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Nonlocal symmetries related to Bäcklund transformation and their applications

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Abstract

Starting from nonlocal symmetries related to Bäcklund transformation (BT), many interesting results can be obtained. Taking the well-known potential KdV (pKdV) equation as an example, a new type of nonlocal symmetry in an elegant and compact form which comes from BT is presented and used to perform research works in two main subjects: the nonlocal symmetry is localized by introducing suitable and simple auxiliary-dependent variables to generate new solutions from old ones and to consider some novel group invariant solutions; some other models both in finite and infinite dimensions are generated under new nonlocal symmetry. The finite-dimensional models are completely integrable in Liouville sense, which are shown equivalent to the results given through the nonlinearization method for Lax pair.

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1. Introduction

With the development of integrable systems and soliton theory, symmetries [1–3] play the more and more important role in nonlinear mathematical physics. Thanks to the classical or nonclassical Lie group method, Lie point symmetries of a differential system can be obtained, from which one can transform given solutions to new ones via finite transformation and construct group invariant solutions by similarity reductions. However, little importance is attached to the existence and applications of nonlocal symmetries [2, 3]. Firstly, seeking nonlocal symmetries in itself is a difficult work to perform [4, 5]. One of our authors (Lou) has made some efforts to obtain infinite many nonlocal symmetries by inverse recursion operators [4, 6], the conformal invariant form (Schwartz form) [7] and Darboux transformation [8, 9]. Moreover, it appears that the nonlocal symmetries are rarely used to construct explicit solutions, since the finite symmetry transformations and similarity reductions cannot be

directly calculated. Naturally, it is necessary to inquire as to whether nonlocal symmetries can be transformed to local ones. The introduction of potential [3] and pseudopotential-type symmetries [10–12] which possess close prolongation extends the applicability of symmetry methods to obtain solutions of differential equations (DEs). In that context, the original given equation(s) can be embedded in some prolonged systems. Hence, these nonlocal symmetries with close prolongation are anticipated [13–15].

On the other hand, to find new integrable models is another important application of symmetry study. A systematic approach has been developed by Cao [16–18] to find finite-dimensional integrable systems by the nonlinearization of Lax pair under certain constraints between potentials and eigenfunctions. In particular, in the study of (1+1)-dimensional soliton equations, various new kinds of confocal involutive systems are constructed by the approach of nonlinearization of eigenvalue problems or constrained flows [19, 20]. It has also been pointed that by restricting a symmetry constraint to the Lax pair of the soliton equation, one can not only obtain the lower dimensional integrable models from higher ones, but can also embed the lower ones into higher dimensional integrable models [7, 9, 21]. Here, alternatively, we are inspired to act the new nonlocal symmetry on the Bäcklund transformation (BT) instead of Lax pair to generate other integrable systems. The related work may be adventurous but full of enormous interest.

In this paper, taking the well-known potential KdV equation (pKdV) as a special example, a new class of nonlocal symmetries is derived from its BT, which may give more interesting applications than those nonlocal symmetries that only include potentials and pseudopotentials. The prolongation of the new nonlocal symmetries is found soon after extending the pKdV equation to an auxiliary system with four dependent variables. Then, the finite symmetry transformation and similarity reductions are computed to give novel exact solutions of the KdV equation. What we want to mention is the process which can once lead to two exact solutions from one due to the BT. Moreover, for the pKdV equation, some other models both in finite and infinite dimensions are obtained. The finite-dimensional systems obtained here are found equivalent to the results given by Cao [17], which have been verified as completely integrable in the Liouville sense. This discovery confirms that these obtained infinite-dimensional models should have many nice integrable properties, which needs our further study.

The paper is organized as follows. In section 2, we present a detailed description about the new nonlocal symmetry of the pKdV equation. In section 3, we extend the nonlocal symmetry to be equivalent to a Lie point symmetry of some auxiliary prolonged system admitting four dependent variables. Then, the finite symmetry transformation and similarity reductions are made to produce exact solutions of the pKdV and then the KdV equation. Section 4 is devoted to constructing various integrable systems by means of the symmetry constraint method. Conclusions and discussions are given in section 5.

2. BT and nonlocal symmetries of the pKdV equation

2.1. BT for the pKdV equation

The well-known KdV equation reads

$$\omega_t + \omega_{xxx} - 6\omega\omega_x = 0, \quad (1)$$

where the subscripts x and t denote partial differentiation. For convenience to deal analytically with a potential function u , introduced by setting $\omega = u_x$, it follows from equation (1) that u would satisfy the equation

$$u_t + u_{xxx} - 3u_x^2 = 0, \quad (2)$$

which is called the pKdV equation.

For equation (2), there exist the following BT [22]:

$$u_x + u_{1,x} = -2\lambda + \frac{(u - u_1)^2}{2}, \tag{3}$$

$$u_t + u_{1,t} = 2u_x^2 + 2u_{1,x}^2 + 2u_x u_{1,x} - (u - u_1)(u_{xx} - u_{1,xx}), \tag{4}$$

with λ being the arbitrary parameter.

Equations (3) and (4) show that if u is a solution of equation (2), so is u_1 , that is to say, they represent a finite symmetry transformation between two exact solutions of equation (2).

On the other hand, equations (3) and (4) can also be viewed as a nonlinear Lax pair of equation (2). For

$$u_{1,x} = -u_x - 2\lambda + \frac{(u - u_1)^2}{2}, \tag{5}$$

$$u_{1,t} = -u_t + 2u_x^2 + 2u_{1,x}^2 + 2u_x u_{1,x} - (u - u_1)(u_{xx} - u_{1,xx}), \tag{6}$$

its compatibility condition $u_{1,xt} = u_{1,tx}$ is exactly equation (2). In fact, both equations (3) and (4) hint that they are all Riccati-type equations about u or u_1 , which can be linearized by the well-known Cole–Hopf transformation

$$u = -2\frac{\psi_x}{\psi}, \quad \text{or} \quad u_1 = -2\frac{\psi_{1,x}}{\psi_1}. \tag{7}$$

Moreover, by virtue of the dependent variable transformation (7), one can convert equation (2) into the following bilinear form:

$$(D_x^4 + D_x D_t)\psi \cdot \psi = 0; \tag{8}$$

meanwhile it leads equations (3) and (4) to

$$(D_x^2 - \lambda)\psi \cdot \psi_1 = 0, \tag{9}$$

$$(D_t + D_x^3 + 3\lambda D_x)\psi \cdot \psi_1 = 0, \tag{10}$$

where the Hirota bilinear operator $D_x^m D_t^n$ is defined by

$$D_x^m D_t^n a \cdot b = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n a(x, t) b(x', t') \Big|_{x'=x, t'=t}.$$

2.2. The nonlocal symmetry from Bäcklund transformation

For equation (2) with its BT (3) and (4), consider the invariant property under

$$\lambda \rightarrow \lambda + \varepsilon\delta, \quad u \rightarrow u + \varepsilon\sigma, \quad u_1 \rightarrow u_1 + \varepsilon\sigma'.$$

That is to say, it is required that σ and σ' should satisfy

$$\begin{aligned} \sigma_t + \sigma_{xxx} - 6u_x\sigma_x &= 0, \\ \sigma'_x + \sigma_x + 2\delta - (\sigma - \sigma')(u - u_1) &= 0, \\ \sigma'_t + \sigma_t - (4u_x + 2u_{1x})\sigma_x - (2u_x + 4u_{1x})\sigma'_x + (\sigma - \sigma')(u_{xx} - u_{1,xx}) \\ &\quad + (u - u_1)(\sigma_{xx} - \sigma'_{xx}) = 0. \end{aligned} \tag{11}$$

We may find substantial possible nonlocal symmetries about u by solving equations (11) and a special case with $\delta = 0$ is presented and studied as follows.

Proposition 1. *The pKdV equation (2) has a new type of nonlocal symmetry given by*

$$\sigma = \exp\left(\int u - u_1 dx\right), \tag{12}$$

where u and u_1 satisfy BT (3) and (4). That means σ given by (12) satisfies the following symmetry equation:

$$\sigma_t + \sigma_{xxx} - 6u_x\sigma_x = 0. \tag{13}$$

Proof. By direct calculation. □

On the other hand, we let the bilinear pKdV equation (8) to be invariant under the transformation $\psi \rightarrow \psi + \varepsilon\sigma_\psi$, which produces the corresponding symmetry equation

$$(D_x^4 + D_x D_t)\sigma_\psi \cdot \psi = 0. \tag{14}$$

The Cole–Hopf transformation $u = -2\frac{\psi_x}{\psi}$ between equation (2) and its bilinear equation (8) determines a symmetry transformation for σ and σ_ψ , saying that

$$\sigma = \frac{2\psi_x\sigma_\psi}{\psi^2} - \frac{2\sigma_{\psi,x}}{\psi}. \tag{15}$$

Inserting equations (12) and (7) into equation (15), we obtain a class of nonlocal symmetry for equation (8):

$$\sigma_\psi = -\frac{\psi}{2} \int \frac{\psi_1^2}{\psi^2} dx. \tag{16}$$

Hence, it gives the following proposition for equation (8).

Proposition 2. *The bilinear pKdV equation (8) has the nonlocal symmetry expressed by (16), where ψ and ψ_1 satisfy bilinear BT (9) and (10).*

Proof. One can directly check that σ_ψ given by (16) satisfies the symmetry equation (14) under the consideration (9) and (10). □

3. Localization of the nonlocal symmetries and explicit solutions

We know that Lie point symmetries [2, 3] can be applied to construct explicit solutions for DEs, whereas the similar calculations seem to be invalid for nonlocal symmetries. So it is anticipated to turn the nonlocal symmetries into local ones, especially into Lie point symmetries. Following this idea, one may extend the original system to a closed prolonged system by introducing some additional dependable variables [13–15].

Fortunately, starting from the nonlocal symmetry (12), the prolongation is found to be closed when another two dependent variables $v \equiv v(x, t)$ and $g \equiv g(x, t)$ are introduced by

$$\begin{aligned} v_x &= u - u_1, & v_t &= 2(u - u_1)(u_x - 2\lambda) - 2u_{xx}, \\ g_x &= e^v, & g_t &= -e^v[2u_x + 8\lambda - (u - u_1)^2]. \end{aligned} \tag{17}$$

Now the prolonged equations (2), (3), (4) and (17) contain four dependable variables u, u_1, v and g , respectively, whose corresponding symmetries are

$$\sigma_u = e^v, \quad \sigma_{u_1} = 0, \quad \sigma_v = g, \quad \sigma_g = \frac{1}{2}g^2. \tag{18}$$

Remark 1. What is more interesting here is that the symmetry σ_g shown in (18) implies that the auxiliary-dependent variable g satisfies

$$g_t = \{g; x\}g_x + 6\lambda g_x, \quad \{g; x\} \equiv \frac{g_{xxx}}{g_x} - \frac{3}{2} \frac{g_{xx}^2}{g_x^2}, \tag{19}$$

which is just the Schwartz form of the KdV (SKdV) equation (1). This may provide us with a new way to seek for the Schwartz forms of DEs, especially for the discrete integrable models, without using the Painlevé analysis.

3.1. Finite symmetry transformation

Due to (18), the symmetry vector of the prolonged system has the form

$$V = e^v \frac{\partial}{\partial u} + 0 \frac{\partial}{\partial u_1} + g \frac{\partial}{\partial v} + \frac{1}{2} g^2 \frac{\partial}{\partial g}. \tag{20}$$

Then, by solving the following initial value problem

$$\begin{aligned} \frac{d\bar{u}}{d\varepsilon} &= e^{\bar{v}}, & \frac{d\bar{u}_1}{d\varepsilon} &= 0, & \frac{d\bar{v}}{d\varepsilon} &= \bar{g}, & \frac{d\bar{g}}{d\varepsilon} &= \frac{1}{2} \bar{g}^2, \\ \bar{u}|_{\varepsilon=0} &= u, & \bar{u}_1|_{\varepsilon=0} &= u_1, & \bar{v}|_{\varepsilon=0} &= v, & \bar{g}|_{\varepsilon=0} &= g, \end{aligned} \tag{21}$$

the finite transformation can be written out as follows:

$$\bar{u} = u + \frac{2\varepsilon}{2 - \varepsilon g} e^v, \quad \bar{u}_1 = u_1, \quad \bar{v} = v + 2 \ln \frac{2}{2 - \varepsilon g}, \quad \bar{g} = \frac{2}{2 - \varepsilon g} g. \tag{22}$$

Remark 2. The original BT (3) and (4) in itself suggest a finite transformation from one solution u to another one u_1 , and then the new BT (22) arrives at a third solution \bar{u} . Actually, transformation (22) is just the so-called Levi transformation [23]. The result of this paper shows the fact that two kinds of BT possess the same infinitesimal form (12).

Now by force of the finite transformation (22), one can obtain a new solution from any initial solution. For example, it is easy to solve an initial solution of prolonged equation system (2), (3), (4) and (17), namely

$$\begin{aligned} u &= c, & u_1 &= c + 2\sqrt{\lambda} \tanh \zeta, & v &= -\ln(\tanh^2 \zeta - 1), \\ g &= \frac{\sinh(2\zeta)}{4\sqrt{\lambda}} - \frac{x}{2} + 6\lambda t + c_0, & \zeta &= \sqrt{\lambda}(-x + 4\lambda t), \end{aligned} \tag{23}$$

where λ , c and c_0 are three arbitrary constants.

Starting from (23), a new solution of equation (2) can be presented immediately via (22):

$$\bar{u} = c - \frac{8\sqrt{\lambda}\varepsilon \cosh^2 \zeta}{8\sqrt{\lambda} - \varepsilon[\sinh(2\zeta) - 2\sqrt{\lambda}(x - 12\lambda t - 2c_0)]}, \tag{24}$$

which then leads to the corresponding solution of the KdV equation

$$\bar{\omega} = \bar{u}_x = 16\lambda\varepsilon \cdot \frac{[\cosh(2\zeta) + 1]\varepsilon + \sqrt{\lambda} \sinh(2\zeta)[4 + \varepsilon(x - 12\lambda t - 2c_0)]}{[8\sqrt{\lambda} - \varepsilon(\sinh(2\zeta) - 2\sqrt{\lambda}(x - 12\lambda t - 2c_0))]^2}, \tag{25}$$

with $\zeta = \sqrt{\lambda}(-x + 4\lambda t)$.

3.2. Similar reductions with group invariant solutions

Alternatively, to find more complete group invariant solutions of equation (2), we will study Lie point symmetries of the whole prolonged equation system. Suppose equations (2), (3), (4) and (17) be invariant under the infinitesimal transformations

$$u \rightarrow u + \varepsilon\sigma, \quad u_1 \rightarrow u_1 + \varepsilon\sigma_1, \quad v \rightarrow v + \varepsilon\sigma_2, \quad g \rightarrow g + \varepsilon\sigma_3,$$

with

$$\begin{aligned} \sigma &= X(x, t, u, u_1, v, g)u_x + T(x, t, u, u_1, v, g)u_t - U(x, t, u, u_1, v, g), \\ \sigma_1 &= X(x, t, u, u_1, v, g)u_{1,x} + T(x, t, u, u_1, v, g)u_{1,t} - U_1(x, t, u, u_1, v, g), \\ \sigma_2 &= X(x, t, u, u_1, v, g)v_x + T(x, t, u, u_1, v, g)v_t - V(x, t, u, u_1, v, g), \\ \sigma_3 &= X(x, t, u, u_1, v, g)g_x + T(x, t, u, u_1, v, g)g_t - G(x, t, u, u_1, v, g). \end{aligned} \tag{26}$$

Then, substituting expressions (26) into the symmetry equations of equations (2), (3), (4) and (17)

$$\begin{aligned} \sigma_t + \sigma_{xxx} - 6u_x\sigma_x &= 0, \\ \sigma_{1,x} + \sigma_x - (\sigma - \sigma_1)(u - u_1) &= 0, \\ \sigma_{1t} - \sigma_{xxx} + 2(u - u_1)\sigma_{xx} + 2(\sigma - \sigma_1)u_{xx} - [4\lambda + (u - u_1)^2 - 2u_x]\sigma_x \\ &\quad + 2(\sigma - \sigma_1)(u - u_1)(2\lambda - u_x) = 0, \\ \sigma_{2,x} - \sigma + \sigma_1 &= 0, \\ \sigma_{2t} + 2\sigma_{xx} + 2(u_1 - u)\sigma_x + 2(\sigma_1 - \sigma)(u_x - 2\lambda) &= 0, \\ \sigma_{3,x} - e^v\sigma_2 &= 0, \\ \sigma_{3t} + 2e^v[\sigma_x + (u_1 - u)(\sigma - \sigma_1) - \frac{1}{2}(u - u_1)^2\sigma_2 + (4\lambda + u_x)\sigma_2] &= 0, \end{aligned} \tag{27}$$

and collecting together the coefficients of partial derivatives of dependent variables, it yields a system of overdetermined linear equations for the infinitesimals X, T, U, U_1, V and G , which can be solved by virtue of *Maple* to give

$$\begin{aligned} X(x, t, u, u_1, v, g) &= c_1(x + 12\lambda t) + c_5, \\ T(x, t, u, u_1, v, g) &= 3c_1t + c_2, \\ U(x, t, u, u_1, v, g) &= -c_1(2\lambda x + u) + 2c_4e^v + c_3, \\ U_1(x, t, u, u_1, v, g) &= -c_1(2\lambda x + u_1) + c_3, \\ V(x, t, u, u_1, v, g) &= -c_1 + 2c_4g + c_6, \\ G(x, t, u, u_1, v, g) &= c_4g^2 + c_6g + c_7, \end{aligned} \tag{28}$$

where $c_i (i = 1, \dots, 7)$ are seven arbitrary constants. When $c_1 = c_2 = c_3 = c_5 = c_6 = c_7 = 0$, the reduced symmetry is just (18).

To give the group invariant solutions, we would like to solve symmetry constraint conditions $\sigma = 0$ and $\sigma_i = 0 (i = 1, 2, 3)$ defined by (26) with (28), which is equivalent to solve the following characteristic equation:

$$\begin{aligned} \frac{dx}{c_1(x + 12\lambda t) + c_5} &= \frac{dt}{3c_1t + c_2} = \frac{du}{-c_1(2\lambda x + u) + 2c_4e^v + c_3} \\ &= \frac{du_1}{-c_1(2\lambda x + u_1) + c_3} = \frac{dv}{-c_1 + 2c_4g + c_6} = \frac{dg}{c_4g^2 + c_6g + c_7}. \end{aligned} \tag{29}$$

Two nontrivial similar reductions under consideration $c_4 \neq 0$ are presented and substantial group invariant solutions are found in the follows.

Case 1. $c_1 \neq 0$ and $c_6^2 - 4c_4c_7 \neq 0$.

Without the loss of generality, we let $c_1 \equiv 1$. For simplicity, we introduce arbitrary constants a_4 and a_7 to replace c_4 and c_7 by $a_4^2 = c_6^2 - 4c_4c_7$ and $a_7 = -a_4^2/(16c_4)$, and then after solving equation (29), we have

$$\begin{aligned}
 u &= -\lambda x + 3\lambda^2 t + c_3 + c_5\lambda - 3c_2\lambda^2 \\
 &\quad + (3t + c_2)^{-\frac{1}{3}} \left[U(\xi) - \frac{a_4}{4a_7} \exp(V(\xi) - G(\xi)) \tanh B \right], \\
 u_1 &= -\lambda x + 3\lambda^2 t + c_3 + c_5\lambda - 3c_2\lambda^2 + \frac{U_1(\xi)}{(3t + c_2)^{\frac{1}{3}}}, \\
 v &= -\frac{1}{3} \ln(3t + c_2) - G(\xi) + V(\xi) - 2 \ln \cosh B, \\
 g &= \frac{8a_7}{a_4} \left(\tanh B + \frac{c_6}{a_4} \right), \tag{30}
 \end{aligned}$$

with $B = a_4(3G(\xi) + \ln(3c_1t + c_2))/6$ and $\xi = (x - 6\lambda t + c_5 - 6c_2\lambda)/(3t + c_2)^{\frac{1}{3}}$.

Here, $U(\xi)$, $U_1(\xi)$, $V(\xi)$, $G(\xi)$ and ξ represent five group invariants and substituting (30) into the prolonged equations system gives the following reduced equations:

$$H_{\xi\xi} = \frac{1}{2} \frac{H_\xi^2}{H} + 4a_7H^2 - \xi H - \frac{a_4^2}{32a_7^2H}, \tag{31}$$

$$U_1(\xi) = \frac{a_7H_\xi^2}{H} - 4a_7^2H^2 + 2a_7\xi H - \frac{\xi^2}{4} - \frac{a_4^2}{16a_7H} \tag{32}$$

$$U(\xi) = U_1(\xi) - \frac{H_\xi}{H}, \quad V(\xi) = G(\xi) - \ln(H), \quad G_\xi(\xi) = \frac{1}{4a_7H},$$

with $H \equiv H(\xi)$. One can see that whence H is solved from equation (31), two new group invariant solutions u and u_1 of equation (2) would be immediately obtained through equations (30) and (32).

Moreover, by making a further transformation [24]

$$H(\xi) = \frac{1}{2a_7} \left(P_\xi + P^2 + \frac{\xi}{2} \right), \quad P \equiv P(\xi), \tag{33}$$

equation (31) can be converted into the second Painlevé equation (P_{II}), reading

$$P_{\xi\xi} = 2P^3 + \xi P + \alpha, \tag{34}$$

with $\alpha = -(a_4 + 1)/2$. Now, every known solution of P_{II} (34) will generate two new group invariant solutions of equation (2), and then two new solutions of the KdV equation (1) denoted as ω_1 and ω_2 can be given directly after one derivative with respect to x for u_1 and u :

$$\omega_1 = \frac{1}{(3t + c_2)^{\frac{2}{3}}} (P_\xi + P^2) - \lambda, \tag{35}$$

$$\omega_2 = \frac{1}{(3t + c_2)^{\frac{2}{3}}} \left[\frac{a_4^2}{2F^2} \tanh^2 R_1 + \left(\frac{2a_4P}{F} - \frac{a_4^2}{F^2} \right) \tanh R_1 - \frac{2a_4P}{F} - P_\xi + P^2 \right] - \lambda, \tag{36}$$

where

$$F \equiv F(\xi) = 2P_\xi + 2P^2 + \xi, \quad R_1 = \frac{1}{6} a_4 [\ln(3t + c_2) + 3G(\xi)], \quad G_\xi(\xi) = \frac{1}{2P_\xi + 2P^2 + \xi},$$

and P satisfies P_{II} (34) with $\alpha = -(a_4 + 1)/2$.

It is known that the generic solutions of P_{II} are meromorphic functions and more information about P_{II} is provided in [25], saying that (1) for every $\alpha = N \in \mathbb{Z}$, there exists a unique rational solution of P_{II}; (2) for every $\alpha = N + \frac{1}{2}$, with $N \in \mathbb{Z}$, there exists a unique

one-parameter family of classical solutions which are expressible in terms of Airy functions; (3) for all other values of α , the solution of P_{II} is transcendental.

For example, when $\alpha = 1$ ($a_4 = -3$), P_{II} (34) possesses a simple rational solution $P(\xi) = -1/\xi$, which leads the solutions (35) and (36) to

$$\tilde{\omega}_1 = \frac{2}{\xi_1^2} - \lambda, \quad \tilde{\omega}_2 = -\lambda + \frac{6\xi_1(\xi_1^3 - 24t - 8c_2 - 2)}{(\xi_1^3 + 12t + 4c_2 + 1)^2}, \quad (37)$$

with $\xi_1 \equiv x - 6\lambda t + c_5 - 6c_2\lambda$.

When $\alpha = \frac{1}{2}$ ($a_4 = -2$), P_{II} (34) has a solution expressed by Airy function

$$P(\xi) = 2^{-\frac{1}{3}} \frac{3\text{Ai}(1, -2^{-\frac{1}{3}}\xi) - \sqrt{3}\text{Bi}(1, -2^{-\frac{1}{3}}\xi)}{3\text{Ai}(-2^{-\frac{1}{3}}\xi) - \sqrt{3}\text{Bi}(-2^{-\frac{1}{3}}\xi)}. \quad (38)$$

For simplicity, we convert equation (38) into the equivalent form

$$P(\xi) = \frac{\sqrt{2}\xi^{\frac{3}{2}}J(\frac{4}{3}, \frac{\sqrt{2}}{3}\xi^{\frac{3}{2}}) - 2J(\frac{1}{3}, \frac{\sqrt{2}}{3}\xi^{\frac{3}{2}})}{\xi J(\frac{1}{3}, \frac{\sqrt{2}}{3}\xi^{\frac{3}{2}})}, \quad (39)$$

where $J(n, \xi)$ is the first kind of Bessel function. Substituting (39) into (35) and (36), two exact solutions of KdV equation are obtained as follows:

$$\omega'_1 = -\lambda + \frac{\xi_1}{2(3t + c_2)} \left(1 + 2\frac{J_2^2}{J_1^2} \right), \quad (40)$$

$$\omega'_2 = \frac{32\xi_1^2 J_1^4}{[4\xi_1^2(J_1^2 + J_2^2) - 3 \cdot 2^{1/3}]^2} - \frac{8\sqrt{2}\xi_1^{3/2} J_1 J_2}{\sqrt{3t + c_2}[4\xi_1^2(J_1^2 + J_2^2) - 3 \cdot 2^{1/3}]} - \frac{\xi_1}{2(3t + c_2)} - \lambda, \quad (41)$$

where we denote $J_1 = J(\frac{1}{3}, \frac{\sqrt{2}}{3} \frac{\xi_1^{3/2}}{\sqrt{3t+c_2}})$, $J_2 = J(-\frac{2}{3}, \frac{\sqrt{2}}{3} \frac{\xi_1^{3/2}}{\sqrt{3t+c_2}})$.

To continue the same procedure, sequences of rational solutions and Bessel (Airy) function solutions for the KdV equation will be easily constructed. Furthermore, by selecting suitable parameters in this kind of similar reduction, we may discover more unknown exact solutions among interaction solitons and Painlevé waves of the KdV equation.

Case 2. $c_1 = 0$ and $c_2 \neq 0$.

Firstly, we replace c_4 and c_5 with a_4 and k by $a_4^2 = c_6^2 - 4c_4c_7$ and $k = c_5/c_2$, and it follows the results from equation (29), saying that

$$\begin{aligned} u &= \frac{c_3}{c_2}t + U(z) + \frac{c_3}{c_2}G(z) - \frac{(a_4^2 - c_6^2)}{a_4c_7}e^{V(z)} \tanh \left[\frac{a_4(t + G(z))}{2c_2} \right], \\ u_1 &= \frac{c_3}{c_2}t + U_1(z), \\ g &= \frac{2c_7}{a_4^2 - c_6^2} \left[c_6 + a_4 \tanh \left(\frac{a_4(t + G(z))}{2c_2} \right) \right], \\ v &= V(z) - 2 \ln \cosh \left[\frac{a_4(t + G(z))}{2c_2} \right] \end{aligned} \quad (42)$$

with $z = x - kt$. Substituting (42) into equations (2), (3), (4) and (17) and redefining the parameters, we note that the new group invariants $U(z)$, $U_1(z)$, $V(z)$ and $G(z)$ are subject to

$$W_z^2 - a_2^2W^4 - a_3W^3 + a_5W^2 - a_7W = 0, \quad (43)$$

$$\begin{aligned}
 U_{1z}(z) &= \frac{a_7}{2W} - \lambda - \frac{a_5}{4}, \\
 U(z) &= U_1(z) + \frac{W_z}{W} + \left(3\lambda^2 + \frac{\lambda a_5}{2} - \frac{a_5^2}{16} + \frac{a_3 a_7}{4} \right) G, \\
 V(z) &= \ln(W), \\
 G_z(z) &= \frac{W}{a_7},
 \end{aligned}
 \tag{44}$$

with $W \equiv W(z)$ and $a_2 = \frac{a_4^2 - c_6^2}{a_4 c_7}$, $a_3 = \frac{(a_4^2 - c_6^2)(c_2 k^2 + 48 c_2 \lambda^2 - 16 c_2 k \lambda - 4 c_3)}{a_3^2 c_7}$, $a_5 = 2k - 12\lambda$, $a_7 = \frac{a_3^2 c_7}{c_2(a_4^2 - c_6^2)}$.

Remark 3. The case $c_1 = 0$ here is interesting. From equation (43), we know that W can be expressed as an elliptic integration and can be written out by means of Jacobi elliptic functions. Whence W is fixed from (43), all the other quantities will be given simply by differentiation or integration. The first equation of (42) implies the important byproduct, the explicit exact interaction between the cnoidal periodic wave and kink soliton.

A simple example of this case appears in the form

$$W(z) = \frac{a_3}{4a_2^2} \left[\operatorname{sn} \left(\frac{a_3 z}{4a_2 n}, n \right) - 1 \right],
 \tag{45}$$

with the constraint conditions $a_5 = \frac{a_3^2(1-5n^2)}{16n^2 a_2^2}$ and $a_7 = \frac{a_3^2(n^2-1)}{32n^2 a_2^2}$ in equation (43), where n is the modulus of the Jacobian elliptic function sn .

Taking the given solution (45) into (42) with (44), two exact solutions of the KdV equation are obtained

$$\omega_3 = \frac{a_3^2(1-n^2)}{16a_2^2 n^2(Y+1)} + \frac{a_3^2(5n^2-1)}{64a_2^2 n^2} - \lambda,
 \tag{46}$$

$$\begin{aligned}
 \omega_4 &= \frac{a_3^2(Y+1)^2}{32a_2^2} \tanh^2 R_2 + \frac{a_3^2 \sqrt{Y^2-1} \sqrt{n^2 Y^2-1}}{16na_2^2} \tanh R_2 \\
 &+ \frac{2a_3^2(Y^2-2Y)}{a_2^2} - \frac{a_3^2(n^2+1)}{n^2 a_2^2} - 64\lambda,
 \end{aligned}
 \tag{47}$$

where we have

$$\begin{aligned}
 R_2 &= \frac{a_3^3(n^2-1)(t+a_6)}{64a_2^3 n^2} + \frac{1}{4} \ln \left(\frac{2n}{2n^2 Y^2 + 2n \sqrt{Y^2-1} \sqrt{n^2 Y^2-1} - n^2 - 1} \right) \\
 &+ \frac{1}{2} n \int_0^Y \frac{1}{\sqrt{1-t^2} \sqrt{1-n^2 t^2}} dt, \quad Y = \operatorname{sn} \left(\frac{a_3(192\lambda a_2^2 n^2 - 5a_3^2 n^2 + a_3^2)t}{128n^3 a_3^2} - \frac{a_3 x}{4a_2 n}, n \right),
 \end{aligned}$$

and a_2, a_3, a_6 and λ are four arbitrary constants.

4. Integrable models from nonlocal symmetry

To find new integrable models is another important application of the symmetry study. The symmetry constraint method is one of the most powerful tools to give out new integrable models from known ones. Usually, people cast the symmetry constraint condition on Lax pair of soliton equations. In this section, we would like to combine the nonlocal symmetry with BT to give some integrable models both in lower and higher dimensions.

Let every pair (u, u_i) ($i = 1, 2, \dots, N$) satisfy the following BT:

$$u_x + u_{i,x} = -2\lambda_i + \frac{(u - u_i)^2}{2}, \tag{48}$$

$$u_t + u_{i,t} = 2u_x^2 + 2u_{i,x}^2 + 2u_x u_{i,x} - (u - u_i)(u_{xx} - u_{i,xx}), \tag{49}$$

and the corresponding nonlocal symmetry of u reads $\sigma^i = \exp(\int u - u_i dx)$ for $i = 1, 2, \dots, N$.

4.1. Finite-dimensional integrable systems

In general, every one symmetry of a higher dimensional model can lead the original one to its lower form. Now, considering

$$u_x = \sum_{i=1}^N a_i \exp\left(\int u - u_i dx\right) \tag{50}$$

as a generalized symmetry constraint condition and acting it on the x -part of the BT (3), we give the finite-dimensional $(N + 1)$ -component integro-differential system

$$\begin{aligned} u_x &= \sum_{i=1}^N a_i \exp\left(\int u - u_i dx\right), \\ u_x + u_{i,x} &= -2\lambda_i + \frac{(u - u_i)^2}{2}, \quad i = 1, 2, \dots, N, \end{aligned} \tag{51}$$

where every a_i and λ_i are arbitrary constants. For further simplification, making $u_i = u - (\ln w_{ix})_x$, then the constraint condition (50) becomes

$$u = \sum_{m=1}^N a_m w_m, \tag{52}$$

which transforms (51) into the following N -component differential system

$$2w_{ixx}w_{ix} - 4\left(\sum_{m=1}^N a_m w_{mx}\right)w_{ix}^2 - w_{ix}^2 - 4\lambda_i w_{ix}^2 = 0, \quad i = 1, 2, \dots, N. \tag{53}$$

Taking $s_i = w_{ix}$, we rewrite equation (53) as

$$2s_{ixx}s_i - 4\left(\sum_{m=1}^N a_m s_m\right)s_i^2 - s_{ix}^2 - 4\lambda_i s_i^2 = 0, \quad i = 1, 2, \dots, N. \tag{54}$$

Making $s_i = b_i q_i^2$, equation (54) is equivalent to the downward integrable system

$$q_{ixx} - \left(\sum_{m=1}^N c_m q_m^2\right)q_i - \lambda_i q_i = 0, \quad i = 1, 2, \dots, N, \tag{55}$$

with $c_i = a_i b_i$ being an arbitrary constant.

On the other hand, by virtue of the same symmetry constraint (50) and the t -part of the BT (4), we can construct another set of the integrable system

$$\begin{aligned} u_x &= \sum_{i=1}^N a_i \exp\left(\int u - u_i dx\right), \\ u_t + u_{i,t} &= 2u_x^2 + 2u_{i,x}^2 + 2u_x u_{i,x} - (u - u_i)(u_{xx} - u_{i,xx}), \quad i = 1, 2, \dots, N. \end{aligned} \tag{56}$$

Considering the similar transformations of dependent variables done in x -part, after a series of tedious substitutions, equation (56) becomes

$$\begin{aligned}
 q_i q_{ixt} - q_{it} q_{ix} + \left(2 \sum_{m=1}^N c_m q_m^2 - 4\lambda_i \right) q_{ix}^2 - 4q_i q_{ix} \sum_{m=1}^N c_m q_m q_{mx} - 2q_i^2 \left(\sum_{m=1}^N c_m q_m^2 \right)^2 \\
 + 2\lambda_i q_i^2 \sum_{m=1}^N c_m q_m^2 + 4\lambda_i^2 q_i^2 + 2q_i^2 \sum_{m=1}^N c_m (q_{mx}^2 + q_m q_{mxx}) = 0,
 \end{aligned}$$

(57)

Substituting equation (55) into account, equation (57) can be integrated once about x to give the N -component integrable system

$$q_{it} = -2 \sum_{m=1}^N c_m q_m q_{mx} q_i + 2 \sum_{m=1}^N c_m q_m^2 q_{ix} - 4\lambda_i q_{ix}, \quad i = 1, 2, \dots, N.$$

(58)

In fact, equations (55) and (58) are essentially the canonical equations (F_0) and (F_1), respectively [16], saying that

$$(F_0) : \quad q_{ix} = p_i, \quad p_{ix} = \left(\sum_{m=1}^N c_m q_m^2 \right) q_i + \lambda_i q_i.$$

(59)

$$(F_1) : \quad \left\{ \begin{aligned}
 q_{it} &= -2 \left(\sum_{m=1}^N c_m p_m q_m \right) q_i + 2 \left(\sum_{m=1}^N c_m q_m^2 \right) p_i - 4\lambda_i p_i, \\
 p_{it} &= 2 \left(\sum_{m=1}^N c_m p_m q_m \right) p_i - 2 \left(\sum_{m=1}^N c_m p_m^2 \right) q_i - 4\lambda_i^2 q_i \\
 &\quad \left\{ -2\lambda_i \left(\sum_{m=1}^N c_m q_m^2 \right) q_i - 2 \left(\sum_{m=1}^N \lambda_m c_m q_m^2 \right) q_i \right\}
 \end{aligned} \right.$$

(60)

It should be stressed here that the finite integrable systems (55) and (58) reobtained via this way are just the remarkable results given by Cao in [17] through the nonlinearization method, both of which have been proved completely integrable in Liouville sense. Thanks to these finite integrable systems (59) and (60), the original high-dimensional KdV equation would be solved.

4.2. Infinite-dimensional integrable systems

For obtaining some higher dimensional integrable models, one may introduce some internal parameters [7, 9, 21]. Here, we would like to use the internal parameter-dependent symmetry constraints on BT to construct two sets of infinite-dimensional integrable systems.

It is obvious that equation (2) is invariant under the internal parameter translation, say y translation, so we can view

$$u_y = \sum_{i=1}^N a_i \exp \left(\int u - u_i dx \right)$$

(61)

as a new symmetry constraint condition.

Firstly, imposing (61) on the x -part of the BT (3) yields a (1+1)-dimensional $(N + 1)$ -component integro-differential system

$$\begin{aligned}
 u_y &= \sum_{i=1}^N a_i \exp\left(\int u - u_i dx\right), \\
 u_x + u_{i,x} &= -2\lambda_i + \frac{(u - u_i)^2}{2}, \quad i = 1, 2, \dots, N,
 \end{aligned}
 \tag{62}$$

where λ_i and a_i ($i = 1, 2, \dots, N$) are constants.

By the transformation $u_i = u - (\ln \phi_{iy})_x$, equation (61) becomes

$$u = \sum_{i=1}^N a_i \phi_i,
 \tag{63}$$

which then converts (62) into the following (1+1)-dimensional N -component differential system:

$$2\phi_{ixy}\phi_{iy} - 4\left(\sum_{m=1}^N a_m \phi_{mx}\right)\phi_{iy}^2 - \phi_{ixy}^2 - 4\lambda_i \phi_{iy}^2 = 0, \quad i = 1, 2, \dots, N.
 \tag{64}$$

Alternatively, combining the constraint condition (61) with the t -part of the BT (4) will produce a (1+2)-dimensional system about x, y and t , reading

$$\begin{aligned}
 u_y &= \sum_{i=1}^N a_i \exp\left(\int u - u_i dx\right), \\
 u_t + u_{i,t} &= 2u_x^2 + 2u_{i,x}^2 + 2u_x u_{i,x} - (u - u_i)(u_{xx} - u_{i,xx}), \quad i = 1, 2, \dots, N.
 \end{aligned}
 \tag{65}$$

By means of (63) and (64), equation (65) is transformed into the following N -component system:

$$\begin{aligned}
 \phi_{ixyt}\phi_{iy} - \phi_{ixy}\phi_{iyt} - 2\left(\sum_{m=1}^N a_m \phi_{mxx}\right)\phi_{iy}\phi_{ixy} + \left(\sum_{m=1}^N a_m \phi_{mx} - 2\lambda_i\right)\phi_{ixy}^2 \\
 + 2\left[\left(\sum_{m=1}^N a_m \phi_{mx}\right)^2 + 2\lambda_i \sum_{m=1}^N a_m \phi_{mx} - \sum_{m=1}^N a_m \phi_{mt} + 4\lambda_i^2\right]\phi_{iy}^2 = 0.
 \end{aligned}
 \tag{66}$$

It should be noted that the integrability of the infinite-dimensional systems (64) and (66) is not quite clear. The finite-dimensional models obtained here are completely integrable, that strongly suggests that these infinite-dimensional models should have many nice integrable properties. It will be of much interest to investigate the integrability of these models in the further work.

5. Conclusion and discussions

In this paper, we have shown that combining nonlocal symmetries with BT can result in many diverse applications. The main new progresses made in this paper in the general aspect of integrable systems are as follows.

- (i) The BTs are used to find nonlocal symmetries.

- (ii) Different types of BTs may possess same infinitesimal forms and then new types of BTs may be obtained from old ones.
- (iii) New integrable hierarchies can be obtained from nonlocal symmetries related to BTs.
- (iv) New finite-dimensional integrable systems can be obtained from BTs and related symmetry constraints and reductions. And then the original high-dimensional model can be solved from lower dimensional ones because of the existence of nonlocal symmetries.
- (v) The exact interaction solutions among solitons and other complicated waves including periodic cnoidal waves and Painlevé waves are revealed which have not yet been found for any integrable models because it is difficult to solve the original BT (or Darboux transformation) problem if the original seed solutions are taken as the cnoidal or Painlevé waves.
- (vi) The localization procedure results in a new way to find Schwartz form of the original model which is obtained usually via Painlevé analysis for the continuous integrable systems. The method may provide a potential method to transform discrete integrable systems to Schwartz forms because usually the BTs of discrete integrable models are known.

The above progresses are realized especially for the potential KdV (pKdV) equation. For pKdV equation, it possesses a new class of nonlocal symmetry resulting from its BT. Since this BT is of Riccati type, more information about its bilinear forms is also learned via the Cole–Hopf transformation.

In order to extend applicability of nonlocal symmetry to obtain explicit solutions of the KdV equation, we introduce another two auxiliary variables v and g to form a prolonged system with u and u_1 , so that the original nonlocal symmetry can be transformed to a Lie point symmetry of the new system. Then what follows are innovative Lie–Bäcklund transformation and two kinds of novel similarity reductions: by virtue of two kinds of BT, the solitary wave solutions of KdV equation are obtained through the transformations of trivial solutions; concerning the complete Lie point symmetries of the prolonged system, we achieve rich group invariant solutions including rational solution hierarchy, Bessel function solution hierarchy and periodic function solutions.

Applying nonlocal symmetry on the BT of the pKdV equation, finite-dimensional integrable systems are given, which are found equivalent to the excellent work done by Cao [17]. Moreover, the introduction of an internal parameter as a new argument helps us to build two sets of infinite-dimensional models.

However, in this paper, it still remains unclear what kind of nonlocal symmetries must have close prolongations and can be applied to construct distinctive exact solutions. The concrete integrability for the given infinite-dimensional models is also unknown. Moreover, one may consider algebraic geometry solutions of the completely integrable finite-dimensional systems to achieve related solutions of the KdV equation. It is quite reasonable and meaningful that these matters merit our further study.

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References

- [1] Rogers C and Shadwick W F 1982 *Bäcklund Transformation and Their Applications* (New York: Academic)
- [2] Olver P J 1993 *Applications of Lie Groups to Differential Equations* (New York: Springer)
- [3] Bluman G W and Kumei S 1989 *Symmetries and Differential Equations* (Berlin: Springer)
- [4] Lou S Y 1993 *Phys. Lett. B* **302** 261
Lou S Y 1993 *Int. J. Mod. Phys. A* (Proc. Suppl.) **3** 531
Lou S Y 1994 *J. Math. Phys.* **35** 2390
- [5] Guthrie G A 1993 *J. Phys. A: Math. Gen.* **26** L905
Guthrie G A and Hickman M S 1993 *J. Math. Phys.* **26** 193
Guthrie G A 1994 *Phys. R. Soc. Lond. A* **446** 107
- [6] Lou S Y 1993 *J. Phys. A: Math. Gen.* **26** L789
Lou S Y 1993 *Phys. Lett. A* **181** 13
Lou S Y 1994 *Phys. Lett. A* **187** 239
Lou S Y 1994 *Solitons Chaos Fractals* **4** 1961
Ruan H Y and Lou S Y 1993 *J. Phys. Soc. Japan* **62** 1917
Han P and Lou S Y 1994 *Acta Phys. Sin.* **43** 1042 (in Chinese)
Lou S Y and Chen W Z 1993 *Phys. Lett. A* **179** 271
- [7] Lou S Y 1997 *J. Phys. A: Math. Phys.* **30** 4803
- [8] Lou S Y and Hu X B 1997 *J. Phys. A: Math. Gen.* **30** L95
- [9] Lou S Y and Hu X B 1997 *J. Math. Phys.* **38** 6401
- [10] Edelen D G 1980 *Isovector Methods for Equations of Balance* (Alphen aam den Rijn: Sijthoff and Noordhoff)
- [11] Krasil'shchik I S and Vinogradov A M 1984 *Acta Appl. Math.* **2** 79
- [12] Krasil'shchik I S and A M 1989 *Acta Appl. Math.* **15** 161
- [13] Galas F 1992 *J. Phys. A: Math. Gen.* **25** L981
- [14] Lou S Y and Hu X B 1993 *Chin. Phys. Lett.* **10** 577
- [15] Lou S Y, Ruan H Y, Chen W Z, Wang Z L and Chen L L 1994 *Chin. Phys. Lett.* **11** 593
- [16] Cao C W 1990 *Sci. China A* **33** 528
- [17] Cao C W 1991 *Acta Math. Sin.* **7** 216
- [18] Cao C W, Wu Y T and Geng X G 1999 *J. Math. Phys.* **40** 3948
- [19] Antonowicz M and Rauch-Wojciechowski S 1991 *J. Phys. A: Math. Gen.* **24** 5043
- [20] Antonowicz M and Rauch-Wojciechowski S 1992 *J. Math. Phys.* **33** 2115
- [21] Lou S Y 1997 *Commun. Theor. Phys.* **27** 249
- [22] Wahlquist H D and Estabrook F B 1973 *Phys. Rev. Lett.* **31** 1386
- [23] Levi D 1988 *Inverse Problems* **4** 165
- [24] Ince E L 1956 *Ordinary Differential Equations* (New York: Dover)
- [25] Umemura H and Watanabe H 1997 *Nagoya Math. J.* **148** 151