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An Approach for Solving Short-Wave Models for Camassa–Holm Equation and Degasperis–Procesi Equation*

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Abstract *In this paper, to construct exact solution of nonlinear partial differential equation, an easy-to-use approach is proposed. By means of the transformation of the independent variables and the travelling wave transformation, the partial differential equation is reduced to an ordinary differential equation. To solve the ordinary differential equation, we assume the soliton solution in the explicit expression and obtain the travelling wave solution. By the transformation back to the original independent variables, the soliton solution of the original partial differential equation is derived. We investigate the short wave model for the Camassa–Holm equation and the Degasperis–Procesi equation respectively. One-cusp soliton solution of the Camassa–Holm equation is obtained. One-loop soliton solution of the Degasperis–Procesi equation is also obtained, the approximation of which in a closed form can be obtained firstly by the Adomian decomposition method. The obtained results in a parametric form coincide perfectly with those given in the present reference. This illustrates the efficiency and reliability of our approach.*

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Key words: Camassa–Holm equation, Degasperis–Procesi equation, one-cusp soliton, one-loop soliton, Adomian decomposition method

1 Introduction

We consider the following partial differential equation (PDE) for a scalar function $u = u(x,t)$

$$\alpha u_x - u_{txx} = \beta u_x u_{xx} + u u_{xxx}\,, \tag{1}$$

where α and β are constant specified later and the subscripts t and x appended to u denote the partial differentiation. Equation (1) stems from the short-wave limit of the PDE,[1]

$$u_t + \alpha u_x - u_{txx} + (\beta+1) u u_x = \beta u_x u_{xx} + u u_{xxx}\,. \tag{2}$$

For the special values of β, equation (2) turns out to be a completely integrable PDE. When $\beta = 2$, it becomes the Camassa–Holm equation (CH)[2−5] and when $\beta = 3$, it reduces to the Degasperis–Procesi equation (DP).[6,7] The recent study reveals equation (1) with $\beta = 2$ or $\beta = 3$ can be used to describe the long-term dynamics of short surface waves.[8−10]

In this paper, we propose an approach based on the transformation of the independent variables and the travelling wave transformation to obtain the exact solution of the PDE. The short wave models for the CH equation and the DP equation are investigated respectively. By our approach, the one-cusp soliton solution of the CH equation is obtained and the one-loop soliton solution of the DP equation is derived, which also can be obtained by employing the Adomian decomposition method (ADM).[11−17] The results we obtained coincide perfectly with those given in Ref. [1].

The paper is organized as follows. In the next section, the approach is described for solving the PDE. In Sec. 3, the short wave model for the CH equation is studied. In Sec. 4, the short wave model for the DP equation is studied and the ADM loop solution of it is shown. Finally, some important conclusions and the further study are presented.

2 Description of Alternate Approach

To obtain the soliton solution of the PDE, integrating Eq. (1) with respect to x once and setting the constant of the integration to be zero yields

$$u_{tx} - \alpha u + \frac{\beta-1}{2} u_x^2 + u u_{xx} = 0\,. \tag{3}$$

Similar to Vakhnenko [18], we introduce the new independent variables X and T, defined by

$$x = qX + W(X,T) + x_0\,, \quad t = T\,, \tag{4}$$

where $u(x,t) = W_T(X,T)$, q is constant associated with α and x_0 is arbitrary constant.

From Eq. (4) it follows that

$$\frac{\partial}{\partial X} = \gamma \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial T} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x}\,, \tag{5}$$

where

$$\gamma = q + W_X\,. \tag{6}$$

From Eq. (5), equation (3) can be written as

$$\theta(W_X, W_T, W_{TTX}) = 0\,. \tag{7}$$

Using the travelling wave transformation

$$W(X,T) = \phi(\xi)\,, \quad \xi = X - cT\,, \tag{8}$$

where c is a constant to be determined later, equation (3) is reduced to a nonlinear ordinary differential equation

$$P(\phi', \phi'', \phi''') = 0\,. \tag{9}$$

We assume that the travelling wave solution can be expressed in the following form

$$\phi(\xi) = \frac{B}{1+\exp(a\xi)}\,, \tag{10}$$

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for $\beta = 2$ and

$$\phi(\xi) = B\exp(a\xi)/[1+\exp(a\xi)]\,, \tag{11}$$

for $\beta = 3$, where a and B are constants to be determined.

Substituting Eq. (10) or Eq. (11) into Eq. (9) and setting all the coefficients of $\exp(ka\xi), k = (1,2,3,\ldots)$ to be zero, we get the over-determined system of linear algebraic equation with respect to c, a, B. By solving the system, we end up with the explicit expressions for c, a, B. According to (10) or (11), we obtain the soliton solution $\phi(\xi) = W(X,T)$ of Eq. (7).

So we get

$$u(x,t) = U(X,T) = W_T(X,T)\,, \tag{12}$$

and with Eq. (4), the solution of Eq. (1) in a parametric form is

$$u(X,t) = U(X,t)\,, \quad x(X,t) = qX + W(X,t) + x_0\,. \tag{13}$$

3 One-cusp Soliton Solution of Short Wave Model for CH Equation

The appropriate form of short wave model for the CH equation is provided by taking $\beta = 2$ and $\alpha = -2\kappa^2$ ($\kappa > 0$) in Eq. (1).[1] We write it in the form

$$u_{txx} + 2\kappa^2 u_x + 2u_x u_{xx} + u u_{xxx} = 0\,. \tag{14}$$

Integrating Eq. (14) with respect to x once and setting the constant of the integration to be zero, we have

$$u_{tx} + 2\kappa^2 u + u_x^2/2 + u u_{xx} = 0\,. \tag{15}$$

To obtain the soliton solution, we introduce the new independent variables X and T,

$$x = X/\kappa + W(X,T) + x_0, \quad t = T\,, \tag{16}$$

where

$$u(x,t) = U(X,T) = W_T(X,T)\,, \tag{17}$$

and x_0 is an arbitrary constant.

From Eq. (16) it follows that

$$\frac{\partial}{\partial X} = \gamma\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial T} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\,, \tag{18}$$

where

$$\gamma = 1/\kappa + W_X\,. \tag{19}$$

we have readily

$$u_x^2 = \gamma^{-2} U_x^2\,. \tag{20}$$

So we find

$$U_{TX} + 2\kappa^2\gamma U - \tfrac{1}{2}\gamma^{-1}U_X^2 = 0\,. \tag{21}$$

Substituting Eq. (17) into Eq. (21), we have

$$2W_{TTX} + 2\kappa W_X W_{TTX} + 4\kappa W_T + 4\kappa^3 W_X^2 W_T + 8\kappa^2 W_X W_T - \kappa W_{TX}^2 = 0\,. \tag{22}$$

Using the travelling wave transformation

$$W(X,T) = \phi(\xi), \quad \xi = X - cT\,, \tag{23}$$

where c is a constant to be determined later and substituting Eq. (23) into Eq. (22), we have

$$2c\phi''' + 2\kappa c\phi\phi''' - 4\kappa\phi' - 4\kappa^3(\phi')^3 - 8\kappa^2(\phi')^2 - \kappa c(\phi'')^2 = 0\,. \tag{24}$$

We assume that

$$\phi(\xi) = B/[1+\exp(a\xi)]\,, \tag{25}$$

where a and B are constants to be determined later.

We substitute Eq. (25) into Eq. (24) and then set all the coefficients of $\exp(ka\xi)$ $(k = 1,2,3,4,5)$ to be zero. We get the system of linear algebraic equations with respect to c, a, and B as follows:

$$\begin{aligned} &4Ba\kappa - 2Ba^3c = 0\,,\\ &4Ba^3c + 16Bak + B^2a^4\kappa c - 8B^2a^2\kappa^2 = 0\,,\\ &12Ba^3c - 16B^2a^2\kappa^2 + 24Ba\kappa + 4B^3a^3\kappa^3 \\ &\quad - 6B^2a^4\kappa c = 0\,. \end{aligned} \tag{26}$$

By solving the above system, we have

$$c = 2\kappa/a^2, \quad B = 4/a\kappa\,, \tag{27}$$

where a is arbitrary constant.

So we have

$$\phi(\xi) = \frac{2}{a\kappa} - \frac{2}{a\kappa}\tanh\Big(\frac{a}{2}\xi\Big)\,, \tag{28}$$

where

$$\xi = X - \frac{2\kappa}{a^2}T\,. \tag{29}$$

From Eqs. (16) and (17) the one-cusp soliton solution of Eq. (14) in a parametric form is

$$\begin{aligned} u(X,t) &= \frac{2}{a^2}\,\mathrm{sech}^2\Big(\frac{a}{2}\xi\Big)\,,\\ x(X,t) &= \frac{X}{\kappa} - \frac{2}{a\kappa}\tanh\Big(\frac{a}{2}\xi\Big) + x_0\,, \end{aligned} \tag{30}$$

where

$$\xi = X - \frac{2\kappa}{a^2}t\,. \tag{31}$$

Figure 1 shows the one-cusp soliton solution of the CH equation for $\kappa = 1$, $a = \sqrt{2}$, $t = 2$, $x_0 = 0$.

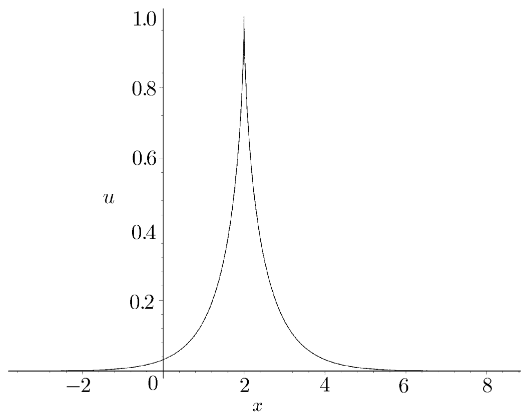

Fig. 1 The one-cusp soliton solution of CH equation for $\kappa = 1$, $a = \sqrt{2}$, $t = 2$, $x_0 = 0$.

Remark the one-cusp soliton solution of short wave model for the CH equation derived above coincides perfectly with those given in Ref. [1]. This illustrates the efficiency and reliability of the proposed approach.

4 One-loop Soliton Solution of Short Wave Model for DP Equation

4.1 *One-Loop Soliton Solution of Short Wave Model for DP Equation by Approach*

The appropriate form of short wave model for the DP equation is provided by taking $\beta = 3$ and $\alpha = -3\kappa^3$ ($\kappa > 0$) in Eq. (1).[1] We write it in the form

$$u_{txx} + 3\kappa^3 u_x + 3u_x u_{xx} + u u_{xxx} = 0\,. \tag{32}$$

Integrating Eq. (32) with respect to x once and setting the constant of the integration to be zero, we have

$$u_{tx} + 3\kappa^3 u + u_x^2 + u u_{xx} = 0\,. \tag{33}$$

To obtain the soliton solution, we introduce the new independent variables X and T

$$x = \frac{X}{\kappa} + W(X,T) + x_0, \quad t = T\,, \tag{34}$$

where

$$u(x,t) = U(X,T) = W_T(X,T)\,, \tag{35}$$

and x_0 is an arbitrary constant.

From Eq. (34) it follows that

$$\frac{\partial}{\partial X} = \gamma\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial T} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\,, \tag{36}$$

where

$$\gamma = \frac{1}{\kappa} + W_X\,. \tag{37}$$

So from Eq. (33) we find

$$U_{TX} + 3\kappa^3\gamma U = 0\,. \tag{38}$$

Substituting Eq. (35) into Eq. (38), we have

$$W_{TTX} + 3\kappa^2 W_T + 3\kappa^3 W_X W_T = 0\,. \tag{39}$$

Using the travelling wave transformation

$$W(X,T) = \phi(\xi), \quad \xi = X - cT\,, \tag{40}$$

where c is a constant to be determined later and substituting Eq. (40) into Eq. (39), we have

$$c\phi''' - 3\kappa^2\phi' - 3\kappa^3(\phi')^2 = 0\,. \tag{41}$$

We assume that

$$\phi(\xi) = \frac{B\exp(a\xi)}{1+\exp(a\xi)}\,, \tag{42}$$

where a and B are constants to be determined later.

We substitute Eq. (42) into Eq. (41) and then set all the coefficients of $\exp(ka\xi)$ $(k = 1,2,3)$ to be zero. We get the system of linear algebraic equations with respect to c, a and B as follows:

$$\begin{aligned} &Ba^3c - 3Ba\kappa^2 = 0\,,\\ &4Ba^3c + 6Ba\kappa^2 + 3B^2a^2\kappa^3 = 0\,. \end{aligned} \tag{43}$$

By solving the above system, we have

$$c = \frac{3\kappa^2}{a^2}, \quad B = -\frac{6}{a\kappa}\,, \tag{44}$$

where a is arbitrary constant.

So we have

$$\phi(\xi) = -\frac{3}{a\kappa} - \frac{3}{a\kappa}\tanh\Bigl(\frac{a}{2}\xi\Bigr)\,, \tag{45}$$

where

$$\xi = X - \frac{3\kappa^2}{a^2}T\,. \tag{46}$$

From Eqs. (34) and (35) the one-loop soliton solution of Eq. (32) in a parametric form is

$$\begin{aligned} u(X,t) &= \frac{9\kappa}{2a^2}\operatorname{sech}^2\Bigl(\frac{a}{2}\xi\Bigr)\,,\\ x(X,t) &= \frac{X}{\kappa} - \frac{3}{a\kappa}\tanh\Bigl(\frac{a}{2}\xi\Bigr) + x_0\,, \end{aligned} \tag{47}$$

where

$$\xi = X - \frac{3\kappa^2}{a^2}t\,. \tag{48}$$

Figure 2 shows the one-loop soliton solution of the DP equation for $\kappa = 1$, $a = (3/2)\sqrt{2}$, $t = 2$, $x_0 = 0$.

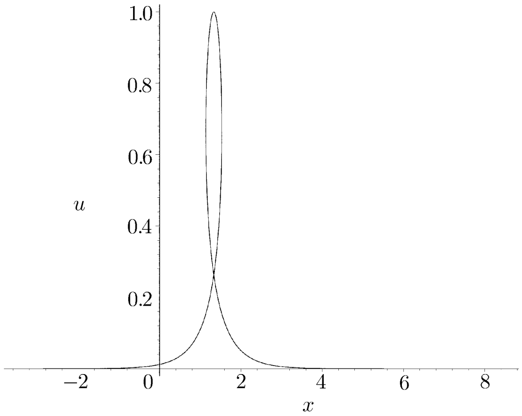

Fig. 2 he one-loop soliton solution of the DP equation for $\kappa = 1$, $a = (3/2)\sqrt{2}$, $t = 2$, $x_0 = 0$.

Remark The one-loop soliton solution of short wave model for the DP equation derived above coincides perfectly with those given in Ref. [1]. This illustrates the efficiency and reliability of the proposed approach.

4.2 *ADM One-Loop Soliton Solution of Short Wave Model for DP Equation*

The short wave model for the DP equation with the linear highest-order derivative with respect to spatial variables and temporal variable is not easy to be solved by the ADM for obtaining the loop soliton solution. We firstly propose the scheme to obtain the ADM one-loop soliton solution. Now we outline the scheme as follows.

To solve the DP equation by the ADM, we consider Eq. (41) with $c = 3\kappa^2/a^2$ and we have

$$\phi''' - a^2\phi' - \kappa a^2(\phi')^2 = 0\,, \tag{49}$$

with the appropriate initial conditions

$$\phi(0) = -3/a\kappa, \quad \phi'(0) = -3/2\kappa, \quad \phi''(0) = 0\,. \tag{50}$$

In an operator form, equation (49) can be written as

$$L\phi = a^2R\phi + \kappa a^2N\phi\,, \tag{51}$$

where the differential operator L is

$$L = \partial^3/\partial\xi^3\,, \tag{52}$$

the inverse operator L^{-1} is

$$L^{-1}(\cdot) = \int_0^\xi\int_0^\xi\int_0^\xi(\cdot)\,\mathrm{d}\xi\,, \tag{53}$$

the remaind linear operator R is

$$R = \partial/\partial\xi\,, \tag{54}$$

the nonlinear operator N is

$$N = \bigl(\partial/\partial\xi\bigr)^2\,. \tag{55}$$

Applying L^{-1} on both sides of Eq. (51), we have

$$\begin{aligned} \phi = {}&\phi(0) + \xi\phi'(0) + \frac{\xi^2}{2}\phi''(0)\\ &+ a^2L^{-1}R\phi + \kappa a^2L^{-1}N\phi\,. \end{aligned} \tag{56}$$

In terms of the ADM, ϕ can be represented as a series

$$\phi = \sum_{n=0}^{\infty}\phi_n\,. \tag{57}$$

So we have

$$\phi = \phi_0 + a^2 L^{-1}\Big(\sum_{n=0}^{\infty} \phi'_n\Big) + \kappa a^2 L^{-1}\Big(\sum_{n=0}^{\infty} A_n\Big), \qquad (58)$$

where A_n's are obtained by writing

$$\nu(\lambda) = \sum_{n=0}^{\infty} \lambda^n \phi_n, \quad N(\nu(\lambda)) = \sum_{n=0}^{\infty} \lambda^n A_n, \qquad (59)$$

where λ is a parameter for convenience. From (59) we deduce

$$A_n = \frac{1}{n!}\Big[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} N(\nu(\lambda))\Big]_{\lambda=0}, \quad n = 0, 1, \ldots \qquad (60)$$

Now, to determine the component ϕ_n $(n \geq 0)$, we employ the recurrence relation

$$\phi_0 = \phi(0) + \xi\phi'(0) + \frac{\xi^2}{2}\phi''(0),$$
$$\phi_{k+1} = a^2 L^{-1}(\phi_k) + \kappa a^2 L^{-1}(A_k), \quad k \geq 0, \qquad (61)$$

where A_k is the so-called Adomian polynomials.

The first few Adomian polynomials A_n are given by

$$A_0 = (\phi_{0x})^2, \quad A_1 = 2\phi_{0x}\phi_{1x},$$
$$A_2 = 2\phi_{0x}\phi_{2x} + (\phi_{1x})^2. \qquad (62)$$

Substituting Eq. (62) into Eq. (61) gives

$$\phi_0 = -\frac{3}{a\kappa} - \frac{3}{2\kappa}, \quad \phi_1 = \frac{a^2}{8\kappa}\xi^3,$$
$$\phi_2 = -\frac{a^4}{80\kappa}\xi^5. \qquad (63)$$

Thus this gives rise to the approximate solution of Eq. (49) in a series form

$$\phi(\xi) = -\frac{3}{a\kappa} - \frac{3}{2\kappa} + \frac{a^2}{8\kappa}\xi^3 - \frac{a^4}{80\kappa}\xi^5 + \cdots, \qquad (64)$$

and in a closed form by

$$\phi(\xi) = -\frac{3}{a\kappa} - \frac{3}{a\kappa}\tanh\Big(\frac{a}{2}\xi\Big), \qquad (65)$$

which is the exact solution of Eq. (49).

Consequently, we have

$$W(X,T) = -\frac{3}{a\kappa} - \frac{3}{a\kappa}\tanh\Big[\frac{a}{2}\Big(X - \frac{3\kappa^2}{a^2}T\Big)\Big], \qquad (66)$$

and

$$W_T(X,T) = \frac{9\kappa}{2a^2}\operatorname{sech}^2\Big[\frac{a}{2}\Big(X - \frac{3\kappa^2}{a^2}T\Big)\Big]. \qquad (67)$$

Thus from Eqs. (66) and (67), the solution of Eq. (32) is given in a parametric form:

$$u(X,t) = \frac{9\kappa}{2a^2}\operatorname{sech}^2\Big(\frac{a}{2}\xi\Big),$$
$$x(X,t) = \frac{X}{\kappa} - \frac{3}{a\kappa}\tanh\Big(\frac{a}{2}\xi\Big) + x_0, \qquad (68)$$

where

$$\xi = X - \frac{3\kappa^2}{a^2}t, \qquad (69)$$

and a and x_0 are constants.

Remark By the above scheme we proposed, the class of the differential equations like DP equation with the linear highest-order derivative with respect to spatial variables and temporal variable can be solved by ADM. By means of the transformation of the independent variables and travelling wave transformation, we reduce the original equation to an ordinary differential equation with the appropriate initial conditions which is easily solved by the ADM with minimal calculation and high accuracy, then we obtain the solution of the given equation by means of the transformation back to the original independent variables.

5 Conclusion

An easy-to-use approach for the nonlinear partial differential equations is proposed to solve short wave model for the CH equation and the DP equation. The special types of soliton solutions of them are derived. The results coincide perfectly with those given in Ref. [1]. Moreover, we firstly employ the ADM for solving the DP equation to obtain the one-loop soliton solution. The proposed approach is based on the transformation of the independent variables and the travelling wave transformation. The efficiency and reliability of the approach are illustrated by the examples. The further work is to extend this approach to solve the more other partial differential equations like them and obtain the special types of soliton solutions of the PDE.

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