begin{aligned} v_{32} = A_0 &+ \frac{-4(\beta^2R\mu_1^2+\alpha+\beta^2+\alpha R\mu_1^2)k^2\mu_1R\sqrt{-R}\coth(\sqrt{-R}\xi)}{-\mu_1\sqrt{-R}\coth(\sqrt{-R}\xi)+1} \\ &\pm \frac{2\beta k^2\sqrt{\beta^2R\mu_1^2+\alpha+\beta^2+\alpha R\mu_1^2}R\mu_1\sqrt{-R}i\mathrm{csch}(\sqrt{-R}\xi)}{-\mu_1\sqrt{-R}\coth(\sqrt{-R}\xi)+1} - \frac{A_2R\coth^2(\sqrt{-R}\xi)}{(-\mu_1\sqrt{-R}\coth(\sqrt{-R}\xi)+1)^2} \\ &\pm \frac{\left(2\beta k^2\sqrt{\beta^2R\mu_1^2+\alpha+\beta^2+\alpha R\mu_1^2}(R\mu_1^2+1)\right)\coth(\sqrt{-R}\xi)R\mathrm{csch}(\sqrt{-R}\xi)}{(-\mu_1\sqrt{-R}\coth(\sqrt{-R}\xi)+1)^2}, \end{aligned} \tag{4.15.2}$$

where $\xi = k(x+\lambda t)$, $A_0 = -(2\beta^2R\mu_1^2+2\alpha R\mu_1^2+\beta^2+\alpha)k^2R$, $A_2 = -4\beta^2k^2R\mu_1^2-2\alpha k^2-2\beta^2k^2-4\alpha k^2R\mu_1^2-2\beta^2k^2R^2\mu_1^4-2\mu_1^4\alpha k^2R^2$, $R<0$, μ_1, k and λ are arbitrary constants.

Family 4

$$u_{41} = \frac{\pm 2\beta^2k\mu_1R \pm 2\alpha kR\mu_1 - \lambda\sqrt{\beta^2+\alpha}}{\sqrt{\beta^2+\alpha}} \pm 2\frac{\sqrt{\beta^2+\alpha}(1+\mu_1^2R)k\sqrt{-R}\tanh(\sqrt{-R}\xi)}{-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi)+1}, \tag{4.16.1}$$

$$\begin{aligned} v_{41} = A_0 &- \frac{4k^2R\mu_1(1+\mu_1^2R)\left(\alpha+\beta^2\pm\beta\sqrt{\beta^2+\alpha}\right)\sqrt{-R}\tanh(\sqrt{-R}\xi)}{-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi)+1} \\ &+ \frac{2k^2(1+\mu_1^2R)^2\left(\alpha+\beta^2\pm\beta\sqrt{\beta^2+\alpha}\right)R\tanh^2(\sqrt{-R}\xi)}{(-\mu_1\sqrt{-R}\tanh(\sqrt{-R}\xi)+1)^2}, \end{aligned} \tag{4.16.2}$$

$$u_{42} = \frac{\pm 2\beta^2k\mu_1R \pm 2\alpha kR\mu_1 - \lambda\sqrt{\beta^2+\alpha}}{\sqrt{\beta^2+\alpha}} \pm 2\frac{\sqrt{\beta^2+\alpha}(1+\mu_1^2R)k\sqrt{-R}\coth(\sqrt{-R}\xi)}{-\mu_1\sqrt{-R}\coth(\sqrt{-R}\xi)+1}, \tag{4.17.1}$$

$$\begin{aligned} v_{42} = A_0 &- \frac{4k^2R\mu_1(1+\mu_1^2R)\left(\alpha+\beta^2\pm\beta\sqrt{\beta^2+\alpha}\right)\sqrt{-R}\coth(\sqrt{-R}\xi)}{-\mu_1\sqrt{-R}\coth(\sqrt{-R}\xi)+1} \\ &+ \frac{2k^2(1+\mu_1^2R)^2\left(\alpha+\beta^2\pm\beta\sqrt{\beta^2+\alpha}\right)R\coth^2(\sqrt{-R}\xi)}{(-\mu_1\sqrt{-R}\coth(\sqrt{-R}\xi)+1)^2}, \end{aligned} \tag{4.17.2}$$

where $\xi = k(x+\lambda t)$, $A_0 = -2\left(\alpha+\beta^2\pm\beta\sqrt{\beta^2+\alpha}\right)(R\mu_1^2+1)k^2R$, $R<0$, μ_1, k and λ are arbitrary constants.

Family 5

$$u_{51} = a_0 + \frac{\pm\sqrt{\beta^2+\alpha}(R+1)k\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm \left(-\sqrt{\beta^2+\alpha}R - \sqrt{\beta^2+\alpha}\right)k\sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi)}{\pm\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm \sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi)+1}, \tag{4.18.1}$$

$$\begin{aligned} v_{51} = A_0 &+ \frac{2Rk^2\left(-\beta^2-\alpha\pm\beta\sqrt{\beta^2+\alpha}\right)(R+1)\sqrt{-R}\tanh(\sqrt{-R}\xi)}{(\pm\sqrt{-R}\tanh(\sqrt{-R}\xi)\pm\sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi)+1)(R-1)} \\ &\pm \frac{2Rk^2\left(-\beta^2-\alpha\pm\beta\sqrt{\beta^2+\alpha}\right)(R+1)\sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi)}{(\pm\sqrt{-R}\tanh(\sqrt{-R}\xi)\pm\sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi)+1)(R-1)} \\ &- \frac{(R+1)^2k^2\left(-\beta^2-\alpha\pm\beta\sqrt{\beta^2+\alpha}\right)R\tanh^2(\sqrt{-R}\xi)}{(\pm\sqrt{-R}\tanh(\sqrt{-R}\xi)\pm\sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi)+1)^2(R-1)} \\ &\pm \frac{(R+1)^2k^2\left(-\beta^2-\alpha\pm\beta\sqrt{\beta^2+\alpha}\right)\tanh(\sqrt{-R}\xi)R\mathrm{isech}(\sqrt{-R}\xi)}{(\pm\sqrt{-R}\tanh(\sqrt{-R}\xi)\pm\sqrt{-R}\mathrm{isech}(\sqrt{-R}\xi)+1)^2(R-1)}, \end{aligned} \tag{4.18.2}$$

$$u_{52} = a_0 + \frac{\pm\sqrt{\beta^2+\alpha}(R+1)k\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \left(-\sqrt{\beta^2+\alpha}R - \sqrt{\beta^2+\alpha}\right)k\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi)}{\pm\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi)+1}, \tag{4.19.1}$$

$$\begin{aligned} v_{52} = A_0 &+ \frac{2Rk^2\left(-\beta^2-\alpha\pm\beta\sqrt{\beta^2+\alpha}\right)(R+1)\sqrt{-R}\coth(\sqrt{-R}\xi)}{\left(\pm\sqrt{-R}\coth(\sqrt{-R}\xi)\pm\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi)+1\right)(R-1)} \\ &\pm \frac{2Rk^2\left(-\beta^2-\alpha\pm\beta\sqrt{\beta^2+\alpha}\right)(R+1)\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi)}{\left(\pm\sqrt{-R}\coth(\sqrt{-R}\xi)\pm\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi)+1\right)(R-1)} \\ &- \frac{(R+1)^2k^2\left(-\beta^2-\alpha\pm\beta\sqrt{\beta^2+\alpha}\right)R\coth^2(\sqrt{-R}\xi)}{\left(\pm\sqrt{-R}\coth(\sqrt{-R}\xi)\pm\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi)+1\right)^2(R-1)} \\ &\pm \frac{(R+1)^2k^2\left(-\beta^2-\alpha\pm\beta\sqrt{\beta^2+\alpha}\right)\coth(\sqrt{-R}\xi)R\mathrm{csch}(\sqrt{-R}\xi)}{\left(\pm\sqrt{-R}\coth(\sqrt{-R}\xi)\pm\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi)+1\right)^2(R-1)}, \end{aligned} \tag{4.19.2}$$

where $\xi = k(x+\lambda t)$, $A_0 = \frac{((\pm 13R-7)(\beta^2+\alpha)^{\frac{3}{2}}\beta \pm (1-R)(\alpha\beta+\beta^3)\sqrt{\beta^2+\alpha}-(2\alpha\beta^2+\alpha^2+\beta^4)(5R+1))k^2R}{-\frac{(5R+1)(R-1)(\beta^2+\alpha)}{(R+1)}}$, $a_0 = \frac{\pm kR\alpha \pm k\beta^2R-\sqrt{\beta^2+\alpha}\lambda}{\sqrt{\beta^2+\alpha}}$, $R<0$, k and λ are arbitrary constants.

Remark 4. The solution (4.10) can be reduced to the solution 4 in [11], when $\alpha+\beta^2>0$, $k=\sqrt{-R}$ and $\mu_2=0$. The solution (4.11) can be reduced to the solution 2 in [11], when $\alpha+\beta^2>0$, $k=\sqrt{-R}$ and $\mu_2=0$. The solution (4.14) can be reduced to the solution 2 in [11], when $\alpha+\beta^2<0$, $k=\sqrt{-R}$ and $\mu_1=0$. The solution (4.15) can be reduced to the solution 4 in [11], the solution (4.16) can be reduced to the solution 1 in [11], the solution (4.17) can be reduced to the solution 3 in [11], when $\alpha+\beta^2>0$, $k=\sqrt{-R}$ and $\mu_1=0$. The other solutions obtained here, to our knowledge, are all new families of exact solutions of the WBK system.

5. Generalized Riccati equation rational expansion method

In this section we would like extend the Riccati equation rational expansion method, i.e., using generalized ansätz to replace (2.4) and solutions of generalized Riccati equation to replace the variables in (2.4).

Consider the $(2+1)$-dimension Burgers system, i.e.,

$$\begin{aligned} &-u_t + uu_y + \alpha v u_x + \beta u_{yy} + \alpha\beta u_{xx} = 0,\\ &u_x - v_y = 0. \end{aligned} \tag{5.1}$$

According to the *Step 1* in Section 2, we make the following travelling wave transformation:

$$u(x,y,t) = U(\xi), \qquad v(x,y,t) = V(\xi), \quad \xi = k(x + ly + \lambda t), \tag{5.2}$$

where k and λ are constants to be determined later, and thus system (5.1) becomes

$$\begin{aligned} &-\lambda U' + lUU' + \alpha VU' + \beta k l^2 U'' + \alpha\beta k U'' = 0,\\ &U' - lV' = 0. \end{aligned} \tag{5.3}$$

According to *Step 2* in Section 2, We suppose that $(2+1)$-dimension Burgers system has the following formal travelling wave solution:

$$\begin{aligned} U(\xi) &= a_0 + \sum_{i=1}^{m_u} \frac{a_i\phi^i(\xi) + b_i\phi^{i-1}(\xi)\phi'(\xi) + c_i\frac{1}{\phi^i(\xi)}}{(\mu_1\phi(\xi) + \mu_2\phi'(\xi) + \mu_3\frac{1}{\phi} + 1)^i},\\ V(\xi) &= A_0 + \sum_{i=1}^{m_v} \frac{A_i\phi^i(\xi) + B_i\phi^{i-1}(\xi)\phi'(\xi) + C_i\frac{1}{\phi^i(\xi)}}{\left(\mu_1\phi(\xi) + \mu_2\phi'(\xi) + \mu_3\frac{1}{\phi} + 1\right)^i}, \end{aligned} \tag{5.4}$$

and the new variable $\phi = \phi(\xi)$ satisfying

$$\phi' - (h_1 + h_2\phi^2) = \frac{\mathrm{d}\phi}{\mathrm{d}\xi} - (h_1 + h_2\phi^2) = 0, \tag{5.5}$$

where R, a_0, a_i, b_i, c_i, A_0, A_i, B_i and C_i $(i = 1,2,\ldots,m_i)$ are constants to be determined later.

For the $(2+1)$-dimension Burgers system, according to *Step 3* in Section 2, by balancing the highest nonlinear terms and the highest-order partial derivative terms in (5.3), gives $m_u = 1$ and $m_v = 1$. So we have

$$\begin{aligned} U(\xi) &= a_0 + \frac{a_1\phi^i(\xi) + b_1\phi'(\xi) + c_i\frac{1}{\phi(\xi)}}{\mu_1\phi(\xi) + \mu_2\phi'(\xi) + \mu_3\frac{1}{\phi(\xi)} + 1},\\ V(\xi) &= A_0 + \frac{A_1\phi^i(\xi) + B_1\phi'(\xi) + C_i\frac{1}{\phi(\xi)}}{\mu_1\phi(\xi) + \mu_2\phi'(\xi) + \mu_3\frac{1}{\phi(\xi)} + 1}. \end{aligned} \tag{5.6}$$

According to *Step 4* in Section 2, with the aid of *Maple*, substituting (5.6) along with (5.5) into (5.3), yields a set of algebraic equations for $\phi^i(\xi)$ $(i = 0,1,\ldots)$. Setting the coefficients of these terms $\phi^i(\xi)$ of the resulting system's numerator to zero yields a set of over-determined algebraic equations.

According to *Step 5* in Section 2, by use of the *Maple* soft package "Charsets" by Dongming Wang, we get the explicit expressions for a_0, a_1, b_1, c_1, A_0, A_1, B_1, C_1, μ_1, μ_2, μ_3, k, l and λ.

We know that the general solutions of Eq. (5.5) are

(1) when $h_1 = \frac{1}{2}$ and $h_2 = -\frac{1}{2}$,

$$\phi(\xi) = \tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi), \qquad \phi(\xi) = \coth(\xi) \pm \mathrm{csch}(\xi), \tag{5.7}$$

(2) when $h_1 = h_2 = \pm\frac{1}{2}$,

$$\phi(\xi) = \sec(\xi) \pm \tan(\xi), \qquad \phi(\xi) = \csc(\xi) \pm \cot(\xi), \tag{5.8}$$

(3) when $h_1 = 1$ and $h_2 = -1$,

$$\phi(\xi) = \tanh(\xi), \qquad \phi(\xi) = \coth(\xi). \tag{5.9}$$

(4) when $h_1 = h_2 = 1$,

$$\phi(\xi) = \tan(\xi), \tag{5.10}$$

(5) when $h_1 = h_2 = -1$,

$$\phi(\xi) = \cot(\xi), \tag{5.11}$$

(6) when $h_1 = 0$ and $h_2 \neq 0$,

$$\phi(\xi) = -\frac{1}{h_2\xi + c_0}. \tag{5.12}$$

Thus according to system (5.2), (5.7)–(5.12) and the conclusions in *Step 5*, we can obtain following rational formal travelling-wave solutions of (2 + 1)-dimension Burgers system.

Note: Here like Section 4, we omit tan-, cot-type and rational solutions and just list new solutions compared with the result in Section 4.

Family 1

$$u_{11} = a_0 + \frac{\frac{l\beta k}{2\mu_2^2}(4\mu_1^2-\mu_2^2)(4\mu_1^2+\mu_2)(\tanh(\xi)\pm i\mathrm{sech}(\xi))^2}{\mu_1(\tanh(\xi)\pm i\mathrm{sech}(\xi))^2+\mu_2(\tanh(\xi)\pm i\mathrm{sech}(\xi))(\mathrm{sech}^2(\xi)\mp i\mathrm{sech}(\xi)\tanh(\xi))+\tanh(\xi)\pm i\mathrm{sech}(\xi)+\mu_3}$$
$$+\frac{l\beta k\mu_1(8\mu_1^2-2\mu_2^2)(\tanh(\xi)\pm i\mathrm{sech}(\xi))(\mathrm{sech}^2(\xi)\mp i\mathrm{sech}(\xi)\tanh(\xi))-\mu_2c_1}{\mu_2(\mu_1(\tanh(\xi)\pm i\mathrm{sech}(\xi))^2+\mu_2(\tanh(\xi)\pm i\mathrm{sech}(\xi))(\mathrm{sech}^2(\xi)\mp i\mathrm{sech}(\xi)\tanh(\xi))+\tanh(\xi)\pm i\mathrm{sech}(\xi)+\mu_3)}, \tag{5.13.1}$$

$$v_{11} = A_0 + \frac{\frac{\beta k}{2\mu_2^2}(4\mu_1^2-\mu_2^2)(4\mu_1^2+\mu_2)(\tanh(\xi)\pm i\mathrm{sech}(\xi))^2}{\mu_1(\tanh(\xi)\pm i\mathrm{sech}(\xi))^2+\mu_2(\tanh(\xi)\pm i\mathrm{sech}(\xi))(\mathrm{sech}^2(\xi)\mp i\mathrm{sech}(\xi)\tanh(\xi))+\tanh(\xi)\pm i\mathrm{sech}(\xi)+\mu_3}$$
$$+\frac{\beta k\mu_1(8\mu_1^2-2\mu_2^2)(\tanh(\xi)\pm i\mathrm{sech}(\xi))(\mathrm{sech}^2(\xi)\mp i\mathrm{sech}(\xi)\tanh(\xi))-\frac{\mu_2}{l}c_1}{\mu_2(\mu_1(\tanh(\xi)\pm i\mathrm{sech}(\xi))^2+\mu_2(\tanh(\xi)\pm i\mathrm{sech}(\xi))(\mathrm{sech}^2(\xi)\mp i\mathrm{sech}(\xi)\tanh(\xi))+\tanh(\xi)\pm i\mathrm{sech}(\xi)+\mu_3)}, \tag{5.13.2}$$

$$u_{12} = a_0 + \frac{\frac{l\beta k}{2\mu_2^2}(4\mu_1^2-\mu_2^2)(4\mu_1^2+\mu_2)(\coth(\xi)\pm \mathrm{csch}(\xi))^2}{\mu_1(\coth(\xi)\pm \mathrm{csch}(\xi))^2+\mu_2(\coth(\xi)\pm \mathrm{csch}(\xi))(\mathrm{csch}^2(\xi)\pm \mathrm{csch}(\xi)\coth(\xi))+\coth(\xi)\pm \mathrm{csch}(\xi)+\mu_3}$$
$$+\frac{l\beta k\mu_1(8\mu_1^2-2\mu_2^2)(\coth(\xi)\pm \mathrm{csch}(\xi))(\mathrm{csch}^2(\xi)\mp i\mathrm{csch}(\xi)\coth(\xi))-\mu_2c_1}{\mu_2(\mu_1(\coth(\xi)\pm \mathrm{csch}(\xi))^2+\mu_2(\coth(\xi)\pm \mathrm{csch}(\xi))(\mathrm{csch}^2(\xi)\pm \mathrm{csch}(\xi)\coth(\xi))+\coth(\xi)\pm \mathrm{csch}(\xi)+\mu_3)}, \tag{5.14.1}$$

$$v_{12} = A_0 + \frac{\frac{\beta k}{2\mu_2^2}(4\mu_1^2-\mu_2^2)(4\mu_1^2+\mu_2)(\coth(\xi)\pm \mathrm{csch}(\xi))^2}{\mu_1(\coth(\xi)\pm \mathrm{csch}(\xi))^2+\mu_2(\coth(\xi)\pm \mathrm{csch}(\xi))(\mathrm{csch}^2(\xi)\pm \mathrm{csch}(\xi)\coth(\xi))+\coth(\xi)\pm \mathrm{csch}(\xi)+\mu_3}$$
$$+\frac{\beta k\mu_1(8\mu_1^2-2\mu_2^2)(\coth(\xi)\pm \mathrm{csch}(\xi))(\mathrm{csch}^2(\xi)\mp i\mathrm{csch}(\xi)\coth(\xi))-\frac{\mu_2}{l}c_1}{\mu_2(\mu_1(\coth(\xi)\pm \mathrm{csch}(\xi))^2+\mu_2(\coth(\xi)\pm \mathrm{csch}(\xi))(\mathrm{csch}^2(\xi)\pm \mathrm{csch}(\xi)\coth(\xi))+\coth(\xi)\pm \mathrm{csch}(\xi)+\mu_3)}, \tag{5.14.2}$$

where $\xi = k(x + ly + \lambda t)$, $a_0 = \frac{2\beta kl^2\mu_1\mu_2^2+2\beta kl^2\mu_2\mu_1-8\mu_1^3\alpha\beta k-8\mu_1^3\beta kl^2+2\alpha\beta k\mu_2\mu_1+2\alpha\beta k\mu_1\mu_2^2-\alpha A_0\mu_2^2+\lambda\mu_2^2}{\mu_2^2 l}$, $\mu_3 = -\frac{\mu_1(8\mu_1^2-\mu_2^2-2\mu_2)}{\mu_2^2}$, $c_1 = -\frac{l\beta k(4\mu_1^2-\mu_2)(8\mu_1^2-\mu_2^2-2\mu_2)(8\mu_1^2-\mu_2^2)}{2\mu_2^4}$, μ_1, μ_2, l and λ are arbitrary constants.

Family 2

$$u_{21} = a_0 + \frac{2l\beta k(-\mu_2^2+\mu_1^2)(2\mu_1^2+\mu_2)\tanh^2(\xi)}{\mu_2^2(\mu_1\tanh^2(\xi)+\mu_2\tanh(\xi)\mathrm{sech}^2(\xi)+\mu_3+\tanh(\xi))} + \frac{4l\beta k\mu_1\mu_2^3(-\mu_2^2+\mu_1^2)\tanh(\xi)\mathrm{sech}^2(\xi) - 2l\beta k(2\mu_1^2-\mu_2)(-\mu_2^2+2\mu_1^2-\mu_2)(-\mu_2^2+\mu_1^2)}{\mu_2^4(\mu_1\tanh^2(\xi)+\mu_2\tanh(\xi)\mathrm{sech}^2(\xi)+\mu_3+\tanh(\xi))}, \tag{5.15.1}$$

$$v_{21} = A_0 + \frac{2\beta k(-\mu_2^2+\mu_1^2)(2\mu_1^2+\mu_2)\tanh^2(\xi)}{\mu_2^2(\mu_1\tanh^2(\xi)+\mu_2\tanh(\xi)\mathrm{sech}^2(\xi)+\mu_3+\tanh(\xi))} + \frac{4\beta k\mu_1\mu_2^3(-\mu_2^2+\mu_1^2)\tanh(\xi)\mathrm{sech}^2(\xi) - 2\beta k(2\mu_1^2-\mu_2)(-\mu_2^2+2\mu_1^2-\mu_2)(-\mu_2^2+\mu_1^2)}{\mu_2^4(\mu_1\tanh^2(\xi)+\mu_2\tanh(\xi)\mathrm{sech}^2(\xi)+\mu_3+\tanh(\xi))}, \tag{5.15.2}$$

$$u_{22} = a_0 + \frac{2l\beta k(-\mu_2^2+\mu_1^2)(2\mu_1^2+\mu_2)\coth^2(\xi)}{\mu_2^2(\mu_1\coth^2(\xi)+\mu_2\coth(\xi)\mathrm{csch}^2(\xi)+\mu_3+\coth(\xi))} + \frac{4l\beta k\mu_1\mu_2^3(-\mu_2^2+\mu_1^2)\coth(\xi)\mathrm{csch}^2(\xi) - 2l\beta k(2\mu_1^2-\mu_2)(-\mu_2^2+2\mu_1^2-\mu_2)(-\mu_2^2+\mu_1^2)}{\mu_2^4(\mu_1\coth^2(\xi)+\mu_2\coth(\xi)\mathrm{csch}^2(\xi)+\mu_3+\coth(\xi))}, \tag{5.16.1}$$

$$v_{22} = A_0 + \frac{2\beta k(-\mu_2^2+\mu_1^2)(2\mu_1^2+\mu_2)\coth^2(\xi)}{\mu_2^2(\mu_1\coth^2(\xi)+\mu_2\coth(\xi)\mathrm{csch}^2(\xi)+\mu_3+\coth(\xi))} + \frac{4\beta k\mu_1\mu_2^3(-\mu_2^2+\mu_1^2)\coth(\xi)\mathrm{csch}^2(\xi) - 2\beta k(2\mu_1^2-\mu_2)(-\mu_2^2+2\mu_1^2-\mu_2)(-\mu_2^2+\mu_1^2)}{\mu_2^4(\mu_1\coth^2(\xi)+\mu_2\coth(\xi)\mathrm{csch}^2(\xi)+\mu_3+\coth(\xi))}, \tag{5.16.2}$$

where $\xi = k(x+ly+\lambda t)$, $a_0 = \frac{4\beta kl^2\mu_1\mu_2^2+2\beta kl^2\mu_2\mu_1-4\mu_1^3\alpha\beta k-4\mu_1^3\beta kl^2+2\alpha\beta k\mu_2\mu_1+4\alpha\beta k\mu_1\mu_2^2-\alpha A_0\mu_2^2+\lambda\mu_2^2}{\mu_2^2 l}$, $\mu_3 = -\frac{\mu_1(-\mu_2^2+2\mu_1^2-\mu_2)}{\mu_2^2}$, μ_1, μ_2, l and λ are arbitrary constants.

6. Summary and conclusions

In summary, we have proposed a unified algebraic method: rational expansion method with symbolic computation, which greatly exceeds the applicability of the existing tanh method in obtaining multiple travelling wave solutions of the NLEEs.

In Section 3, when $\mu_1 = \mu_2 = 0$, our method just reduces to the Jacobi elliptic function expansion method [10].

In Section 4, when $\mu_1 = \mu_2 = 0$, our method just reduces to the tanh-method [6,7,9,10]; when $\mu_1 = 0$, our method just reduce to the projective Riccati equation expansion method [14–16].

As we can see in Section 5, we can naturally extend our method to generalized form, such as: we can replace system (2.4) by

$$U_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum_{r_1+\cdots+r_n=j} a^j_{r_1\cdots r_n} F_1^{r_1}\cdots F_n^{r_n}}{\sum_{l_1+\cdots+l_n=j} \mu^j_{l_1\cdots l_n} F_1^{l_1}\cdots F_n^{l_n} + 1}, \tag{6.1}$$

where a_{i0}, $a^j_{r_1\cdots r_n}$, $\mu^j_{l_1\cdots l_n}$ and ξ are differentiable function to be determined and $\frac{\mathrm{d}F_i}{\mathrm{d}\xi} = K_i(F_1,\ldots,F_n)$, where K_i are polynomial of F_i. It is clearly to see that (6.1) is also satisfying solving the recurrent relation or derivative relation for the terms of polynomial for computation closed.

The feature of our proposed method is that, we firstly propose a more general ansätz, in which the closed-form solutions of NLEEs that can be expressed as a finite series of some Jacobi elliptic function or solutions of subequation (like the Riccati equation) in rational form. The method provides us with new and more general travelling wave solutions that cannot be obtained by the tanh method, Jacobi elliptic function expansion methods, the homogeneous balance method and projective Riccati equation expansion method. This method is also computerizable. The key idea is to decompose NLEE system into a system of algebraic equations or an ordinary differential equation, which allows us to perform complicated and tedious algebraic computation by using symbolic computation. Finally, we point out that our method also can be applied to a large class of either integrable or non-integrable nonlinear coupled systems. The details for these cases will be investigated in our future works.

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