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2008 Chinese Phys. B 17 3953

(http://iopscience.iop.org/1674-1056/17/11/005)

Vol 17 No 11, November 2008
1674-1056/2008/17(11)/3953-12

Chinese Physics B

Adomian decomposition method and Padé approximants for solving the Blaszak–Marciniak lattice*

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(Received 28 January 2008; revised manuscript received 29 February 2008)

The Adomian decomposition method (ADM) and Padé approximants are combined to solve the well-known Blaszak–Marciniak lattice, which has rich mathematical structures and many important applications in physics and mathematics. In some cases, the truncated series solution of ADM is adequate only in a small region when the exact solution is not reached. To overcome the drawback, the Padé approximants, which have the advantage in turning the polynomials approximation into a rational function, are applied to the series solution to improve the accuracy and enlarge the convergence domain. By using the ADM-Padé technique, the soliton solutions of the Blaszak–Marciniak lattice are constructed with better accuracy and better convergence than by using the ADM alone. Numerical and figurative illustrations show that it is a promising tool for solving nonlinear problems.

Keywords: Adomian decomposition method, Padé approximants, Blaszak–Marciniak lattice, soliton solution

PACC: 0340K, 0220

1. Introduction

The differential–difference equations (DDEs) are always important in applications. They enter as models in many biological chains, are encountered frequently in queuing problems and appear as discretizations of field theory. So both as themselves and as approximations of continuous problems, they play a very important role in fields of mathematics, physics, biology, and engineering. Since the work of Fermi *et al* in the 1950s,[1] much research work on DDEs has been carried out. Levi and Yamilov[2] analysed the condition for the existence of higher symmetries for a class of DDEs, and many researchers analysed the properties of solutions of DDEs.[3–6] The information about integrable DDEs can be found in Refs.[7–15].

The Adomian decomposition method (ADM)[16–18] has been applied to a large class of linear and nonlinear problems in mathematics, physics, biology and chemistry, etc. It is powerful for obtaining approximate solutions or even closed-form analytical solutions of differential equations[19–30] without linearization or perturbation and provides an efficient numerical solution with minimal calculations. In some cases, the series solution of ADM converges in a limited interval, and outside it, its error is high. To overcome the drawback, the Padé approximants, which often show superior performance over series approximation, are applied to the series solution of ADM to improve the accuracy and enlarge the convergence domain. Recently, combining the ADM and the Padé approximants (ADM–Padé) has attracted much attention of many mathematicians and physicists to solve the DDEs and PDEs. Abassy *et al*[31] solved Burgers' and good Boussinesq equations. Basto *et al*[32] approximated the theoretical solution of Burgers equation. Wazwaz solved Thomas–Fermi equation[33] and approximated Volterra's population model.[34]

In this paper, the Blaszak–Marciniak three-field and four-field lattices, which were derived as an application of γ-matrix formalism to the algebra of shift operators of dynamical systems,[13] are considered separately. The ADM–Padé technique has been successfully implemented to solve them by converting the series solutions into the diagonal Padé approximants. The soliton solutions are obtained with higher accuracy and faster convergence than by using the ADM alone. Numerical and figurative illustrations show

*Project supported by the National Key Basic Research Project of China (Grant No 2004CB318000), the National Natural Science Foundation of China (Grant Nos 10771072 and 10735030), and Shanghai Leading Academic Discipline Project of China (Grant No B412).

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that it is a promising tool for solving nonlinear problems.

The rest of the present paper is organized as follows. In Section 2, the ADM–Padé technique for solving the discrete differential–difference equations is outlined. In Section 3, the Blaszak–Marciniak three-field lattice is studied. In Section 4, the Blaszak–Marciniak four-field lattice is investigated. Finally, the important conclusions and the further work are presented in Section 5.

2. The description of ADM–Padé technique

2.1. The description of ADM for solving the DDEs

For the purpose of illustrating the decomposition method, we consider a system of nonlinear differential–difference equations as follows:

$$\begin{aligned} &L_i(u_i(n,t)) + R_i(u_1(n,t), u_1(n+1,t), u_1(n-1,t), ..., u_k(n,t), u_k(n+1,t), u_k(n-1,t), ...)\\ &+ N_i(u_1(n,t), u_1(n+1,t), u_1(n-1,t), ..., u_k(n,t), u_k(n+1,t), u_k(n-1,t), ...) = g_i(n,t), \end{aligned} \tag{1}$$

where $u_i(n,t)$ is the unknown function of discrete spatial variable n and temporal variable t, L_i is the highest-order derivative that is assumed to be invertible, R_i is the remainder linear operator, N_i is the nonlinear operator, and g_i is the source term. So we have

$$\begin{aligned} L_i(u_i(n,t)) =& g_i(n,t) - R_i(u_1(n,t), u_1(n+1,t), u_1(n-1,t), ..., u_k(n,t), u_k(n+1,t), u_k(n-1,t), ...)\\ &- N_i(u_1(n,t), u_1(n+1,t), u_1(n-1,t), ..., u_k(n,t), u_k(n+1,t), u_k(n-1,t), ...). \end{aligned} \tag{2}$$

Applying the inverse operator L_i^{-1} to both sides of Eq.(2) gives

$$\begin{aligned} L_i^{-1}L_i(u_i(n,t)) =& L_i^{-1}g_i(n,t) - L_i^{-1}R_i(u_1(n,t), u_1(n+1,t), u_1(n-1,t), ..., u_k(n,t), u_k(n+1,t), u_k(n-1,t), ...)\\ &- L_i^{-1}N_i(u_1(n,t), u_1(n+1,t), u_1(n-1,t), ..., u_k(n,t), u_k(n+1,t), u_k(n-1,t), ...). \end{aligned} \tag{3}$$

Using the initial conditions, we obtain

$$\begin{aligned} u_i(n,t) =& f_i(n,t) - L_i^{-1}R_i(u_1(n,t), u_1(n+1,t), u_1(n-1,t), ..., u_k(n,t), u_k(n+1,t), u_k(n-1,t), ...)\\ &- L_i^{-1}N_i(u_1(n,t), u_1(n+1,t), u_1(n-1,t), ..., u_k(n,t), u_k(n+1,t), u_k(n-1,t), ...), \end{aligned} \tag{4}$$

where the function f_i represents the term arising from integrating the source term g_i and from using the given initial conditions or boundary conditions. Then, $u_i(n,t)(1 \le i \le k)$ can be represented as a series

$$u_i(n,t) = \sum_{m=0}^{\infty} u_{i,m}(n,t). \tag{5}$$

The nonlinear term $N_i(1 \le i \le k)$ will be decomposed by the infinite series of the Adomian polynomials

$$\begin{aligned} &N_i(u_1(n,t), u_1(n+1,t), u_1(n-1,t), ..., u_k(n,t), u_k(n+1,t), u_k(n-1,t), ...)\\ &= \sum_{m=0}^{\infty} A_{i,m}(u_1(n,t), u_1(n+1,t), u_1(n-1,t), ..., u_k(n,t), u_k(n+1,t), u_k(n-1,t), ...), \end{aligned} \tag{6}$$

where $A_{i,m}$ terms are obtained by writing

$$\nu_i(\lambda) = \sum_{m=0}^{\infty} u_{i,m}(n,t)$$

and

$$N_i(\nu_i(\lambda)) = \sum_{m=0}^{\infty} \lambda^m A_{i,m} \tag{7}$$

with λ being a parameter for convenience, and they are deduced to be

$$A_{i,m} = \frac{1}{m!}\left[\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m} N_i(\nu_1(\lambda), \nu_2(\lambda), ..., \nu_k(\lambda))\right]_{\lambda=0}. \tag{8}$$

To determine the components $u_{i,m}(1 \leq i \leq k)$, we employ the recursive relation

$$u_{i,0}(n,t) = f_1(n,t),$$

and

$$u_{i,m+1}(n,t) = -L_i^{-1} R_i(u_1(n,t), u_1(n+1,t), u_1(n-1,t), ..., u_k(n,t), u_k(n+1,t), u_k(n-1,t), ...) - L_i^{-1} A_{i,m}. \tag{9}$$

So the following expression:

$$u_{i,r}(n,t) = \sum_{m=0}^{r} u_{i,m}(n,t) \tag{10}$$

denotes the r-term approximation to $u_i(n,t)(1 \leq i \leq k)$.

2.2. The Padé approximants to the series solution

A truncated series solution $u(n,t)$ of order at least $L+M$ at t can be obtained by using the ADM, and it can be used to obtain the Padé $[L+M](n,t)$ approximate solution for $u(n,t)$.

We denote L, M Padé approximants to $f(z)$ by

$$[L/M] = \frac{P_L(z)}{Q_M(z)}, \tag{11}$$

where $P_L(z)$ is a polynomial of degree at most L and $Q_M(z)(Q_M(z) \neq 0)$ is a polynomial of degree at most M. The former power series is

$$f(z) = \sum_{k=0}^{\infty} c_k z^k, \tag{12}$$

and we write the $P_L(z)$ and $Q_M(z)$ as

$$\begin{aligned} P_L(z) &= p_0 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots + p_L z^L, \\ Q_M(z) &= q_0 + q_1 z + q_2 z^2 + q_3 z^3 + \ldots + q_M z^M, \end{aligned} \tag{13}$$

so

$$f(z) - \frac{P_L(z)}{Q_M(z)} = O(z^{L+M+1}) \quad \text{as} \quad z \to 0, \tag{14}$$

and the coefficients of $P_L(z)$ and $Q_M(z)$ are determined by the equation. From Eq.(14), we have

$$f(z)Q_M(z) - P_L(z) = O(z^{L+M+1}), \tag{15}$$

which is a system of $L+M+1$ homogeneous equations with $L+M+2$ unknown quantities. We impose the normalization condition

$$Q_M(0) = 1. \tag{16}$$

We can write out Eq.(15) as

$$\begin{aligned} &c_{L+1} + c_L q_1 + \ldots + c_{L-M+1} q_M = 0, \\ &c_{L+2} + c_{L+1} q_1 + \ldots + c_{L-M+2} q_M = 0, \\ &\qquad \vdots \\ &c_{L+M} + c_{L+M-1} q_1 + \ldots + c_L q_M = 0, \end{aligned} \tag{17}$$

$$\begin{aligned} &c_0 = p_0, \\ &c_1 + c_0 q_1 = p_1, \\ &c_2 + c_1 q_1 + c_0 q_2 = p_2, \\ &\qquad \vdots \\ &c_L + c_{L-1} q_1 + \ldots + c_0 q_L = p_L. \end{aligned} \tag{18}$$

From Eq.(17), we can obtain the $q_i(1 \leq i \leq M)$. Once the values of $q_1,\ q_2, \ldots, q_M$ are all known, Eq.(18) gives an explicit formula for the unknown quantities $p_1, p_2, \ldots, p_L$. Since the diagonal approximants like [2/2], [3/3], [4/4], [5/5] or [6/6] have the most accurate approximation, we will construct only the diagonal approximants by built-in utilities of Maple in the following sections.

3. The soliton solution of the Blaszak–Marciniak three-field lattice

We consider the Blaszak–Marciniak three-field lattice as follows:[13]

$$\begin{aligned} a_t(n,t) &= c(n+1,t) - c(n-1,t), \\ b_t(n,t) &= a(n-1,t)c(n-1,t) - a(n,t)c(n,t), \\ c_t(n,t) &= c(n,t)(b(n,t) - b(n+1,t)) \end{aligned} \tag{19}$$

subject to the initial condition

$$\begin{aligned} a(n,0) &= f_1(n), \\ b(n,0) &= f_2(n), \\ c(n,0) &= f_3(n). \end{aligned} \tag{20}$$

We rewrite Eq. (19) in the operator form:

$$\begin{aligned} L_t a(n,t) &= c(n+1,t) - c(n-1,t), \\ L_t b(n,t) &= a(n-1,t)c(n-1,t) - a(n,t)c(n,t), \\ L_t c(n,t) &= c(n,t)(b(n,t) - b(n+1,t)), \end{aligned} \tag{21}$$

where L_t is a first order differential operator and L_t^{-1} is an integrate operator defined as

$$L_t^{-1} \equiv \int_0^t (\cdot)\mathrm{d}t. \tag{22}$$

Operating L_t^{-1} on both sides of Eq.(21) and using the initial conditions, we obtain

$$\begin{aligned} a(n,t) =& f_1(n) + L_t^{-1}(c(n+1,t) - c(n-1,t)), \\ b(n,t) =& f_2(n) + L_t^{-1}(a(n-1,t)c(n-1,t) \\ & - a(n,t)c(n,t)), \\ c(n,t) =& f_3(n) + L_t^{-1}(c(n,t)(b(n,t) \\ & - b(n+1,t))), \end{aligned} \tag{23}$$

therefore

$$\begin{aligned} a(n,t) =& f_1(n) + L_t^{-1}(c(n+1,t) - c(n-1,t)), \\ b(n,t) =& f_2(n) + L_t^{-1}(M(a(n-1,t),c(n-1,t)) \\ & - N(a(n,t),c(n,t))), \\ c(n,t) =& f_3(n) + L_t^{-1}(P(c(n,t),b(n,t)) \\ & - Q(c(n,t),b(n+1,t))), \end{aligned} \tag{24}$$

where

$$\begin{aligned} M(a(n-1,t),c(n-1,t)) &= a(n-1,t)c(n-1,t), \\ N(a(n,t),c(n,t)) &= a(n,t)c(n,t), \\ P(c(n,t),b(n,t)) &= c(n,t)b(n,t), \\ Q(c(n,t),b(n+1,t)) &= c(n,t)b(n+1,t) \end{aligned}$$

can be expressed in terms of Adomian polynomial as follows:

$$\begin{aligned} M(a(n-1,t),c(n-1,t)) &= \sum_{m=0}^{\infty} A_m, \\ N(a(n,t),c(n,t)) &= \sum_{m=0}^{\infty} B_m, \\ P(c(n,t),b(n,t)) &= \sum_{m=0}^{\infty} C_m, \\ Q(c(n,t),b(n+1,t)) &= \sum_{m=0}^{\infty} D_m. \end{aligned} \tag{25}$$

We assume the expressions of $a(n,t)$, $b(n,t)$, $c(n,t)$ to be in the decomposition form as follows:

$$\begin{aligned} a(n,t) &= \sum_{m=0}^{\infty} a_m(n,t), \\ b(n,t) &= \sum_{m=0}^{\infty} b_m(n,t), \\ c(n,t) &= \sum_{m=0}^{\infty} c_m(n,t). \end{aligned} \tag{26}$$

According to Eq.(8), we can have the first few components of Adomian polynomial as follows:

$$\begin{aligned} A_0 =& a_0(n-1,t)c_0(n-1,t), \\ B_0 =& a_0(n,t)c_0(n,t), \\ C_0 =& c_0(n,t)b_0(n,t), \\ D_0 =& c_0(n,t)b_0(n+1,t), \\ A_1 =& a_0(n-1,t)c_1(n-1,t) + a_1(n-1,t)c_0(n-1,t), \\ B_1 =& a_0(n,t)c_1(n,t) + a_1(n,t)c_0(n,t), \\ C_1 =& c_0(n,t)b_1(n,t) + c_1(n,t)b_0(n,t), \\ D_1 =& c_0(n,t)b_1(n+1,t) + c_1(n,t)b_0(n+1,t), \\ A_2 =& a_0(n-1,t)c_2(n-1,t) + a_1(n-1,t)c_1(n-1,t) \\ & + a_2(n-1,t)c_0(n-1,t) \\ B_2 =& a_0(n,t)c_2(n,t) + a_1(n,t)c_1(n,t) \\ & + a_2(n,t)c_0(n,t), \\ C_2 =& c_0(n,t)b_2(n,t) + c_1(n,t)b_1(n,t) \\ & + c_2(n,t)b_0(n,t), \\ D_2 =& c_0(n,t)b_2(n+1,t) + c_1(n,t)b_1(n+1,t) \\ & + c_2(n,t)b_0(n+1,t). \end{aligned}$$

To determine the components $a_m(n,t)$, $b_m(n,t)$, and $c_m(n,t)$, we employ the recursive relations

$$\begin{aligned}
&a_0(n,t) = f_1(n),\\
&a_m(n,t) = L_t^{-1}(c_{m-1}(n+1,t) - c_{m-1}(n-1,t)),\\
&b_0(n,t) = f_2(n),\\
&b_m(n,t) = L_t^{-1}(A_{m-1} - B_{m-1}),\\
&c_0(n,t) = f_3(n),\\
&c_m(n,t) = L_t^{-1}(C_{m-1} - D_{m-1}).
\end{aligned} \tag{27}$$

So the rth-order approximate solution is evaluated as follows:

$$\begin{aligned}
a_r(n,t) &= \sum_{m=0}^{r} a_m(n,t),\\
b_r(n,t) &= \sum_{m=0}^{r} b_m(n,t),\\
c_r(n,t) &= \sum_{m=0}^{r} c_m(n,t).
\end{aligned} \tag{28}$$

Now, we will give the soliton solution of the Blaszak–Marciniak three-field lattice. We suppose that Eq.(19) has the following initial condition:

$$\begin{aligned}
a(n,0) =&(0.8431942046\mathrm{exp}(0.7n + 0.255768733)(1 + \exp(0.7n + 0.255768733))\\
&- 0.8431942046\mathrm{exp}(0.7n + 0.255768733)^2)/(1 + \exp(0.7n - 0.4442312668))/(1 + \exp(0.7n\\
&+ 0.955768733)),\\
b(n,0) =&(0.9182560670\mathrm{exp}(0.7n - 0.4442312668)/(1 + \exp(0.7n + 0.255768733))\\
&- 0.9182560670(1 + \exp(0.7n - 0.4442312688))/(1 + \exp(0.7n + 0.255768733))^2\mathrm{exp}(0.7n\\
&+ 0.255768733))/(1 + \exp(0.7n - 0.4442312688))(1 + \exp(0.7n + 0.255768733)),
\end{aligned}$$

and

$$\begin{aligned}
c(n,0) =&(1 + \exp(0.7n - 0.4442312688))(1 + \exp(0.7n + 0.955768733))/(1 + \exp(0.7n\\
&+ 0.255768733))^2,
\end{aligned} \tag{29}$$

so we obtain

$$\begin{aligned}
a_0(n,t) &= a(n,0),\\
b_0(n,t) &= b(n,0),
\end{aligned}$$

and

$$c_0(n,t) = c(n,0). \tag{30}$$

The remaining components $a_m(n,t)$, $b_m(n,t)$, and $c_m(n,t)$ can be determined by recursive relations (27). Therefore, we can obtain the sixth-order approximation

$$\begin{aligned}
a_6(n,t) &= \sum_{m=0}^{6} a_m(n,t),\\
b_6(n,t) &= \sum_{m=0}^{6} b_m(n,t),
\end{aligned}$$

and

$$c_6(n,t) = \sum_{m=0}^{6} c_m(n,t).$$

The exact solutions of the problem are[14]

$$\begin{aligned}
a(n,t) &= \frac{D_t^2 g(n+1,t,z)\cdot g(n+1,t,z)}{2g(n,t,z)g(n+2,t,z)},\\
b(n,t) &= \left(\ln\frac{g(n,t,z)}{g(n+1,t,z)}\right)_t,
\end{aligned}$$

and

$$c(n,t) = \frac{g(n,t,z)g(n+2,t,z)}{g^2(n+1,t,z)}, \tag{31}$$

where $g(n,t,z) = 1+\exp(\eta)$, $\eta = pn+qz+rt+\eta^0$, $q = \lambda(1-\mathrm{e}^{-p})$, $r = \lambda^{-1}(\mathrm{e}^{-p}-1)$, with $\lambda = \left(\frac{\mathrm{e}^{2p}}{1+\mathrm{e}^p}\right)^{1/3}$, p and η^0 being constants, z being an auxiliary variable, and $p = 0.7$, $z = 1$ and $\eta^0 = -1$ chosen. Figure 1 shows that the series solutions of ADM have a good approximation in a small interval of convergence, and outside it their error becomes higher.

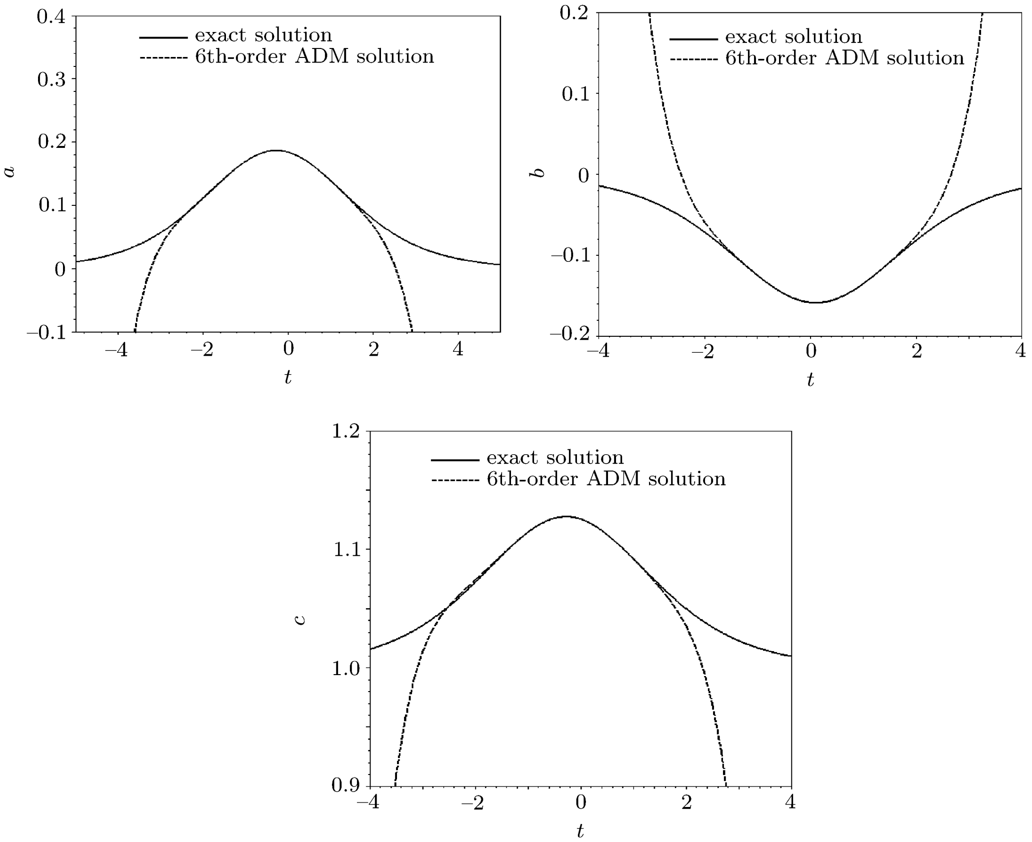


Fig.1. A comparison between the ADM solutions and the exact solutions of the Blaszak–Marciniak three-field lattice at $n = 0$.

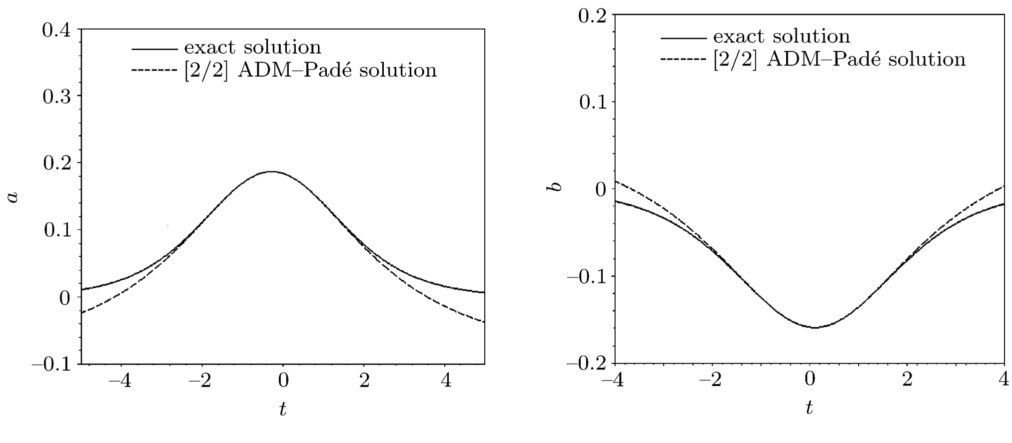
Using the ADM–Padé approximation at $n = 0$, the rational approximations [2/2] is

$$
\begin{aligned}
a[2/2](0,t) =& \frac{0.1842599155 - 0.007367100502t - 0.01238274749t^2}{1 + 0.06378729452t + 0.1189682357t^2}, \\
b[2/2](0,t) =& \frac{-0.1587321893 - 0.00229590129t + 0.01108748115t^2}{1 - 0.02747360966t + 0.1344824454t^2},
\end{aligned}
$$

and

$$
c[2/2](0,t) = \frac{1.125520474 + 0.07179392356t + 0.133901181t^2}{1 + 0.07681249462t + 0.1423364854t^2}. \tag{32}
$$

Figures 2 and 3 show that the ADM–Padé technique can enlarge the convergence domain of the series solution at $n = 0$. It is clear that the interval of convergence has increased by the ADM–Padé technique.



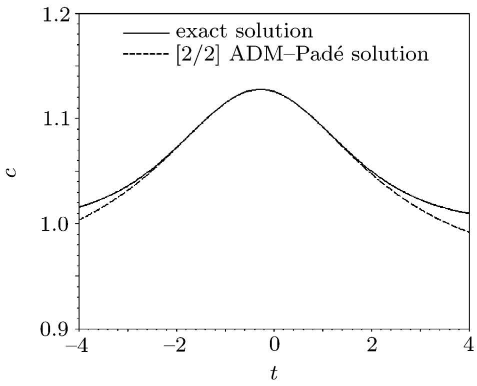


Fig.2. A comparison between the [2/2] ADM–Padé solutions and the exact solutions of the Blaszak–Marciniak three-field lattice at $n = 0$.

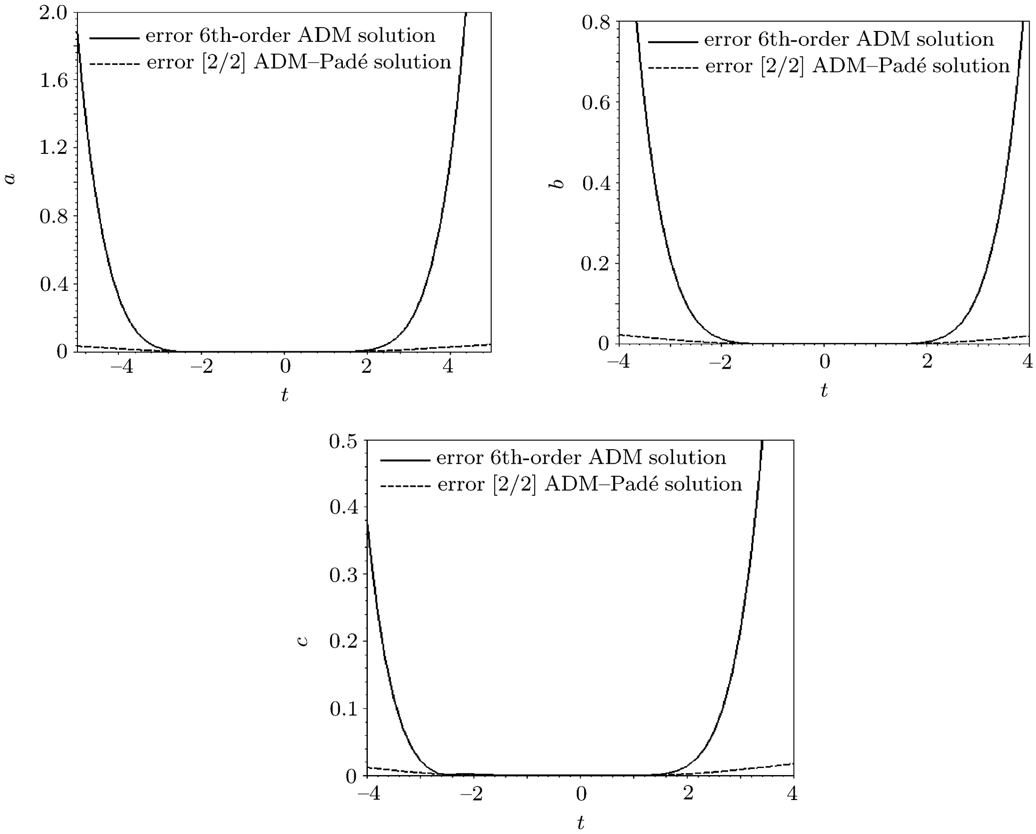


Fig.3. A comparison between the error of the ADM solutions and the error of the [2/2] ADM–Padé solutions of the Blaszak–Marciniak three-field lattice at $n = 0$.

Remark We chose the initial value problem associated with the Blaszak–Marciniak three-field lattice on the trial-and-error basis of the exact solutions obtained by Ref.[14] to illustrate the efficiency of ADM–Padé technique.

4. The soliton solution of the Blaszak–Marciniak four-field lattice

We consider the Blaszak–Marciniak four-field lattice as follows:[13]

$$
\begin{aligned}
&u_t(n,t) = u(n,t)(v(n,t) - v(n-1,t)),\\
&v_t(n,t) = w(n,t)u(n+1,t) - u(n,t)w(n-1,t),\\
&w_t(n,t) = q(n,t)u(n+2,t) - u(n,t)q(n-1,t),\\
&q_t(n,t) = u(n+3,t) - u(n,t)
\end{aligned}
\tag{33}
$$

subject to the initial condition

$$
\begin{aligned}
u(n,0) &= f_1(n)\\
v(n,0) &= f_2(n)\\
w(n,0) &= f_3(n)\\
q(n,0) &= f_4(n).
\end{aligned}
\tag{34}
$$

We rewrite Eq.(33) in the operator form:

$$
\begin{aligned}
&L_t u(n,t) = u(n,t)v(n,t) - u(n,t)v(n-1,t),\\
&L_t v(n,t) = w(n,t)u(n+1,t) - u(n,t)w(n-1,t),\\
&L_t w(n,t) = q(n,t)u(n+2,t) - u(n,t)q(n-1,t),\\
&L_t q(n,t) = u(n+3,t) - u(n,t),
\end{aligned}
\tag{35}
$$

where L_t is a first order differential operator and L_t^{-1} is a integrate operator defined as

$$
L_t^{-1} \equiv \int_0^t (\cdot)dt. \tag{36}
$$

Operating L_t^{-1} on both sides of Eq.(35) and using the initial conditions, we obtain

$$
\begin{aligned}
u(n,t) =& f_1(n) + L_t^{-1}(u(n,t)v(n,t)\\
&- u(n,t)v(n-1,t)),\\
v(n,t) =& f_2(n) + L_t^{-1}(w(n,t)u(n+1,t)\\
&- u(n,t)w(n-1,t)),\\
w(n,t) =& f_3(n) + L_t^{-1}(q(n,t)u(n+2,t)\\
&- u(n,t)q(n-1,t)),\\
q(n,t) =& f_4(n) + L_t^{-1}(u(n+3,t)\\
&- u(n,t)).
\end{aligned}
\tag{37}
$$

Therefore

$$
\begin{aligned}
u(n,t) =& f_1(n) + L_t^{-1}(M(u(n,t),v(n,t))\\
&- N(u(n,t),v(n-1,t))),\\
v(n,t) =& f_2(n) + L_t^{-1}(P(w(n,t),u(n+1,t))\\
&- Q(u(n,t),w(n-1,t))),\\
w(n,t) =& f_3(n) + L_t^{-1}(R(q(n,t),u(n+2,t))\\
&- S(u(n,t),q(n-1,t))),\\
q(n,t) =& f_4(n) + L_t^{-1}(u(n+3,t)\\
&- u(n,t)),
\end{aligned}
\tag{38}
$$

where

$$
\begin{aligned}
&M(u(n,t),v(n,t)) = u(n,t)v(n,t),\\
&N(u(n,t),v(n-1,t)) = u(n,t)v(n-1,t),\\
&P(w(n,t),u(n+1,t)) = w(n,t)u(n+1,t),\\
&Q(u(n,t),w(n-1,t)) = u(n,t)w(n-1,t),\\
&R(q(n,t),u(n+2,t)) = q(n,t)u(n+2,t),\\
&S(u(n,t),q(n-1,t)) = u(n,t)q(n-1,t)
\end{aligned}
\tag{39}
$$

can be expressed in terms of Adomian polynomial as follows:

$$
\begin{aligned}
M(u(n,t),v(n,t)) &= \sum_{m=0}^{\infty} A_m,\\
N(u(n,t),v(n-1,t)) &= \sum_{m=0}^{\infty} B_m,\\
P(w(n,t),u(n+1,t)) &= \sum_{m=0}^{\infty} C_m,\\
Q(u(n,t),w(n-1,t)) &= \sum_{m=0}^{\infty} D_m,\\
R(q(n,t),u(n+2,t)) &= \sum_{m=0}^{\infty} E_m,\\
S(u(n,t),q(n-1,t)) &= \sum_{m=0}^{\infty} F_m.
\end{aligned}
\tag{40}
$$

We assume the expressions of $u(n,t)$, $v(n,t)$, $w(n,t)$, and $q(n,t)$ to be in the decomposition form as follows:

$$
\begin{aligned}
u(n,t) &= \sum_{m=0}^{\infty} u_m(n,t),\\
v(n,t) &= \sum_{m=0}^{\infty} v_m(n,t),\\
w(n,t) &= \sum_{m=0}^{\infty} w_m(n,t),\\
q(n,t) &= \sum_{m=0}^{\infty} q_m(n,t).
\end{aligned}
\tag{41}
$$

According to Eq.(8), we can have the first few components of Adomian polynomial as follows:

$$
\begin{aligned}
A_0 =& u_0(n,t)v_0(n,t),\\
B_0 =& u_0(n,t)v_0(n-1,t)),\\
C_0 =& w_0(n,t)u_0(n+1,t),\\
D_0 =& u_0(n,t)w_0(n-1,t),\\
E_0 =& q_0(n,t)u_0(n+2,t)\\
F_0 =& u_0(n,t)q_0(n-1,t),\\
A_1 =& u_0(n,t)v_1(n,t) + u_1(n,t)v_0(n,t),\\
B_1 =& u_0(n,t)v_1(n-1,t) + u_1(n,t)v_0(n-1,t),\\
C_1 =& w_0(n,t)u_1(n+1,t) + w_1(n,t)u_0(n+1,t),\\
D_1 =& u_0(n,t)w_1(n-1,t) + u_1(n,t)w_0(n-1,t),\\
E_1 =& q_0(n,t)u_1(n+2,t) + q_1(n,t)u_0(n+2,t),\\
F_1 =& u_0(n,t)q_1(n-1,t) + u_1(n,t)q_0(n-1,t),
\end{aligned}
$$

$$\begin{aligned}
A_2 =&u_0(n,t)v_2(n,t)+u_1(n,t)v_1(n,t)\\
&+u_2(n,t)v_0(n,t),\\
B_2 =&u_0(n,t)v_2(n-1,t)+u_1(n,t)v_1(n-1,t)\\
&+u_2(n,t)v_0(n-1,t),\\
C_2 =&w_0(n,t)u_2(n+1,t)+w_1(n,t)u_1(n+1,t)\\
&+w_2(n,t)u_0(n+1,t),\\
D_2 =&u_0(n,t)w_2(n-1,t)+u_1(n,t)w_1(n-1,t)\\
&+u_2(n,t)w_0(n-1,t),\\
E_2 =&q_0(n,t)u_2(n+2,t)+q_1(n,t)u_1(n+2,t)\\
&+q_2(n,t)u_0(n+2,t),\\
F_2 =&u_0(n,t)q_2(n-1,t)+u_1(n,t)q_1(n-1,t)\\
&+u_2(n,t)q_0(n-1,t).
\end{aligned}$$

To determine the components $u_m(n,t)$, $v_m(n,t)$, $w_m(n,t)$, and $q_m(n,t)$, we employ the recursive relations

$$\begin{aligned}
&u_0(n,t)=f_1(n),\\
&u_m(n,t)=L_t^{-1}(A_{m-1}-B_{m-1}),\\
&v_0(n,t)=f_2(n),\\
&v_m(n,t)=L_t^{-1}(C_{m-1}-D_{m-1}),\\
&w_0(n,t)=f_3(n),\\
&w_m(n,t)=L_t^{-1}(E_{m-1}-F_{m-1}),\\
&q_0(n,t)=f_4(n),\\
&q_m(n,t)=L_t^{-1}(u_{m-1}(n+3,t)-u_{m-1}(n,t)).
\end{aligned}\tag{42}$$

So the rth-order approximate solution is evaluated as follows:

$$\begin{aligned}
u_r(n,t)&=\sum_{m=0}^{r}u_m(n,t),\\
v_r(n,t)&=\sum_{m=0}^{r}v_m(n,t),\\
w_r(n,t)&=\sum_{m=0}^{r}w_m(n,t),\\
q_r(n,t)&=\sum_{m=0}^{r}q_m(n,t).
\end{aligned}$$

Now, we will give the soliton solution of the Blaszak–Marciniak four-field lattice. We suppose that Eq.(33) has the following initial condition:

$$\begin{aligned}
u(n,0) =&(1+\exp(0.7n-1.935545602))(1+\exp(0.7n-3.335545602))/(1+\exp(0.7n-2.635545602))^2,\\
v(n,0) =&(0.9778363838\exp(0.7n-1.935545602)/(1+\exp(0.7n-2.635545602))\\
&-0.9778363838(1+\exp(0.7n-1.935545602))/(1+\exp(0.7n-2.635545602))^2\\
&\times\exp(0.7n-2.635545602))/(1+\exp(0.7n-1.935545602))(1+\exp(0.7n-2.635545602),\\
w(n,0) =&(0.9561639935\exp(0.7n-1.935545602)(1+\exp(0.7n-1.935545602))-0.9561639935\exp(0.7n\\
&-1.935545602)^2)/(1+\exp(0.7n-2.635545602))/(1+\exp(0.7n-1.235545602)),\\
q(n,0) =&(0.5219053203\exp(0.7n-0.535545602)/(1+\exp(0.7n-2.635545602))\\
&-0.5219053203(1+\exp(0.7n-0.535545602))/(1+\exp(0.7n-2.635545602))^2\exp(0.7n\\
&-2.635545602))/(1+\exp(0.7n-0.535545602))(1+\exp(0.7n-2.635545602)).
\end{aligned}\tag{43}$$

So we obtain

$$\begin{aligned}
u_0(n,t)&=u(n,0),\\
v_0(n,t)&=v(n,0),\\
w_0(n,t)&=w(n,0),\\
q_0(n,t)&=q(n,0).
\end{aligned}\tag{44}$$

The remaining components $u_m(n,t)$, $v_m(n,t)$, $w_m(n,t)$, and $q_m(n,t)$ can be determined by recursive relations (42). Therefore, we can obtain the sixth-order approximation as follows:

$$\begin{aligned}
u_6(n,t)&=\sum_{m=0}^{6}u_m(n,t),\\
v_6(n,t)&=\sum_{m=0}^{6}v_m(n,t),\\
w_6(n,t)&=\sum_{m=0}^{6}w_m(n,t),\\
q_6(n,t)&=\sum_{m=0}^{6}q_m(n,t).
\end{aligned}$$

The exact solution of this problem is[15]

$$
\begin{aligned}
u(n,t) &= \frac{g(n+1,t,y,z)g(n-1,t,y,z)}{g^2(n,t,y,z)},\\
v(n,t) &= \left(\ln\frac{g(n+1,t,y,z)}{g(n,t,y,z)}\right)_t,\\
w(n,t) &= \frac{1}{2}\frac{D_t^2 g(n+1,t,y,z)\cdot g(n+1,t,y,z)}{g(n+2,t,y,z)g(n,t,y,z)},\\
q(n,t) &= \left(\ln\frac{g(n+3,t,y,z)}{g(n,t,y,z)}\right)_z,
\end{aligned}
\tag{45}
$$

where $g(n,t,y,z) = 1+\exp(\eta)$, $\eta = pn+qt+rz+sy+\eta^0$, $q = \lambda(1-\mathrm{e}^{-p})$, $r = \lambda^{-1}(\mathrm{e}^p-1)$, $s = \lambda^2(1-\mathrm{e}^{2p})$, and $\lambda^4 = \mathrm{e}^p(1+\mathrm{e}^p+\mathrm{e}^{2p})$, with p and η^0 being constants, z and y being auxiliary variables, and $p = 0.7$, $\lambda \approx 1.94$, $z = 1$, $y = 1$, and $\eta^0 = -6$ chosen. Figure 4 shows that the series solutions of ADM have a good approximation in a small interval of convergence, and outside their error is high.

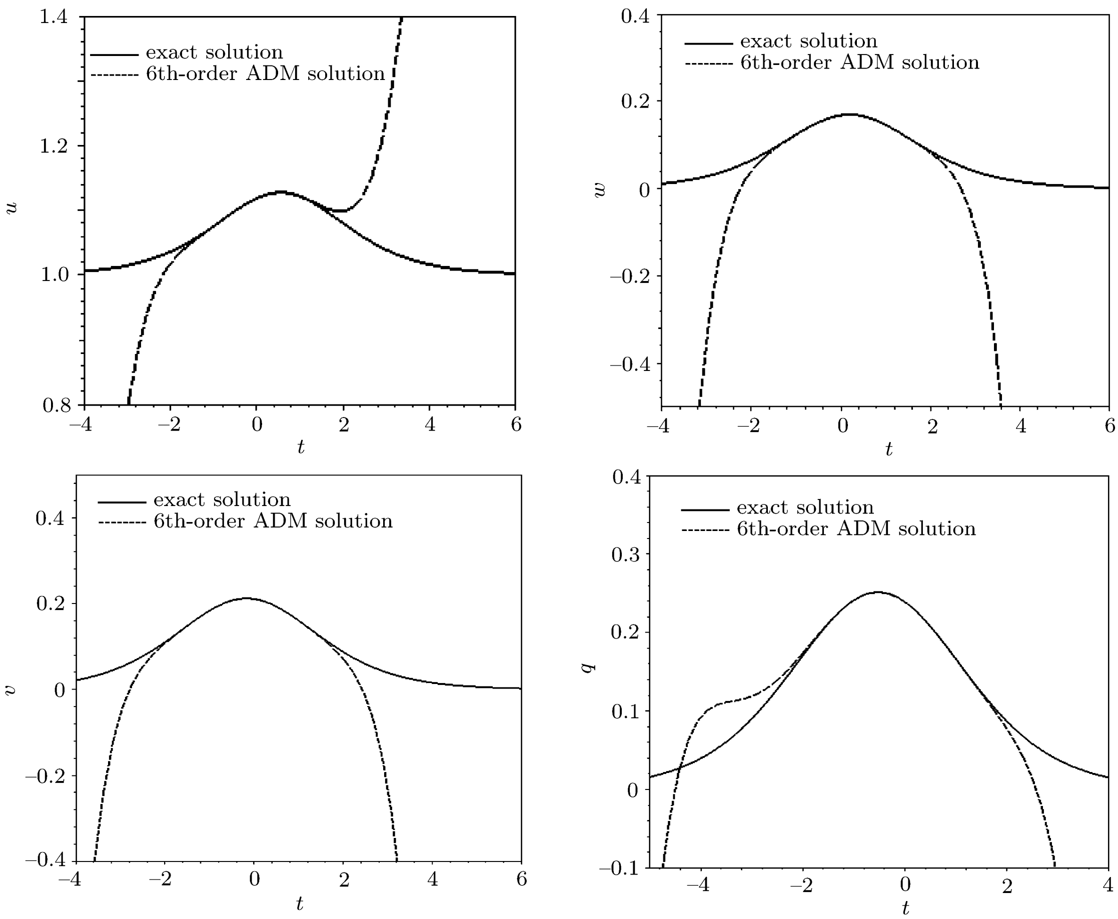


Fig.4. A comparison between the ADM solutions and the exact solutions of the Blaszak–Marciniak four-field lattice at $n = 3$.

Using the ADM–Padé approximation at $n = 3$, the rational approximations [2/2] is

$$
\begin{aligned}
u[2/2](3,t) &= \frac{1.118856571 - 0.149496319t + 0.1589109181t^2}{1 - 0.1607841443t + 0.1665800746t^2},\\
v[2/2](3,t) &= \frac{0.1679891548 + 0.005137555869t - 0.01307824036t^2}{1 - 0.05718246955t + 0.1533758035t^2},\\
w[2/2](3,t) &= \frac{0.2107274354 - 0.005698375769t - 0.01648810819t^2}{1 + 0.04415956808t + 0.133414314t^2},\\
q[2/2](3,t) &= \frac{0.23891114 - 0.02167042777t - 0.01569265655t^2}{1 + 0.1011621778t + 0.1150656397t^2}.
\end{aligned}
\tag{46}
$$

Figures 5 and 6 show that the ADM–Padé technique can enlarge the convergence domain of the series solution at $n = 3$. It is clear that the interval of convergence has increased by the ADM–Padé technique.

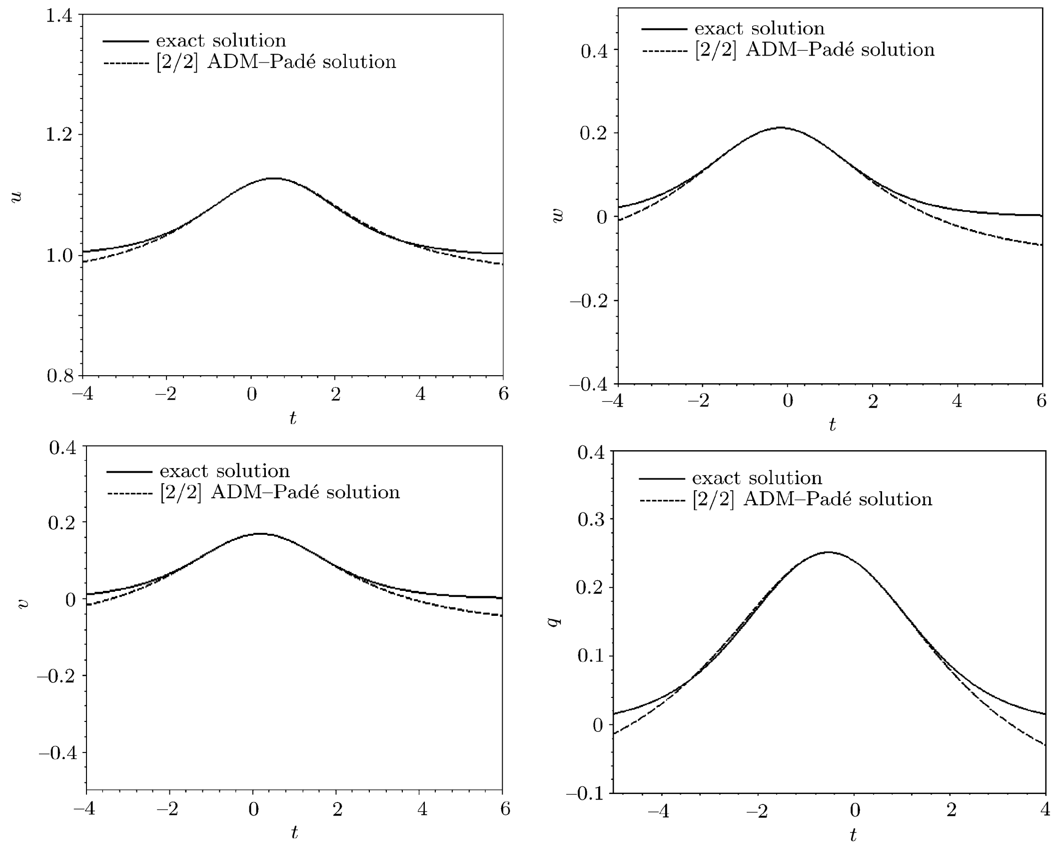


Fig.5. A comparison between the [2/2] ADM–Padé solutions and the exact solutions of the Blaszak–Marciniak four-field lattice at $n = 3$.

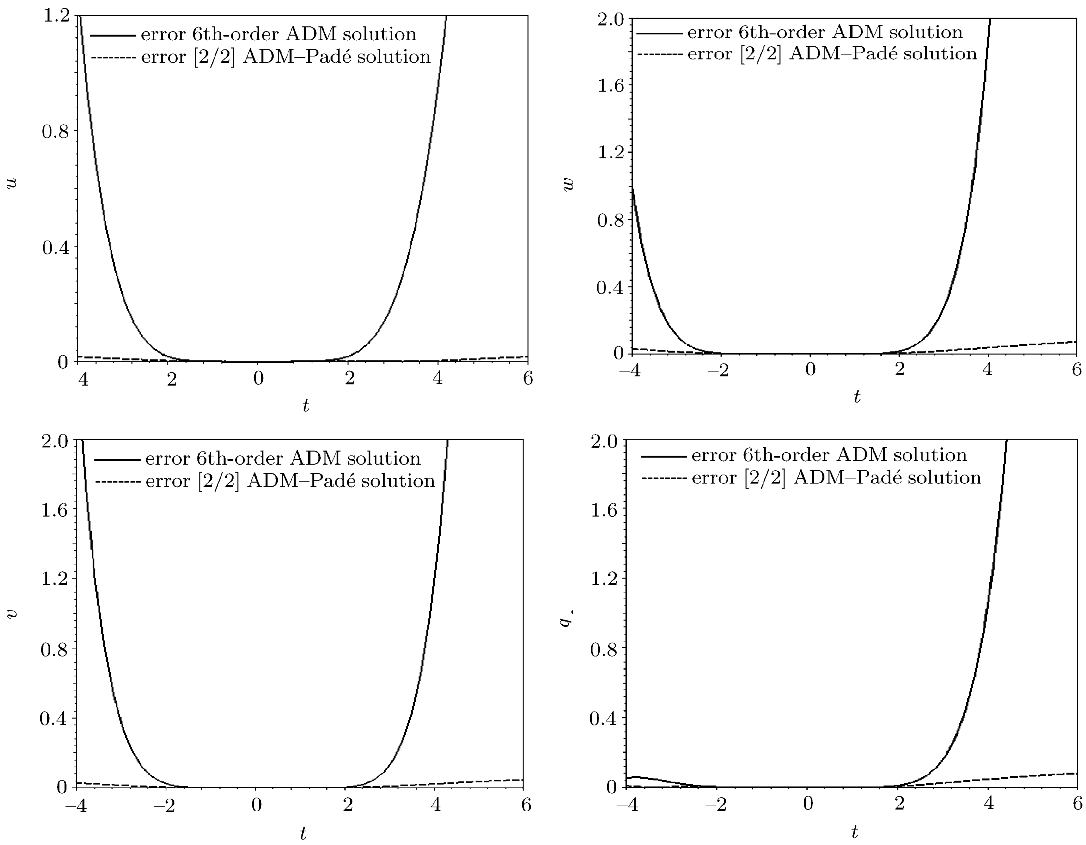


Fig.6. A comparison of between the error the ADM solutions and the error of the [2/2] ADM–Padé solutions of the Blaszak–Marciniak four-field lattice at $n = 3$.

Remark We have chosen the initial value problem associated with the Blaszak–Marciniak four-field lattice on the trial-and-error basis of the exact solutions obtained by Ref.[15] to illustrate the efficiency of the ADM–Padé technique.

5. Conclusions

The Blaszak–Marciniak lattices under consideration are integrable discrete soliton systems with rich mathematical structures such as infinitely many conservation laws, bi-Hamiltonian structure, Abelian symmetry algebras of infinite dimensions, nonlinear superposition formula, etc,[10−15] which are valuable for applying dynamical systems.[13] We solve the Blaszak–Marciniak three-field and four-field lattices by using the ADM–Padé technique that can improve the accuracy and enlarge the convergence domain of the truncated series solution of ADM and achieve the soliton solutions that accord well with the exact solutions obtained from Refs.[14,15] with minimal calculations and avoidance of any unrealistic assumption. Numerical solutions and figures illustrate that the technique is a promising tool for solving nonlinear problems. In the future work, we will apply this technique to approximating the multi-soliton solutions of differential–difference equations.

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