Nonlinear Analysis: Real World Applications 45 (2019) 918–941

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

High-order soliton matrices for Sasa–Satsuma equation via local Riemann–Hilbert problem

Bo Yang, Yong Chen *

Shanghai Key Laboratory of Trustworthy Computing, East China Normal University, Shanghai, 200062, People's Republic of China

A R T I C L E I N F O

Article history:
Received 1 May 2016
Received in revised form 4 August 2018
Accepted 6 August 2018
Available online xxxx

Keywords:
Riemann–Hilbert problem
Darboux transformation
High-order soliton solution
Asymptotic analysis

A B S T R A C T

A study of high-order soliton matrices for Sasa–Satsuma equation in the framework of the Riemann–Hilbert problem approach is presented. Through a standard dressing procedure, soliton matrices for simple zeros and elementary high-order zeros in the Riemann–Hilbert problem for Sasa–Satsuma equation are constructed, respectively. It is noted that pairs of zeros are simultaneously tackled in the situation of the high-order zeros, which is different from other NLS-type equation. Furthermore, the generalized Darboux transformation for Sasa–Satsuma equation is also presented. Moreover, collision dynamics along with the asymptotic behavior for the two-solitons are analyzed, and long time asymptotic estimations for the high-order one-soliton are concretely calculated. In this case, two double-humped solitons with nearly equal velocities and amplitudes can be observed.

1. Introduction

It is known to us all that several important nonlinear partial differential equations (PDEs) in mathematical physics are integrable with rich mathematical structures and extensive physics applications. In particular, it is always possible to find explicit solutions to these equations, such as they often have multi-soliton solutions. Among these integral PDEs, the nonlinear Schrödinger (NLS) equation:

$$iq_T + \frac{1}{2}q_{XX} + |q|^2 q = 0, \tag{1}$$

has been considered as the most important mathematical model. Eq. (1) has various applications in a wide range of physical systems such as water waves [1,2], nonlinear optics [3,4], solid-state physics and plasma physics [5]. This equation can be used to model optical solitons in fibers. However, several phenomena observed in the experiment cannot be explained by NLS equation, as the short soliton pulses get shorter,

* Corresponding author.
E-mail address: ychen@sei.ecnu.edu.cn (Y. Chen).

https://doi.org/10.1016/j.nonrwa.2018.08.004

some additional effects become important. In order to explain such phenomena, Kodama and Hasegawa [6,7] proposed a high-order NLS equation

$$iq_T + \frac{1}{2}q_{XX} + |q|^2 q + i\varepsilon \left[\beta_1 q_{XXX} + \beta_2 |q|^2 q_X + \beta_3 q(|q|^2)_X\right] = 0,\ \varepsilon = \pm 1. \tag{2}$$

This equation has clear and important physical significance. It was proposed [6,7] to describe the propagation of femtosecond pulses in optical fibers, or to model the propagation and interaction of the ultrashort pulses in the sub-picosecond or femtosecond regime. In this equation, u represents the slowly varying envelope of the electric field, while X and T are the normalized distance along the direction of the propagation and retarded time, respectively. β_1, β_2 and β_3 are the real parameters with respect to the third-order dispersion, self-steepening and stimulated Raman scattering, respectively.

In general, the integral PDEs can be analyzed by means of inverse scattering transformation (IST) method. However, Eq. (2) is not completely integrable unless certain restrictions are imposed on β_1, β_2 and β_3. Until now, we have the following four integrable cases:

The Kaup–Newell [8] derivative NLS equation($\beta_1 : \beta_2 : \beta_3 = 0 : 1 : 1$),
The derivative NLS equation-type II [9]($\beta_1 : \beta_2 : \beta_3 = 0 : 1 : 0$),
The Hirota [10] NLS equation($\beta_1 : \beta_2 : \beta_3 = 1 : 6 : 0$),
The Sasa–Satsuma [11] NLS equation($\beta_1 : \beta_2 : \beta_3 = 1 : 6 : 3$).

In particular, Ref. ¡ce:cross-refs refid=”b8 b9 b10 b11”¿[11]¡/ce:cross-refs¿ considers the following equation:

$$iq_T + \frac{1}{2}q_{XX} + |q|^2 q + i\varepsilon \left[q_{XXX} + 6|q|^2 q_X + 3q(|q|^2)_X\right] = 0, \tag{3}$$

with the variable transformations having been introduced:

$$u(x,t) = q(X,T)\exp\left\{\frac{-i}{6\varepsilon}\left(X - \frac{T}{18\varepsilon}\right)\right\},\ t = T,\ x = X - \frac{T}{12\varepsilon}.$$

Then Eq. (3) can be reduced to a complex modified KdV-type equation:

$$u_t + \varepsilon\left\{u_{xxx} + 6|u|^2 u_x + 3u(|u|^2)_x\right\} = 0. \tag{4}$$

Eq. (4) is generally known as the Sasa–Satsuma (S–S) equation. Two interesting features about this equation are that its solitons are embedded inside the continuous spectrum of the equation [12], and their shapes can be double humped for a wide range of soliton parameters [11]. The initial value problem for the local Sasa–Satsuma equation has been solved earlier using inverse scattering transform. The inverse problem is solved via the Gel'fand–Levitan–Marchenko (GLM) equation and the N-soliton solution is constructed [11]. The squared eigenfunctions for S–S equation were calculated [12], and the initial–boundary value (IBV) problem for S–S equation on the half line was also investigated via the unified transform method [13]. Besides, the Hirota's bilinear approach [14] and the Darboux transformation [15] were also imposed separately on this equation to obtain several types of soliton solutions. Moreover, rogue wave solutions for this equation were also investigated [16–19].

The inverse scattering method is a powerful tool to solve the initial value problem for nonlinear integrable PDEs, and it is the poles of the reflection coefficient, or the zeros of the Riemann–Hilbert problem (RHP), that give rise to the soliton solutions. For the KdV equation, because the Lax pair is self-adjoint operator, the discrete spectrum only produce simple poles. However, for the focusing NLS equation, the Lax operator is no longer self adjoint, thus it can produce multiple poles, which leads to the high-order soliton solution. Being an important kind of exact solution of the NLS-type equation, the high-order soliton has wide applications, it can describe a weak bound state of solitons and may appear in the study of train propagation of solitons with nearly equal velocities and amplitudes but having a particular chirp [20]. Soliton matrices corresponding to arbitrary number of high-order zeros for the RHP with arbitrary matrix dimension were derived for integrable nonlinear equations [21,22]. The high-order soliton formula of Landau–Lifshitz (L–L) equation was constructed through the generalized Darboux transformation combined with inverse scattering method [23].

In this article, we study general soliton matrices for S–S equation via the RHP approach, which corresponds to simple zeros and high-order zeros of RHP. N-soliton solution for S–S equation was already given via different approaches [11,24]. The major procedures of the RHP approach are inherited from the idea proposed in [21]. Owing to the symmetry properties of Jost solution and scattering data, the corresponding zeros in the RHP for S–S equation appear in pairs. In the case of simple zeros, we construct the soliton matrices for S–S equation via the RHP formulation along with dressing procedure [25–28]. Besides, we give the form of DT for S–S equation via a rigorous proof. The properties for one-soliton are studied while the collision dynamics for two-solitons are further analyzed. In the case of the elementary high-order zeros, the high-order soliton matrices for S–S equation are derived and the asymptotic estimations for the high-order one-soliton solution are calculated. An interesting novel phenomenon for this solution is the observation of two double-humped solitons with nearly equal velocities and amplitudes, which indicates more sophisticated structures and more physical importance for this equation.

This paper is organized as follows. In Section 2, the inverse scattering theory is established for the 3×3 spectral problem, and the corresponding matrix Riemann–Hilbert problem is formulated. In Section 3, the N-soliton formula for S–S equation is derived by considering the simple zeros in the RHP. Then the Darboux transformation is naturally constructed with a proof on its vitality. In Section 4, the high-order soliton matrices and the generalized Darboux transformation is constructed and the explicit high-order N-soliton formula is obtained, which corresponds to the elementary high-order zeros in the RHP. The final section contains some remarks and discussions on the nonlocal deformation of local S–S equation.

2. Inverse scattering theory for Sasa–Satsuma equation

In this section, we consider the scattering and inverse scattering problem for Sasa–Satsuma equation. Here, we consider the focusing case and take $\varepsilon = 1$ in Eq. (4).

2.1. Scattering theory of the spectral problem

Considering the following Sasa–Satsuma equation

$$u_t + u_{xxx} + 6|u|^2 u_x + 3u(|u|^2)_x = 0, \quad (x, t \in \mathbf{R}) \tag{5}$$

which is the compatibility condition of the following spectral problem [11] :

$$\Phi_x = U\Phi, \quad \Phi_t = V\Phi, \tag{6}$$

with 3×3 matrices U and V in the forms of:

$$\begin{aligned} U &= -ik\sigma_3 + Q, \\ V &= -4i\varepsilon k^3\sigma_3 + V_1, \end{aligned}$$

where Q and V_1 are matrix functions, $\sigma_3 = diag(1, 1, -1)$,

$$Q = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & \overline{u} \\ -\overline{u} & -u & 0 \end{pmatrix} \tag{7}$$

is the matrix of potential $u = u(x,t)$, the overline "$-$" stands complex conjugation, k is the spectral parameter, and V_1 has the form:

$$\begin{aligned}
V_1 &= k^2 V_1^{(2)} + k^1 V_1^{(1)} + V_1^{(0)},\\
V_1^{(2)} &= 4Q, V_1^{(1)} = 2i\sigma_3(Q_x - Q^2),\\
V_1^{(0)} &= -4|u|^2 Q - Q_{xx} + [Q_x, Q].
\end{aligned}$$

The scattering problem is the spatial part of system (6), i.e.

$$(-\partial_x + Q)\,\Psi = ik\sigma_3\Psi. \tag{8}$$

Supposing $u(x) = u(x,0)$ decays to zero sufficiently fast when $|x| \to \infty$.

More precisely, $u(x,t)$ belong to the weighted Sobolev space $H_1^1(\mathbb{R})$:

$$H_1^1(\mathbb{R}) = \left\{ f(x) \mid f, f_x, xf \in L^2(\mathbb{R}) \right\},$$

so that the direct problem can be well-posed.

Introducing a new matrix function:

$$J(x,t) = \Psi E_1^{-1}, \quad E_1 = e^{-ik\sigma_3 x - 4ik^3\sigma_3 t}, \tag{9}$$

E_1 is a solution of spectral equation (6) at $x \to \pm\infty$, then spectral problem (6) becomes:

$$J_x = -ik[\sigma_3, J] + QJ, \tag{10a}$$

$$J_t = -4ik^3[\sigma_3, J] + V_1 J, \tag{10b}$$

where $[\ ,\ ]$ is the common commutator. Introducing the matrix Jost solutions $J_\pm(x,k)$ of Eq. (10a) with the asymptotic boundary condition:

$$J_\pm(x,k) \to \mathbb{I}, \text{ when } x \to \pm\infty. \tag{11}$$

Here, $\mathbb{I}$ is the 3×3 unit matrix, it is noted that Jost solutions with condition (11) solve the following Volterra type integral equations:

$$J_\pm(x,k) = \mathbb{I} + \int_{\pm\infty}^{x} dy e^{-ik\sigma_3(x-y)} Q(y) J_\pm(y,k) e^{ik\sigma_3(x-y)}. \tag{12}$$

Let $J_\pm^{[k]}$ be the kth columns of matrices $J_\pm$, then $J_\pm$ can be divided into $J_\pm = (J_\pm^{[1]}, J_\pm^{[2]}, J_\pm^{[3]})$. The properties of the Jost solution $J_\pm(x,k)$ can be summarized as the following:

Properties 1. *Supposing $Q \in L^1(\mathbb{R})$, then solution $\left(J_-^{[1]}, J_-^{[2]}, J_+^{[3]}\right)$ is analytic in $\mathbb{C}_+ = \{k|Imk > 0\}$, while $\left(J_+^{[1]}, J_+^{[2]}, J_-^{[3]}\right)$ is analytic in $\mathbb{C}_- = \{k|Imk < 0\}$. And they are all continuous on the real line.*

Proof. From the integral equation (12), we have

$$J_-^{[1]}(x;k) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \int_{\pm\infty}^{x} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{2ik(x-y)} \end{pmatrix} Q(y) J_-^{[1]}(y;k) dy. \tag{13}$$

Making an estimation from equation (13),

$$|J_-^{[1]}(x;k)| \le 1 + \int_{-\infty}^{x} |Q(y) \parallel J_-^{[1]}(y;k)| dy. \tag{14}$$

Introducing the following series:

$$J_{-}^{[1]}(x;k)=g_0+\sum_{n=1}^{+\infty}g_n(x;k), \tag{15}$$

where,

$$g_0=\begin{pmatrix}1\\0\\0\end{pmatrix},\ g_{k+1}=\int_{\pm\infty}^{x}\begin{pmatrix}1&0&0\\0&1&0\\0&0&e^{2ik(x-y)}\end{pmatrix}Q(y)g_k(y;k)dy,$$

it can be seen that

$$|g_1(x;k)|\le\int_{-\infty}^{x}|Q(y)|dy,$$

then it follows that

$$|g_n(x;k)|\le\frac{1}{n!}\left(\int_{-\infty}^{x}|Q(y)|dy\right)^n.$$

Hence we have the estimation

$$|J_{-}^{[1]}(x;k)|\le\sum_{n=0}^{+\infty}\frac{1}{n!}\left(\int_{-\infty}^{x}|Q(y)|dy\right)^n=\exp\left(\int_{-\infty}^{x}|Q(y)|dy\right).$$

The above estimate implies that series (15) converges uniformly in $\mathbb{C}_+$, so that solution $J_{-}^{[1]}$ is analytical in the upper half plane and can be continuously extended to the real line. Besides, the uniqueness of the solution can be proved by inequality (14) with the Gronwall inequality. Same results can be also obtained for solution $J_{+}^{[1]}, J_{\pm}^{[2]}, J_{\pm}^{[3]}$. This completes the proof. □

Being the solutions of spectral problem (2), matrix function J_-E and J_+E are linearly interconnected by the 3×3 scattering matrix $S(k)$:

$$J_-(x,k)E=J_+(x,k)ES(k),\ \text{for } k\in\mathbb{R}, \tag{16}$$

where, $E=-ikx\sigma_3$, and $S(k)=(s_{ij})_{3\times 3}$. It is noted that $s_{11}(k)$, $s_{12}(k)$, $s_{21}(k)$ and $s_{22}(k)$ can be analytical continuation to upper half plane $\mathbb{C}_+$, and $s_{33}(k)$ allows analytical extension to $\mathbb{C}_-$. Other elements in $S(k)$ may not be well defined for $k\in\mathbb{C}_-\cup\mathbb{C}_+$.

In fact, the symmetry properties for the Jost solution and scattering matrix have already been given in [12], so we just revisited them again and listed these results in the following:

Firstly, the Jost solutions satisfy the involution property:

$$J_{\pm}^{\dagger}(x,\overline{k})=J_{\pm}^{-1}(x,k). \tag{17}$$

Secondly, the Jost solution has another important symmetry:

$$J_{\pm}(x,k)=\sigma\overline{J_{\pm}}(x,-\overline{k})\sigma,\ \sigma=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}. \tag{18}$$

Next, it is naturally obtained from relation (16) that the scattering matrix $S(k)$ obeys the same symmetric properties

$$S^{\dagger}(\overline{k})=S^{-1}(k),\quad \overline{S(-k)}=\sigma S(\overline{k})\sigma. \tag{19}$$

Some relations can be obtained from (19), which will play an important role in our later analysis.

In order to construct the Riemann–Hilbert problem, we define the matrix function

$$\Phi_+(x,k)=\left(J_{-}^{[1]},J_{-}^{[2]},J_{+}^{[3]}\right). \tag{20}$$

It can be shown from integral equation (12) that the large-k asymptotic behavior is

$$\Phi_+(x,k) \to \mathbb{I}, \text{ as } k \to \infty \text{ in } \mathbb{C}_+, \tag{21}$$

By the involution property, we can define the analytic counterpart of function $\Phi_+(x,k)$ in $\mathbb{C}_-$. Let $(J_\pm^{-1})_{[k]}$ be the kth row of $J_\pm^{-1}$:

$$J_\pm^{-1} = \begin{pmatrix} (J_\pm^{-1})_{[1]} \\ (J_\pm^{-1})_{[2]} \\ (J_\pm^{-1})_{[3]} \end{pmatrix}.$$

then we define

$$\Phi_-^{-1}(x,k) \triangleq \Phi_+^\dagger(x,\overline{k}) = \begin{pmatrix} (J_-^{-1})_{[1]} \\ (J_-^{-1})_{[2]} \\ (J_+^{-1})_{[3]} \end{pmatrix}, \tag{22}$$

which is analytic in $\mathbb{C}_-$, and the large-k asymptotic behavior for this function is

$$\Phi_-^{-1}(x,k) \to \mathbb{I}, \text{ as } k \to \infty \text{ in } \mathbb{C}_-. \tag{23}$$

2.2. Matrix Riemann-Hilbert problem

In this section, we construct the Riemann–Hilbert problem, it is noted that function $\Phi_+(x,k)$ can be expressed in terms of the Jost functions and elements of the scattering matrix on the line:

$$\Phi_+(x,k) = J_+ES_+E^{-1}, \text{ where } S_+ = \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{21} & s_{22} & 0 \\ s_{31} & s_{32} & 1 \end{pmatrix}, \ k \in \mathbb{R}. \tag{24}$$

Similarly, by relation (16) we have:

$$\Phi_+(x,k) = J_-ES_-E^{-1}, \text{ where } S_- = \begin{pmatrix} 1 & 0 & r_{13} \\ 0 & 1 & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}, \ k \in \mathbb{R}. \tag{25}$$

Furthermore, by the involution property, when k is on the real axis:

$$\Phi_-^{-1}(x,k) = \Phi_+^\dagger(x,k) = ES_-^\dagger E^{-1}J_-^{-1}(x,k); \tag{26}$$

or, we can directly obtain from relation $J_+^{-1} = ESE^{-1}J_-^{-1}$ that:

$$\Phi_-^{-1}(x,k) = E\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s_{13} & s_{23} & s_{33} \end{pmatrix}E^{-1}J_-^{-1}(x,k).$$

Combining (24) with (26) we have:

$$\Phi_-^{-1}\Phi_+(x,k) = G = EG_0(k)E^{-1}, \quad \mathrm{Im}k = 0,$$

where,

$$G_0(k) \equiv S_+^\dagger S_+ = \begin{pmatrix} 1 & 0 & \overline{s_{31}} \\ 0 & 1 & \overline{s_{32}} \\ s_{31} & s_{32} & 1 \end{pmatrix}, \ k \in \mathbb{R},$$

Thus, we formulated a local matrix RH problem on the real line. Here, the "local" RH problem involves the determination of a function analytic in given sectors of the complex plane, from the knowledge of the jumps of this function across the boundaries of the given sectors. On the contrary, in the case of a "non-local" RH problem, a function "loses" its analyticity only on certain contours, or, in the case of the $\bar{\partial}$ problem, the function loses its analyticity in a certain two-dimensional domain of the complex plane [29].

Firstly, we consider the regular Riemann–Hilbert problem, i.e.: $\det \Phi_-^{-1}(x,k) = \overline{r_{33}(k)} = s_{33}(k) \neq 0$ and $\det \Phi_+(x,k) = r_{33}(k) \neq 0$. By the *Plemelj–Sokhotski Formula*, the unique solution $\Phi_+(x,k)$ of the RH problem can be expressed in terms of its boundary values on the contour Γ with the help of Cauchy-type integrals:

$$\Phi(x,k) = I + \frac{1}{2\pi i}\int_\Gamma \frac{\Phi_-(x,\xi)\widehat{G}(\xi)}{\xi - k}d\xi, \quad k \in \mathbb{C}_+ \cup \mathbb{C}_-,$$

where, $\widehat{G}(\xi) = G(\xi) - I$, and $\Phi_-(x,k)$ solves the integral equation:

$$\Phi_-(x,k) = I + \lim_{\tilde{k}\to k_-} \frac{1}{2\pi i}\int_\Gamma \frac{\Phi_-(x,k')\widehat{G}(k')}{k' - \tilde{k}}dk', \quad k \in \mathbb{R},$$

and $\tilde{k} \to k_-$ means the limit taken in $\mathbb{C}_-$.

Next, we consider the Riemann–Hilbert problem with finite simple zeros. From the symmetry condition of $S(k)$, we can suppose finite number of zeros for $r_{33}(k)$ are $\left\{k_j,\ -\overline{k}_j \in \mathbb{C}_+,\right\}_{j=1}^N$, and zeros for $s_{33}(k)$ are $\left\{-k_j,\ \overline{k}_j \in \mathbb{C}_-,\right\}_{j=1}^N$. In this situation, both $\ker(\Phi_+(k_j))$ and $\ker(\Phi_-^{-1}(\overline{k}_j))$ are spanned by one-dimensional column vector $|v_j\rangle$ and row vector $\langle v_j|$, respectively. In other words, the geometric multiplicity for these two matrices is one.

Now we construct a matrix function which could remove all the zeros of this RH problem. For this purpose, we introduce the rational matrix function:

$$T_j = \left(\mathbb{I} + \frac{\overline{k}_j - k_j}{k + k_j}\mathbb{P}_{-j^*}\right)\left(\mathbb{I} + \frac{\overline{k}_j - k_j}{k - \overline{k}_j}\mathbb{P}_j\right), \tag{27}$$

and its inverse matrix:

$$T_j^{-1} = \left(\mathbb{I} + \frac{k_j - \overline{k}_j}{k - k_j}\mathbb{P}_j\right)\left(\mathbb{I} + \frac{k_j - \overline{k}_j}{k + \overline{k}_j}\mathbb{P}_{-j^*}\right), \tag{28}$$

where $\mathbb{P}_j$ and $\mathbb{P}_{-j^*}$ are both rank one projectors:

$$\mathbb{P}_j = \frac{|v_j\rangle\langle v_j|}{\langle v_j|v_j\rangle},\ |v_i\rangle \in \mathrm{Ker}\left(\Phi_+ T_1^{-1}\cdots T_{i-1}^{-1}(k_i)\right),\ \langle v_j| = |v_j\rangle^\dagger,$$
$$\mathbb{P}_{-j^*} = \frac{|v_{-j^*}\rangle\langle v_{-j^*}|}{\langle v_{-j^*}|v_{-j^*}\rangle},\ |v_{-j^*}\rangle \in \mathrm{Ker}\left(\Phi_+ T_1^{-1}\cdots T_{i-1}^{-1}\chi_i^{-1}(-\overline{k}_i)\right),\ \langle v_{-j^*}| = |v_{-j^*}\rangle^\dagger.$$

Therefore, if one is introducing the matrix function:

$$\Gamma = T_N T_{N-1}\cdots T_1,$$

then $\Gamma(x,k)$ cancels all the zeros of $\Phi_\pm$, and the analytic solutions can be represented as:

$$\Phi_+ = \phi_+\Gamma, \quad \Phi_-^{-1} = \Gamma^{-1}\phi_-^{-1}.$$

Here, ϕ_+ and ϕ_-^{-1} are meromorphic 3×3 matrix functions in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively, with finite number of poles and specified residues. Therefore, all the zeros of Riemann–Hilbert problem have been eliminated and we can formulate a regular RH problem:

$$\phi_-^{-1}(x,k)\phi_+(x,k) = \Gamma(x,k)EG_0(k)E^{-1}\Gamma^{-1}(x,k), \tag{29}$$

with boundary condition:

$$\phi_\pm(x,k) = \Phi_\pm(x,k)\Gamma^{-1}(x,k) \longrightarrow \mathbb{I},\ \text{as } k \to \infty. \tag{30}$$

2.3. The inverse problem

In this section, we attempt to recover the revelent potential function $u(x,t), \overline{u}(x,t)$ from the Jost solutions. In fact, this inverse problem can be solved by expanding the related Jost function $\Phi_+(x,k)$ as $k \to \infty$, i.e.:

$$\Phi_+(x,k) = \mathbb{I} + \Phi_+^{(1)}(x)\frac{1}{k} + \Phi_+^{(2)}(x)\frac{1}{k^2} + \mathcal{O}(1/k^3), \quad k \sim \infty.$$

Substituting this expansion into the spectral equation (10a) and comparing the coefficients with the equal powers of spectral parameter k, we obtain:

$$k^0: \quad Q = i[\sigma_3, \Phi_+^{(1)}]. \tag{31}$$

$$k^{-1}: \quad \partial_x \Phi_+^{(1)} = -i[\sigma_3, \Phi_+^{(2)}] + Q\Phi_+^{(1)}. \tag{32}$$

The first equation gives us the reconstruction formula for the potential function, and the second one is useful in the estimation of the leading-order asymptotic expansion, which was calculated in Ref. [12].

Therefore, in order to solve Sasa–Satsuma equation, we need to find the specified analytic solution $\Phi_+(x,k)$, so that potential matrix can be reconstructed from leading term $\Phi_+^{(1)}(x)$.

2.4. Time evolution of scattering data

In this section, we consider the scattering data evolution. In the above discussion, we often omit the time variable t, actually, it should be added. The scattering relation (16) becomes:

$$J_-(x,t,k)E = J_+(x,t,k)ES(k;t).$$

Noticing that J_- must satisfy the spectral Eq. (10b), i.e.:

$$J_{-,t} = -4ik^3[\sigma_3, J_-] + V_1 J_-. \tag{33}$$

Then we can show

$$\left(E_1^{-1}J_-E_1\right)_t = E_1^{-1}V_1J_-E_1. \tag{34}$$

Since $u(x,t) \in H^{1,1}$, we have $V_1 \to 0$ as $|x| \to \infty$. So, evaluating (34) when $x \to +\infty$ we obtain:

$$\lim_{x\to+\infty} \left(E_1^{-1}J_+ES(k;t)E^{-1}E_1\right)_t = 0,$$

which leads

$$\left(e^{4ik^3t\sigma_3}S(k;t)e^{-4ik^3t\sigma_3}\right)_t = 0.$$

Specifically, we have:

$$\begin{aligned} &s_{11,t} = s_{12,t} = s_{21,t} = s_{22,t} = s_{33,t} = 0. \\ &s_{13}(t;k) = s_{13}(0;k)e^{-8ik^3t}; \; s_{23}(t;k) = s_{23}(0;k)e^{-8ik^3t}; \\ &s_{31}(t;k) = s_{31}(0;k)e^{-8ik^3t}; \; s_{32}(t;k) = s_{32}(0;k)e^{-8ik^3t}. \end{aligned}$$

Then the Riemann–Hilbert problem becomes

$$\Phi_-^{-1}\Phi_+ = G(x,t;k) = E_1 \begin{pmatrix} 1 & 0 & r_{13}(0;k) \\ 0 & 1 & r_{23}(0;k) \\ s_{31}(0;k) & s_{32}(0;k) & 1 \end{pmatrix} E_1^{-1}, \; k \in \mathbb{R},$$

with the boundary condition,

$$\Phi_\pm(x,k) \longrightarrow \mathbb{I}, \text{ as } k \to \infty.$$

Furthermore, after some calculation, we find that $J_+(x,t,k)$ also satisfies Eq. (10b):

$$J_{+,t} = -4ik^3[\sigma_3, J_+] + V_1 J_+.$$

Therefore, the analytic matrix functions $\Phi_\pm(x,k)$ indeed solve the temporal part of the spectral equation, and the scattering data needed to solve this RH problem and reconstruct the potential matrix are: $\left\{s_{31}(k;t), s_{32}(k;t), \{\pm k_j, \pm\overline{k}_j\}_{j=1}^N, |v_j\rangle\right\}$.

So far, the inverse scattering transform for Sasa–Satsuma equation has been completed.

3. N-soliton solutions

Now we prepare to construct the N-soliton solutions formula for Sasa–Satsuma equation. It is well known that the soliton solutions correspond to the *Reflectionless potential*, which means the vanishing of scattering coefficients, i.e.: $s_{31} = s_{32} = 0$. Then jump matrix $G(x,t;k)$ becomes $\mathbb{I}$, and we intend to solve the RH problem:

$$\phi_-^{-1}(x,k)\phi_+(x,k) = \mathbb{I}, \quad k \in \mathbb{R},$$

with boundary condition:

$$\phi_\pm(x,k) \longrightarrow \mathbb{I}, \text{ as } k \to \infty.$$

Without loss of generality, we can assume a trivial solution for this RH problem is: $\phi_\pm = \mathbb{I}$. As a result, we have $\Phi_+ = \Gamma$, and matrix Γ is called the "*Dressing*" factor. The asymptotic expansion for the dressing matrix at $k = \infty$ is:

$$\Gamma(x,t;k) = \mathbb{I} + \Gamma^{(1)}(x,t)\frac{1}{k} + \Gamma^{(2)}(x,t)\frac{1}{k^2} + \mathcal{O}(\frac{1}{k^3}), \tag{35}$$

and now the reconstruction formula (31) can be written as

$$Q = i[\sigma_3, \Gamma^{(1)}(x,t)]. \tag{36}$$

In the following, we need to calculate the explicit expression for $\Gamma^{(1)}(x,t)$ so that we will be able to find solutions for Sasa–Satsuma equation. It is noted that the rational matrix function $\Gamma(x,t)$ can be decomposed into the sum of simple fractions:

$$\Gamma(x,t;k) = \mathbb{I} + \sum_{l=1}^{N}\left(\frac{\overline{k_l} - k_l}{k - \overline{k_l}}|x_l\rangle\langle y_l| + \frac{\overline{k_l} - k_l}{k + k_l}|x_{-l^*}\rangle\langle y_{-l^*}|\right),$$
$$\Gamma^{-1}(x,t;k) = \mathbb{I} + \sum_{j=1}^{N}\left(\frac{k_j - \overline{k_j}}{k - k_j}|y_j\rangle\langle x_j| + \frac{k_j - \overline{k_j}}{k + \overline{k_j}}|y_{-j^*}\rangle\langle x_{-j^*}|\right).$$

Next, we consider the identity $\Gamma(x,t;k)\Gamma^{-1}(x,t;k) \equiv \mathbb{I}$ at $k = k_j, -\overline{k_j}$ where we should pose:

$$\Gamma(x,t;k_j)|y_j\rangle\langle x_j| = 0,$$
$$\Gamma(x,t;-\overline{k_j})|y_{-j^*}\rangle\langle x_{-j^*}| = 0,$$

then the singularity of the identity at k_j and $-\overline{k_j}$ can be removed. For convenience, we denote: $k_{2n-1} = k_n$, $k_{2n} = -\overline{k_n}$, and $|x_{2n-1}\rangle\langle y_{2n-l}|=|x_n\rangle\langle y_n|$, $|x_{2n}\rangle\langle y_{2n}|=|x_{-n^*}\rangle\langle y_{-n^*}|$, $n = 1,2,\ldots,N$. Supposing $\langle x_j|y_j\rangle \neq 0$, $\langle x_{-j^*}|y_{-j^*}\rangle \neq 0$, then we obtain:

$$|y_j\rangle = \sum_{l=1}^{2N}|x_l\rangle\langle y_l|\frac{k_l - \overline{k_l}}{k_j - \overline{k_l}}|y_j\rangle, \quad j = 1,2,\ldots,2N. \tag{37}$$

Introducing $2N \times 2N$ matrices:

$$X = (|x_1\rangle, |x_2\rangle, \dots, |x_{2N}\rangle)\,, \; Y = (|y_1\rangle, |y_2\rangle, \dots, |y_{2N}\rangle)\,,$$
$$\Lambda = diag(k_1 - \overline{k_1}, \dots, k_{2N} - \overline{k}_{2N}), \; M = \{M_{l,j}\}_{2N\times 2N} = \left\{ \langle y_l| \frac{1}{k_j - \overline{k_l}} |y_j\rangle \right\}_{2N\times 2N}.$$

Then (37) can be reformulated as:

$$Y = X\Lambda M, \text{ or } X\Lambda = YM^{-1}, \text{ i.e.:}$$
$$(k_l - \overline{k_l})|x_l\rangle = \sum_{j=1}^{2N} (M^{-1})_{j,l} |y_j\rangle, \quad l = 1, 2, \dots, 2N.$$

Introducing notation $|j\rangle \equiv |y_j\rangle$, the dressing matrix becomes:

$$\Gamma(x,t;k) = \mathbb{I} - \sum_{j,l=1}^{2N} \left(\frac{1}{k - \overline{k}_l} |j\rangle (M^{-1})_{j,l} \langle l| \right). \tag{38}$$

Therefore, from the asymptotic expansion (35) we have:

$$\Gamma^{(1)}(x,t;k) = -\sum_{j,l=1}^{2N} \left(|j\rangle (M^{-1})_{j,l} \langle l| \right).$$

In addition, the explicit expression of vector $|y_j\rangle$ can be calculated by using the condition $\Gamma(x,t;k_j)|y_j\rangle \langle x_j| = 0$, i.e., $\Phi_+(x,t;k_j)|y_j\rangle = 0, \; j = 1, 2, \dots, 2N$, we can differentiate this equation in x and t and get:

$$|y_j\rangle_x = -ik_j\sigma_3|y_j\rangle + \mu(x)|y_j\rangle,$$
$$|y_j\rangle_t = -4ik_j^3\sigma_3|y_j\rangle + \nu(t)|y_j\rangle,$$

where $\mu(x)$ and $\nu(t)$ are arbitrary functions, so we have:

$$|y_j\rangle = \exp[-ik\sigma_3 x - 4ik^3\sigma_3 t]|y_{j0}\rangle \exp[\int_{x_0}^{x} \mu(\xi)d\xi + \int_{t_0}^{t} \nu(\xi)d\xi],$$

and $|y_{j0}\rangle$ is a constant vector. A proper choice of function $\mu(x)$ and $\nu(t)$ may make the calculation simpler as we will show later. Next, taking relevant matrix entries from reconstruction formula (36) we have:

$$u(x,t) = 2i\Gamma_{1,3}^{(1)}(x,t) = 2i\Gamma_{3,2}^{(1)}(x,t). \tag{39}$$

Furthermore, with some simple algebra operation, we get a more compact formula:

$$u(x,t) = 2i\frac{|M_1^3|}{|M|} = 2i\frac{|M_3^2|}{|M|}, \tag{40}$$

where,

$$M_1^3 = \begin{pmatrix} M_{11} & \cdots & M_{1,2N} & \langle 1|_3 \\ \vdots & \ddots & \vdots & \vdots \\ M_{2N,1} & \cdots & M_{2N,2N} & \langle 2N|_3 \\ |1\rangle_1 & \cdots & |2N\rangle_1 & 0 \end{pmatrix}, \; M_3^2 = \begin{pmatrix} M_{11} & \cdots & M_{1,2N} & \langle 1|_2 \\ \vdots & \ddots & \vdots & \vdots \\ M_{2N,1} & \cdots & M_{2N,2N} & \langle 2N|_2 \\ |1\rangle_3 & \cdots & |2N\rangle_3 & 0 \end{pmatrix}.$$

Hence, to ensure the vitality of potential $u(x,t)$, we must make sure that $\det(M_1) = \det(M_2)$, and this can be proved via some simple linear algebra technique. To get the explicit N-Soliton solutions, we may take

$$|2j-1\rangle = \begin{pmatrix} \alpha_j \exp[z_j + i\phi_j] \\ \beta_j \exp[z_j + i\phi_j] \\ \gamma_j \exp[-z_j - i\phi_j] \end{pmatrix}, \quad |2j\rangle = \begin{pmatrix} \overline{\beta_j} \exp[z_j - i\phi_j] \\ \overline{\alpha_j} \exp[z_j - i\phi_j] \\ \overline{\gamma_j} \exp[-z_j + i\phi_j] \end{pmatrix},$$

where, $z_j = \eta_j[x + 4(\eta_j^2 - 3\xi_j^2)t]$, $\phi_j = -\xi_j[x + 4(\xi_j^2 - 3\eta_j^2)t]$, and $k_j = \xi_j + i\eta_j$ $(j = 1, 2, \ldots, N)$ are discrete spectrum, $\alpha_j, \beta_j, \gamma_j$ are arbitrary complex numbers. Then the general N-soliton solution for Sasa–Satsuma equation can be obtained via formula (40). Firstly, taking $N = 1$ in (40) we can get the 1-st order solution:

$$u_1 = \frac{c_1 e^{2z_1+2i\phi_1} + c_2 e^{2z_1+6i\phi_1} + c_3 e^{6z_1+2i\phi_1} + c_4 e^{6z_1+6i\phi_1}}{d_1 e^{4z_1} + d_2 e^{4i\phi_1} + d_3 e^{4z_1+4i\phi_1} + d_4 e^{4z_1+8i\phi_1} + d_5 e^{8z_1+4i\phi_1}}, \tag{41}$$

where,

$$\begin{aligned}
&d_1 = (\beta\overline{\alpha} + \alpha\overline{\beta})\eta^2,\ d_2 = -(\alpha^2 - \beta^2)(\overline{\alpha}^2 - \overline{\beta}^2)\eta^2 - (|\alpha|^2 + |\beta|^2)^2\xi^2,\\
&d_3 = -2(|\alpha|^2 + |\beta|^2)(\eta^2 + \xi^2),\ d_4 = (\beta\overline{\alpha} + \alpha\overline{\beta})\eta^2,\ d_5 = -\xi^2.\\
&c_1 = 4\overline{\alpha}(\beta^2 - \alpha^2)\eta^3 + 4i\beta(\beta\overline{\alpha} + \alpha\overline{\beta})\eta^2\xi - 4\alpha(|\alpha|^2 + |\beta|^2)\eta\xi^2,\\
&c_2 = 4\overline{\beta}(\alpha^2 - \beta^2)\eta^3 - 4i\alpha(\beta\overline{\alpha} + \alpha\overline{\beta})\eta^2\xi - 4\beta(|\alpha|^2 + |\beta|^2)\eta\xi^2,\\
&c_3 = -4i\alpha\eta^2\xi - 4\alpha\eta\xi^2\ , c_4 = 4i\beta\eta^2\xi - 4\beta\eta\xi^2.
\end{aligned}$$

The real part z_1 and the image part ϕ_1 in solution (41) cannot be completely separated, and there are oscillation terms $e^{ik\phi_1}(k = 2, 4, 6, 8)$ that emerge in (41), so it may bring about the periodical behavior of the solution. With some simple simplification, we have:

$$u_1 = e^{-2z_1-2i\phi_1}\left(\frac{c_1 + c_3 e^{4z_1} + c_2 e^{4i\phi_1} + c_4 e^{4z_1+4i\phi_1}}{d_3 + d_2 e^{-4z_1} + d_5 e^{4z_1} + 2d_4\cos 4\phi_1}\right). \tag{42}$$

Hence, from this formula, we may find that the oscillation frequency is decided by $4\phi_1$, to be specific, the periodic solution u_1 oscillates with the frequency $\omega_1 = -4\xi_1$ in the x-axis direction and $\omega_2 = 16\xi_1(3\eta_1^2 - \xi_1^2)$ in the t-axis direction, or, in other words, its periodical behavior occurring along the line: $x - 4(\xi_j^2 - 3\eta_j^2)t = 0$.

We might as well name u_1 a "breather" solution, and its asymptotic expression at large z_1 has the form:

$$u_1 \sim \begin{cases} \left(\frac{c_3}{d_5}e^{-2i\phi_1} + \frac{c_4}{d_5}e^{2i\phi_1}\right)e^{-2z_1}, & z_1 \sim +\infty \\ \left(\frac{c_1}{d_2}e^{-2i\phi_1} + \frac{c_2}{d_2}e^{2i\phi_1}\right)e^{2z_1}, & z_1 \sim -\infty \end{cases}$$

and the phase difference for u_1 at its limits:

$$\begin{aligned}
&\arg(u_1(z_1 \sim -\infty)) - \arg(u_1(z_1 \sim +\infty)) = \arg(r_1 e^{i\theta_1}) - \arg(r_2 e^{i\theta_2}),\\
&\text{where, } \theta_1 = \arg\left(\frac{c_3}{d_5}e^{-2i\phi_1} + \frac{c_4}{d_5}e^{2i\phi_1}\right),\ \theta_2 = \arg\left(\frac{c_1}{d_2}e^{-2i\phi_1} + \frac{c_2}{d_2}e^{2i\phi_1}\right).
\end{aligned}$$

In general, N-solitons interactions can be obtained if we take N distinct spectral parameters $k_1, k_2, \ldots, k_N \in \mathbb{C}_+$.

Next, we conduct an investigation into the Darboux transformation for Sasa–Satsuma equation. First of all, the elementary DT for spectral problem (6) has the form:

$$G_1 = \Xi_2\Xi_1,\ \Xi_i = \left(\mathbb{I} + \frac{\overline{k_i} - k_i}{k - \overline{k_i}}\mathbb{P}_i^{(i-1)}\right),\ \mathbb{P}_i^{(i-1)} = \frac{|\chi_i^{(i-1)}\rangle\langle\chi_i^{(i-1)}|}{\langle\chi_i^{(i-1)}|\chi_i^{(i-1)}\rangle}, \tag{43}$$

where

$$|\chi_1^{(0)}\rangle = \Xi_0|y_1\rangle|_{k=k_1},\ |\chi_2^{(1)}\rangle = \Xi_1\Xi_0|y_2\rangle|_{k=k_2},\ \Xi_0 = \mathbb{I},$$

vector $|y_i\rangle$ are special solutions of Lax-pair (6) at $k = k_i$, and $k_2 = -\overline{k_1}$. The new potential matrix $Q^{[1]}$ under this transformation becomes:

$$Q^{[1]} = Q + i\left[\sigma_3, \sum_{i=1}^{2}(\overline{k_i} - k_i)\mathbb{P}_i^{(i-1)}\right], \tag{44}$$

and potential $u^{[1]}$ solves Sasa–Satsuma equation. This two-fold DT can keep the forms of Lax-pair U, V invariant and meanwhile retain the reduction, i.e, $Q_{[1]}$ satisfies the symmetry conditions (17) and (18).

According to the elementary DT (43), we establish the following theorem [23]:

Theorem 1. *Assume we have $2N$ distinct spectral parameters $k_1, k_2, \ldots, k_{2N} \in \mathbb{C}_+$ with the corresponding eigenfunction matrices $|y_1\rangle, |y_2\rangle, \ldots, |y_{2N}\rangle$, which are taken in the special forms. Besides, we need $k_{2n} = -\overline{k}_{2n-1}$, then the N-fold Darboux transformation for Sasa–Satsuma equation can be represented as:*

$$T_N = \mathbb{I} - (|y_1\rangle, |y_2\rangle, \ldots, |y_{2N}\rangle) M^{-1} (k - \overline{S})^{-1} \begin{pmatrix} \langle y_1| \\ \langle y_2| \\ \vdots \\ \langle y_{2N}| \end{pmatrix},$$

$$with,\ M = \left\{ \frac{\langle y_i | y_j \rangle}{k_j - \overline{k_i}} \right\}_{1 \le i,j \le 2N},\ S = diag(k_1, k_2, \ldots., k_{2N}).$$

Proof. The Darboux transformation can be constructed by N-times iteration of elementary DT, that is:

$$T_N = G_N G_{N-1} \ldots G_1,$$

where,

$$G_k = \Xi_{2k} \Xi_{2k-1},\ \Xi_i = \left(\mathbb{I} + \frac{\overline{k_i} - k_i}{k - \overline{k_i}} \mathbb{P}_i^{(i-1)} \right),\ \mathbb{P}_i^{(i-1)} = \frac{|\chi_i^{(i-1)}\rangle\langle\chi_i^{(i-1)}|}{\langle\chi_i^{(i-1)}|\chi_i^{(i-1)}\rangle},$$
$$|\chi_i^{(i-1)}\rangle = \Xi_{i-1}\Xi_{i-2} \ldots \Xi_1 \Xi_0 |y_i\rangle|_{k=k_i},\ \Xi_0 = \mathbb{I},\ i = 1, 2, \ldots, 2N.$$

The above Darboux transformation T_N can be decomposed into the following linear fraction transformation, which we may denote as $\widehat{T}_N$ for convenience:

$$\widehat{T}_N = I + \sum_{i=1}^{2N} \frac{\widehat{P}_i}{k - \overline{k_i}},$$

where $\widehat{P}_i$ are 3×3 rank one matrices, thus we can suppose $\widehat{P}_i = |x_i\rangle\langle y_i|$, then the involution property gives the inverse transformation:

$$\widehat{T}_N^{-1}(k) = \widehat{T}_N^{\dagger}(\overline{k}) = I + \sum_{i=1}^{2N} \frac{|y_i\rangle\langle x_i|}{k - k_i}.$$

Considering the identity $\widehat{T}_N(k)\widehat{T}_N^{-1}(k) = I$ at point k_i, we have:

$$|y_j\rangle = \sum_{i=1}^{2N} |x_i\rangle \frac{\langle y_j | y_i \rangle}{k_j - \overline{k_i}},\quad j = 1, 2, \ldots, 2N.$$

In addition, because $|y_i\rangle \in \mathrm{Ker}(T_N(k_i))$, and as a matter of fact, one can further prove that:

$$\mathrm{rank}(T_N(k_i)) = 2,\quad i = 1, 2, \ldots, 2N,$$

thus we can suppose the kernel for $\widehat{T}_N$ at $k = k_i$ is

$$\mathrm{Ker}(\widehat{T}_N(k_i)) = |y_i\rangle,\ i = 1, 2, \ldots, 2N,$$

where $|y_i\rangle$ solves the spectral Eq. (6) at $k = k_i$. At last, by simple linear algebra, we obtain the N-fold Darboux transformation for Sasa–Satsuma equation in another form. This completes the proof. □

In general, the transformation between potential matrices is in the compact form:

$$Q^{[N]} = Q - i[\sigma_3, \sum_{i,j=1}^{2N} \left(|y_i\rangle (M^{-1})_{i,j} \langle y_j| \right)]. \tag{45}$$

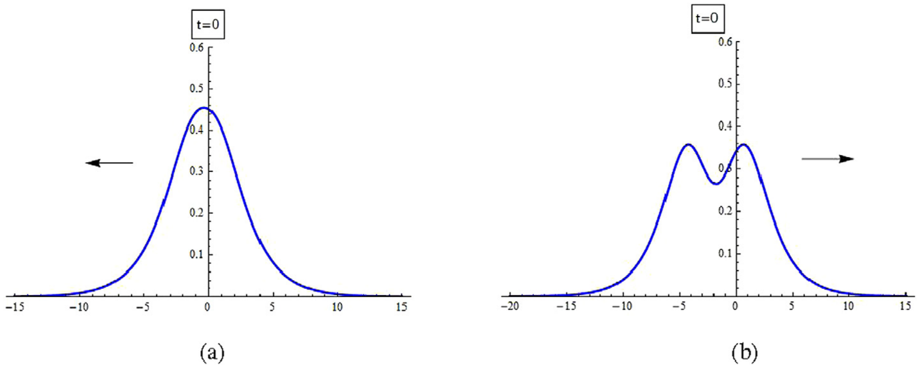

Fig. 1. (a) One-soliton solution with single hump at time $t = 0$. The parameter: $\xi = \frac{1}{4}$, $\eta = \frac{1}{4}$; $p_1 = 1$; (b) One-soliton solution with double humps at $t = 0$. The parameter: $\xi = \frac{1}{24}$, $\eta = \frac{1}{4}$; $p_1 = 1$.

Moreover, it is noted that the reconstruction formula (36) is exactly corresponding to the case when Q is taken as zero matrix in formula (45). And some new solutions of S–S equation can be also generated via (45). For this purpose, we choose the eigenfunction as

$$|y_s\rangle = \exp(\theta_s\sigma_3)\,|y_{s,0}\rangle \exp(-\theta_s)\,,\ \theta_s = -ik_s x - 4ik_s^3 t,\ s \in \mathbb{N}^+, \tag{46}$$

where the constant vector is $|y_{s,0}\rangle = (p_s, q_s, 1)^T$.

Set $N = 1$ in formula (45) and consider the case when $p_1 q_1 = 0$. Choosing $q_1 = 0$ and denoting $\kappa = (\overline{k_1} + k_1)/2\overline{k_1}$, then the bright one soliton is explicitly written as:

$$u_1(x,t) = 2ip_1(\overline{k_1} - k_1)\frac{|p_1|^2 e^{2\theta_1} + \kappa e^{-2\overline{\theta_1}}}{|\kappa|^2 e^{-2(\theta_1+\overline{\theta_1})} + |p_1|^4 e^{2(\theta_1+\overline{\theta_1})} + 2|p_1|^2}. \tag{47}$$

Taking $p_1 = e^{-2\eta_1 x_0 + i\sigma_0}$, where x_0 and σ_0 are arbitrary real constants, $u_1(x,t)$ is written in the form of a traveling solitary wave:

$$\begin{aligned} &u_1(x,t) = \psi(x - v_1 t - x_0)e^{i(-2\xi_1 x - \lambda_1 t + \sigma_0)}, \\ &v_1 = 4(\eta_1^2 - 3\xi_1^2),\ \lambda_1 = 8\xi_1(\xi_1^2 - 3\eta_1^2). \end{aligned}$$

with function ψ defined as

$$\psi(x) = 4\eta_1 \frac{e^{2\eta_1 x} + \kappa e^{-2\eta_1 x}}{e^{4\eta_1 x} + 2 + |\kappa|^2 e^{-4\eta_1 x}},$$

and the intensity profile for $\psi(x)$ is

$$|\psi(x)|^2 = 16\eta_1^2 \frac{e^{4\eta_1 x} + 2|\kappa|^2 + |\kappa|^2 e^{-4\eta_1 x}}{(e^{4\eta_1 x} + 2 + |\kappa|^2 e^{-4\eta_1 x})^2}.$$

An interesting property of this soliton which is shown in Fig. 1 is that its shape can be single or double humped depending on the parameter $|\kappa|$. It is seen that this soliton has two intensity peaks, which is quite unusual in integrable systems where single-soliton solutions are often single humped.

In the following analysis, one can take $t = 0$, $x_0 = 0$, thus the intensity profile becomes:

$$|u_1(x,0)|^2 = (16\eta_1^2|\kappa|^{-1})\frac{e^{4\eta_1 x - \log|\kappa|} + e^{-4\eta_1 x + \log|\kappa|} + 2|\kappa|}{\left(e^{4\eta_1 x - \log|\kappa|} + e^{-4\eta_1 x + \log|\kappa|} + 2|\kappa|^{-1}\right)^2}.$$

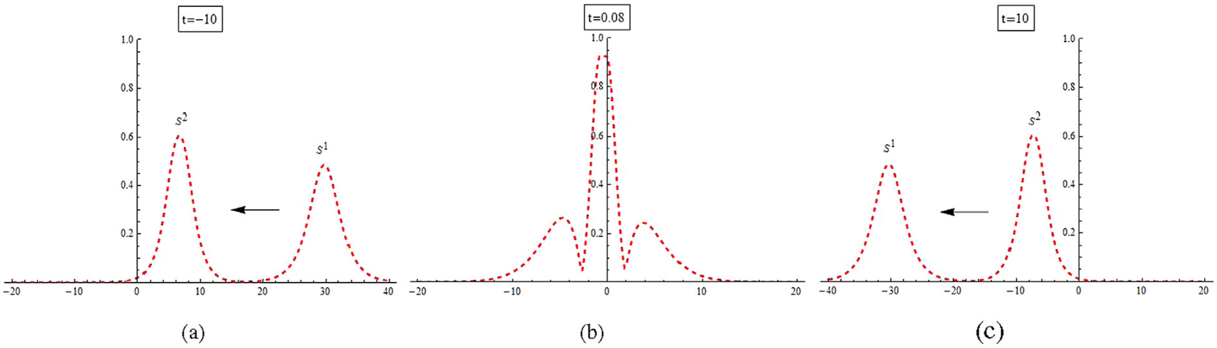

Fig. 2. Two single-humped soliton solutions interaction at $t = -10, 0.08, 10$, respectively. The parameter: $\xi_1 = \frac{1}{2}$, $\eta_1 = \frac{1}{4}$; $\xi_2 = \frac{1}{3}$, $\eta_2 = \frac{1}{3}$; $p_1 = p_2 = 1$.

This is a central symmetry function and its center point is located at $a = \log|\kappa|/4\eta$, which satisfies the extremum condition:

$$\partial_x |u_1(a,0)|^2 = 0.$$

Furthermore, the second order derivative at $x = a$ is calculated as:

$$\partial_x^2 |u_1(a,0)|^2 = -\frac{128\eta^4|\kappa|(2|\kappa|-1)}{(1+|\kappa|)^2}.$$

Thus, it is easy to see that when $2|\kappa| - 1\langle 0, 0 <|\kappa| < 0.5$, $x = a$ is the minimum point, which corresponds to the double humped soliton. When $2|\kappa| - 1 > 0, |\kappa| > 0.5$, $x = a$ is the maximum point, which is exactly the single humped soliton.

In addition, if $p_1 q_1 \neq 0$, these single-soliton solutions are no longer solitary waves. Instead, they become spatially localized and temporally periodic bound states (Mihalache et al. (1993a)) which has been shown in expression (39).

Taking $N = 2$ with $q_1 = q_2 = 0$ in formula (45), thus the two-soliton solution of S–S equation has the form of: $u_2(x,t) = 2i\Omega_1/\Omega_0$ with,

$$\begin{aligned}
\Omega_1 &= \delta_{1,1}e^{4\theta_1+4\theta_2+2\theta_3+4\theta_4} + \delta_{1,2}e^{2\theta_1+4\theta_2+4\theta_3+4\theta_4} + \delta_{2,1}e^{2\theta_1+2\theta_2+2\theta_3+4\theta_4} + \delta_{2,2}e^{2\theta_1+4\theta_2+2\theta_3+2\theta_4} \\
&\quad + \delta_{3,1}e^{4\theta_1+4\theta_2+2\theta_4} + \delta_{3,2}e^{4\theta_1+4\theta_2+2\theta_4} + \delta_{3,3}e^{4\theta_1+4\theta_2+2\theta_4} + \delta_{3,4}e^{2\theta_2+4\theta_3+4\theta_4} + \delta_{3,5}e^{2\theta_1+2\theta_2+2\theta_4} \\
&\quad + \delta_{3,6}e^{2\theta_2+2\theta_3+2\theta_4} + \delta_{4,1}e^{2\theta_1+4\theta_2} + \delta_{4,2}e^{2\theta_1+4\theta_4} + \delta_{4,3}e^{2\theta_3+4\theta_2} + \delta_{4,4}e^{2\theta_3+4\theta_4} + \delta_{5,1}e^{2\theta_2} + \delta_{5,2}e^{2\theta_4}, \\
\Omega_0 &= \rho_{1,1}e^{4\theta_1+4\theta_2+4\theta_3+4\theta_4} + \rho_{1,2}e^{2\theta_1+2\theta_2+4\theta_3+4\theta_4} + \rho_{1,3}e^{4\theta_1+4\theta_2+2\theta_3+4\theta_4} + \rho_{1,4}e^{4\theta_1+2\theta_2+2\theta_3+4\theta_4} \\
&\quad + \rho_{1,5}e^{2\theta_1+4\theta_2+4\theta_3+2\theta_4} + \rho_{1,6}e^{2\theta_1+2\theta_2+2\theta_3+2\theta_4} + \rho_{2,1}e^{2\theta_1+4\theta_2+2\theta_3} + \rho_{2,2}e^{4\theta_1+2\theta_2+2\theta_4} \\
&\quad + \rho_{2,3}e^{2\theta_1+2\theta_3+4\theta_4} + \rho_{2,4}e^{2\theta_2+4\theta_3+2\theta_4} + \rho_{3,1}e^{2\theta_1+2\theta_2} + \rho_{3,2}e^{2\theta_1+2\theta_4} + \rho_{3,3}e^{2\theta_2+2\theta_3} \\
&\quad + \rho_{3,4}e^{2\theta_3+2\theta_4} + \rho_{3,5}e^{4\theta_1+4\theta_2} + \rho_{3,6}e^{4\theta_2+4\theta_3} + \rho_{3,7}e^{4\theta_1+4\theta_4} + \rho_{3,8}e^{4\theta_3+4\theta_4} + \rho_0,
\end{aligned}$$

where, $\theta_j = ik_j x + 4ik_j^3 t$, $k_2 = -\overline{k_1}$, $k_4 = -\overline{k_3}$. The coefficients of these exponential terms are constituted of p_1, p_2 and $\{k_j\}_{j=1}^4$. However, it is cumbersome to write all of them down here and they can be directly calculated via the computer.

In the following, according to different choices of spectral parameters of $\{k_j\}_{j=1}^4$, we plot figures of the shape function $|u_2(x,t)|$ at certain moments $t = t_0$, which displays the time evolution of the two-soliton solution collisions. Fig. 2 shows the collision between two single-humped solitons. In this case, soliton S^1 is running after S^2. Fig. 3 depicts the collision between one single-humped soliton and one double-humped soliton from apposite directions, while Fig. 4 shows two double-humped soliton collisions. It is the

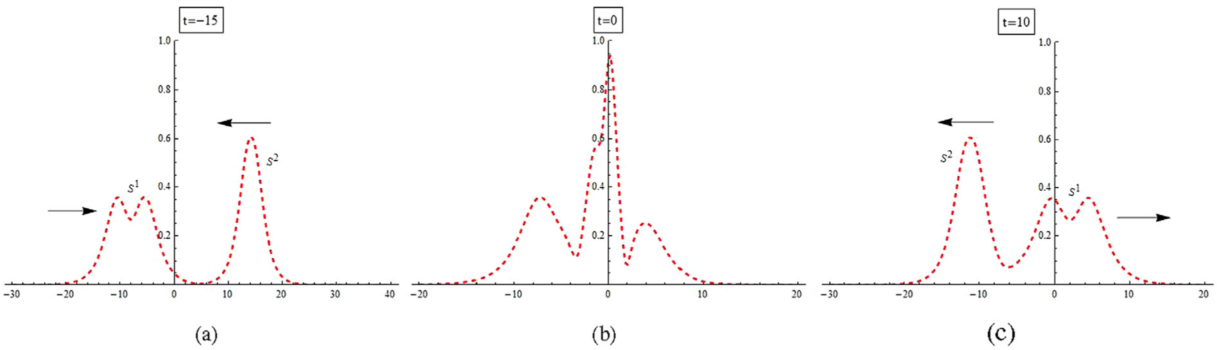

Fig. 3. One singel-humped soliton and one double-humped soliton solution interaction at $t = -15, 0, 10$, respectively. The parameter: $\xi_1 = \frac{1}{24}$, $\eta_1 = \frac{1}{4}$; $\xi_2 = \frac{1}{3}$, $\eta_2 = \frac{1}{3}$; $p_1 = 1$.

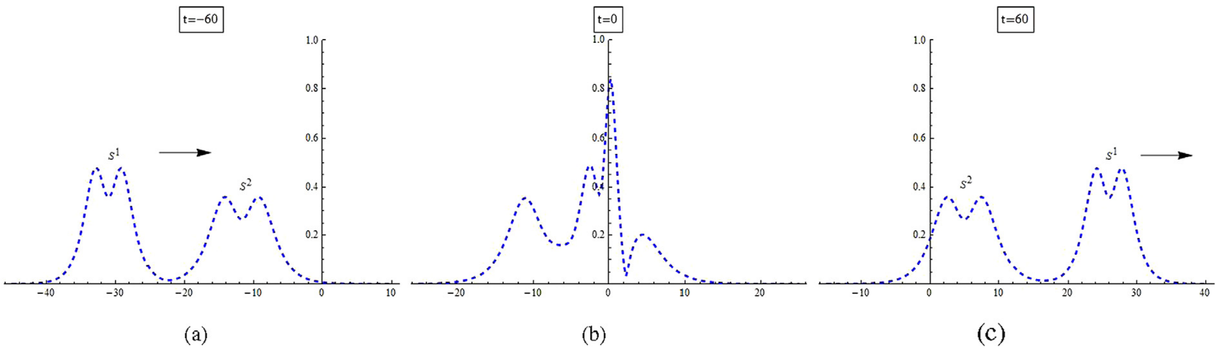

Fig. 4. Two double-humped soliton solutions interaction at $t = -60, 0, 60$, respectively. The parameter: $\xi_1 = \frac{1}{24}$, $\eta_1 = \frac{1}{4}$; $\xi_2 = \frac{1}{18}$, $\eta_2 = \frac{1}{3}$; $p_1 = 1$.

properties of elastic collision that these interacting solitons like particles can retain their respective velocities, amplitudes, widths before and after the collision, except for a phase shift.

Next, using the asymptotic analysis technique [30], we intend to investigate the collision dynamics of these bright two-soliton solutions. First of all, because $\theta_{2i-1} + \theta_{2i} = -2\eta_i(x - 4\nu_i t)$, $\nu_i = 3\xi_i^2 - \eta_i^2$, it yields:

$$(\theta_1 + \theta_2)\eta_2 - (\theta_3 + \theta_4)\eta_1 = 8\eta_1\eta_2(\nu_1 - \nu_2)t.$$

In accordance with the results of inverse scattering transformation, one needs $\eta_i > 0$. And without losing generality, we assume that $\nu_1 > \nu_2$, where $4\nu_i$ stands for the velocity of soliton solution. Then we have asymptotic expressions of $u_2(x,t)$ under different asymptotic states of $\theta_1 + \theta_2$ and $\theta_3 + \theta_4$.

(i) Before collision (as $t \to -\infty$)

(a) If $\theta_1 + \theta_2 \sim 0$, then $\theta_3 + \theta_4 \sim -\infty$:

$$u_2(x,t) \sim S^{1-} = 2i\frac{\delta_{4,1}e^{2\theta_2} + \delta_{5,1}e^{-2\theta_1}}{\rho_{3,5}e^{2\theta_1+2\theta_2} + \rho_{3,1} + \rho_0 e^{-2\theta_1-2\theta_2}}; \tag{48a}$$

(b) If $\theta_3 + \theta_4 \sim 0$, then $\theta_1 + \theta_2 \sim +\infty$:

$$u_2(x,t) \sim S^{2-} = 2i\frac{\delta_{1,1}e^{2\theta_4} + \delta_{3,1}e^{-2\theta_3}}{\rho_{1,1}e^{2\theta_3+2\theta_4} + \rho_{1,3} + \rho_{3,5}e^{-2\theta_3-2\theta_4}}. \tag{48b}$$

(ii) After collision (as $t \to +\infty$)

(a) If $\theta_1 + \theta_2 \sim 0$, then $\theta_3 + \theta_4 \sim +\infty$:

$$u_2(x,t) \sim S^{1+} = 2i\frac{\delta_{1,2}e^{2\theta_2} + \delta_{3,4}e^{-2\theta_1}}{\rho_{1,1}e^{2\theta_1+2\theta_2} + \rho_{1,2} + \rho_{3,8}e^{-2\theta_1-2\theta_2}}; \tag{49a}$$

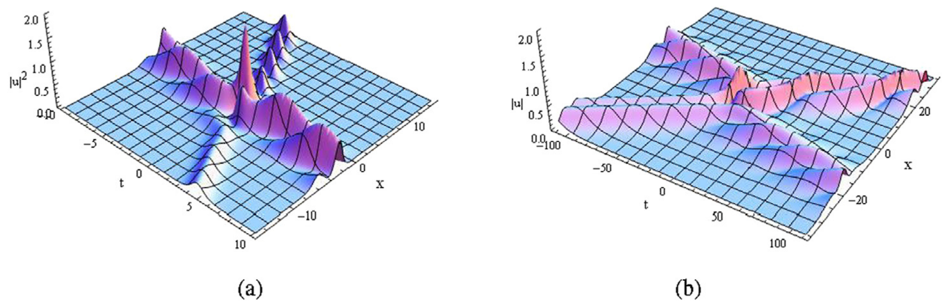

Fig. 5. (a). One single-hump soliton becoming into a bound state when interacting with another bound state. The parameter: $\xi_1 = \frac{1}{2},\ \eta_1 = \frac{1}{4},\ \xi_2 = \frac{1}{4},\ \eta_2 = \frac{1}{3};\ p_1 = p_2 = 1,\ q2 = 1$; (b). Collision of one double-hump soliton becoming into a bound state, where $\xi_1 = \frac{1}{20},\ \eta_1 = \frac{1}{4},\ \xi_2 = \frac{1}{8},\ \eta_2 = \frac{1}{6};\ p_1 = p_2 = 1,\ q2 = 1$.

(b) If $\theta_3 + \theta_4 \sim 0$, then $\theta_1 + \theta_2 \sim -\infty$:

$$u_2(x,t) \sim S^{2+} = 2i\frac{\delta_{4,4}e^{2\theta_4} + \delta_{5,2}e^{-2\theta_3}}{\rho_{3,8}e^{2\theta_3+2\theta_4} + \rho_{3,4} + \rho_0 e^{-2\theta_3-2\theta_4}}. \tag{49b}$$

It is noted that the asymptotic solutions $S^{1\pm}$ and $S^{2\pm}$ can be also written as the function of traveling solitary wave. And the respective velocity for $S^{1\pm}$ and $S^{2\pm}$ is ν_1 and ν_2, which remains unchanged before and after the collision.

In addition, when q_1 and q_2 are not equal to zero. The shape changing phenomena will emerge in the collision of S–S solitons, which have also been found in [15] via the DT construction technique. That is, one soliton (or breather) may become a breather (or soliton) when interacting with another breather, and we have shown these interesting phenomena in Fig. 5.

4. Soliton matrices for high-order zeros

In this case, following the discussion of simple zeros, we consider the high-order zeros in Riemann–Hilbert problem of S–S equation. First of all, we let functions $\Phi_+(k)$ and $\Phi_-^{-1}(k)$ from above RHP have only one pair of zero of order n, i.e. $\{k_1, -\overline{k_1}\}$ and $\{\overline{k_1}, -k_1\}$:

$$\det \Phi_-^{-1} = (k - \overline{k_1})^n (k + k_1)^n \overline{\varphi}(k),\ \det \Phi_+ = (k + \overline{k_1})^n (k - k_1)^n \varphi(k), \tag{50}$$

where $\varphi(k_1), \varphi(-\overline{k_1}) \neq 0,\ \overline{\varphi}(\overline{k_1}), \overline{\varphi}(-k_1) \neq 0$. Following the idea proposed in Ref. [21], we first consider the elementary zero case under the assumption that the geometric multiplicity of $\{k_1, -\overline{k_1}\}$ and $\{\overline{k_1}, -k_1\}$ has the same number. Hence, one needs to construct the dressing matrix $\Gamma(k)$ whose determinant is $\frac{(k-k_1)^n(k+\overline{k_1})^n}{(k+k_1)^n(k-\overline{k_1})^n}$.

As a special case, we first consider the elementary zeros which have geometric multiplicity 1. In this case, Γ is constituted of n elementary dressing factors, i.e.: $\Gamma = \Gamma_n \Gamma_{n-1} \dots \Gamma_1,\ \Gamma_i = \widetilde{\chi}_i(k)\chi_i(k)$, where,

$$\chi_i(k) = \mathbb{I} + \frac{\overline{k_1} - k_1}{k - \overline{k_1}}\mathbb{P}_i,\ \mathbb{P}_i = \frac{|v_i\rangle\langle\overline{v_i}|}{\langle\overline{v_i}|v_i\rangle},\ |v_i\rangle \in \mathrm{Ker}\left(\Phi_+\Gamma_1^{-1}\cdots\Gamma_{i-1}^{-1}(k_1)\right),$$
$$\widetilde{\chi}_i(k) = \mathbb{I} + \frac{\overline{k_1} - k_1}{k + k_1}\widetilde{\mathbb{P}}_i,\ \widetilde{\mathbb{P}}_i = \frac{|\widetilde{v}_i\rangle\langle\widetilde{\overline{v}_i}|}{\langle\widetilde{\overline{v}_i}|\widetilde{v}_i\rangle},\ |\widetilde{v}_i\rangle \in \mathrm{Ker}\left(\Phi_+\Gamma_1^{-1}\cdots\Gamma_{i-1}^{-1}\chi_i^{-1}(-\overline{k}_1)\right).$$

In addition, if we let $\widetilde{\Phi}_+(k) = \Phi_+(k)\Gamma_1^{-1}(k)$ and $\widetilde{\Phi}_-^{-1}(k) = \Gamma_1(k)\Phi_-^{-1}(k)$, then it is proved that matrices $\widetilde{\Phi}_+(k)$ and $\widetilde{\Phi}_-^{-1}(k)$ are still holomorphic in the respective half plans of $\mathbb{C}$. Moreover, $\{k_1, -\overline{k}_1\}$ and $\{\overline{k}_1, -k_1\}$ are still a pair of zeros of det $\widetilde{\Phi}_+(k)$ and det $\widetilde{\Phi}_-^{-1}(k)$, respectively. Thus, $\Gamma(k)^{-1}$ cancels all the high-order zeros for det $\Phi_+(k)$.

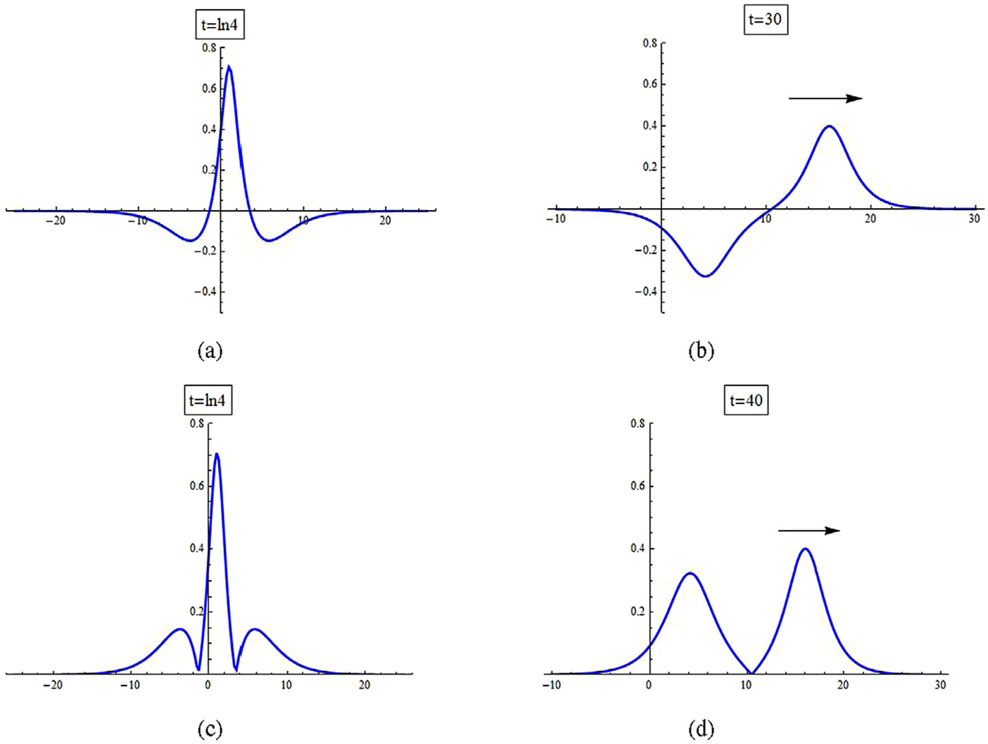

Fig. 6. The single-humped high-order soliton evolution: (a)–(b). Transverse plot of solution $u_1(x,t)$ at $t = ln4, 30$, respectively; (c)–(d). Transverse plot of solution $|u_1(x,t)|$ at $t = ln4, 40$, respectively; The parameter: $\xi_1 = 0, \eta_1 = \frac{1}{4},\ p_1 = 1$.

Moreover, it is necessary to reformulate the dressing factor into summation of fractions, then we derive the soliton matrix $\Gamma(k)$ and its inverse for a pair of an elementary high-order zero. The results can be formulated in the following lemma.

Lemma 1. *Consider a pair of an elementary high-order zero of order n: $\{k_1, -\overline{k_1}\}$ in $\mathbb{C}_+$ and $\{\overline{k_1}, -k_1\}$ in $\mathbb{C}_-$. Then the corresponding soliton matrix and its inverse can be cast in the following form:*

$$\Gamma^{-1}(k) = I + (|p_1\rangle, \cdots, |\widetilde{p}_n\rangle)\, \mathcal{D}(k) \begin{pmatrix} \langle q_n| \\ \vdots \\ \langle \widetilde{q}_1| \end{pmatrix}, \tag{51a}$$

$$\Gamma(k) = I + \left(|\overline{q}_n\rangle, \cdots, |\overline{\widetilde{q}_1}\rangle\right) \overline{\mathcal{D}}(k) \begin{pmatrix} \langle \overline{p}_1| \\ \vdots \\ \langle \overline{\widetilde{p}_n}| \end{pmatrix}, \tag{51b}$$

where $\mathcal{D}(k)$ and $\overline{\mathcal{D}}(k)$ are $2n \times 2n$ block matrices,

$$\mathcal{D}(k) = \begin{pmatrix} \mathcal{T}^+(k-k_1) & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathcal{T}^+(k+\overline{k}_1) \end{pmatrix}, \quad \overline{\mathcal{D}}(k) = \begin{pmatrix} \mathcal{T}^-(k-\overline{k}_1) & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathcal{T}^-(k+k_1) \end{pmatrix},$$

$\mathcal{T}^+(s)$ and $\mathcal{T}^-(s)$ are upper-triangular and lower-triangular Toeplitz matrices defined as:

$$\mathcal{T}^+(s) = \begin{pmatrix} s^{-1} & s^{-2} & \cdots & s^{-n} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & s^{-1} & s^{-2} \\ 0 & \cdots & 0 & s^{-1} \end{pmatrix}, \quad \mathcal{T}^-(s) = \begin{pmatrix} s^{-1} & 0 & \cdots & 0 \\ s^{-2} & s^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ s^{-n} & \cdots & s^{-2} & s^{-1} \end{pmatrix}.$$

This lemma can be proved by induction as in Ref. [21]. Besides, we notice that in the expressions for $\Gamma^{-1}(k)$ (51a) and $\Gamma(k)$ (51b), only half of the vector parameters, i.e.: $|p_1\rangle, \cdots, |p_n\rangle, |\widetilde{p}_1\rangle, \cdots, |\widetilde{p}_n\rangle$ and $\langle\overline{p}_1|, \cdots, \langle\overline{p}_n|, \langle\widetilde{\overline{p}_1}|, \cdots, \langle\widetilde{\overline{p}_n}|$ are independent. In fact, the rest of the vector parameters in (51) can be derived by calculating the poles of each order in the identity $\Gamma(k)\Gamma^{-1}(k) = I$ at $k = k_1$ and $k = k_2 = -\overline{k_1}$:

$$\Gamma(k_1)\begin{pmatrix} |p_1\rangle \\ \vdots \\ |p_n\rangle \end{pmatrix} = 0, \quad \Gamma(-\overline{k}_1)\begin{pmatrix} |\widetilde{p}_1\rangle \\ \vdots \\ |\widetilde{p}_n\rangle \end{pmatrix} = 0, \tag{52}$$

where,

$$\Gamma(k) = \begin{pmatrix} \Gamma(k) & 0 & \cdots & 0 \\ \frac{d}{dk}\Gamma(k) & \Gamma(k) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \frac{1}{(n-1)!}\frac{d^{n-1}}{dk^{n-1}}\Gamma(k) & \cdots & \frac{d}{dk}\Gamma(k) & \Gamma(k) \end{pmatrix}.$$

Hence, in terms of the independent vector parameters, results (51) can be formulated in a more compact form as in [21], and here we just avoid these overlapped parts. In the following, we derive this compact formula via the method of generalized Darboux transformation (gDT). We intend to investigate the relation between dressing matrices and Darboux transformation for S–S equation in the high-order zero case. The essence of Darboux transformation is a kind of gauge transformation. Following the scheme proposed in [23], we can construct the gDT for S–S equation as well.

The elementary form of DT has already been constructed in formula (43), then it is obvious to notice that: $G_1(k_1+\epsilon)|y_1(k_1+\epsilon)\rangle = 0$ and $G_1(k_2+\varepsilon)|y_2(k_2+\varepsilon)\rangle = 0$. Denoting $|\chi_1^{[0]}(k_1)\rangle = |y_1(k_1)\rangle, |\chi_2^{[0]}(k_2)\rangle = |y_2(k_2)\rangle$, and considering the following limitation:

$$|\chi_1^{[1]}(k_1)\rangle \triangleq \lim_{\epsilon\to 0}\frac{G_1(k_1+\epsilon)|\chi_1^{[0]}(k_1+\epsilon)\rangle}{\epsilon} = \frac{d}{dk}\left[G_1(k)|\chi_1^{[0]}(k)\rangle\right]_{k=k_1},$$

then $|\chi_1^{[1]}\rangle$ can be used to construct the next step Darboux transformation, i.e.:

$$\Xi_1^{[1]}(k) = \left(\mathbb{I} + \frac{\overline{k}_1 - k_1}{k - \overline{k}_1}\mathbb{P}_1^{[1]}\right), \ \mathbb{P}_1^{[1]} = \frac{|\chi_1^{[1]}\rangle\langle\chi_1^{[1]}|}{\langle\chi_1^{[1]}|\chi_1^{[1]}\rangle}. \tag{53}$$

In the following step, we introduce

$$\begin{aligned} &|\chi_2^{[1]}(k_2)\rangle = \Xi_1^{[1]}(k_2)|y_2^{[1]}(k_2)\rangle, \\ &|y_2^{[1]}(k_2)\rangle \triangleq \lim_{\varepsilon\to 0}\frac{G_1(k_2+\varepsilon)|\chi_2^{[0]}(k_2+\varepsilon)\rangle}{\varepsilon} = \frac{d}{dk}\left[G_1(k)|\chi_2^{[0]}(k)\rangle\right]_{k=k_2}, \end{aligned}$$

so $\chi_2^{[1]}$ can be used in the next step DT:

$$\Xi_2^{[1]}(k) = \left(\mathbb{I} + \frac{\overline{k}_2 - k_2}{k - \overline{k}_2}\mathbb{P}_2^{[1]}\right), \ \mathbb{P}_2^{[1]} = \frac{|\chi_2^{[1]}\rangle\langle\chi_2^{[1]}|}{\langle\chi_2^{[1]}|\chi_2^{[1]}\rangle}, \tag{54}$$

then we arrive at the 2nd step gDT:

$$G_1^{[1]}(k) = \Xi_2^{[1]}(k)\Xi_1^{[1]}(k). \tag{55}$$

Generally, continuing this process we obtain:

$$|\chi_1^{[N]}\rangle = \lim_{\epsilon\to 0}\frac{G_1^{[N-1]}...G_1^{[1]}G_1^{[0]}(k_1+\epsilon)|\chi_1^{[0]}(k_1+\epsilon)\rangle}{\epsilon^N},$$

$$|y_2^{[N]}\rangle = \lim_{\varepsilon\to 0} \frac{G_1^{[N-1]}...G_1^{[1]}G_1^{[0]}(k_2+\varepsilon)|\chi_2^{[0]}(k_2+\varepsilon)\rangle}{\varepsilon^N},$$

the N-times generalized Darboux matrix can be represented as:

$$T_N(k) = G_1^{[N-1]}...G_1^{[1]}G_1^{[0]}(k),\ G_1^{[i]} = \Xi_2^{[i]}\Xi_1^{[i]}, \tag{56}$$

where,

$$\Xi_s^{[i]} = \mathbb{I} + \frac{\overline{k_s}-k_s}{k-\overline{k_s}}P_s^{[i]},\ P_s^{[i]} = \frac{|\chi_s^{[i]}\rangle\langle\chi_s^{[i]}|}{\langle\chi_s^{[i]}|\chi_s^{[i]}\rangle},\ s=1,2.$$

$$|\chi_1^{[i]}\rangle = \lim_{\epsilon\to 0}\frac{G_1^{[i-1]}...G_1^{[0]}(k_1+\epsilon)|\chi_1^{[0]}(k_1+\epsilon)\rangle}{\epsilon^i},$$

$$|\chi_2^{[i]}(k_2)\rangle = \Xi_1^{[i]}(k_2)|y_2^{[i]}\rangle,\ |y_2^{[i]}\rangle = \lim_{\varepsilon\to 0}\frac{G_1^{[i-1]}...G_1^{[0]}(k_2+\varepsilon)|\chi_2^{[0]}(k_2+\varepsilon)\rangle}{\varepsilon^i}.$$

In addition, the transformation between different potential matrices is:

$$Q^{[N]} = Q + i\left[\sigma_3, \sum_{j=0}^{N-1}(\overline{k_1}-k_1)P_1^{[j]} + (\overline{k_2}-k_2)P_2^{[j]}\right]. \tag{57}$$

In this expression, $P_2^{[i]}$ and $P_1^{[i]}$ are rank-one matrices, so $G_1^{[i]}(k)$ can be also decomposed into the summation of simple fraction, that means the multiple product form of T_N can be directly simplified by the conclusion of Lemma 1. In other words, the above generalized Darboux matrix for S–S equation can be given in the following theorem:

Theorem 2 (*[22], Lemma 4*)**.** *In the case of one pair of elementary high-order zero, the generalized Darboux matrix for S–S equation can be represented as:*

$$T_N = I - YM^{-1}\overline{\mathcal{D}}(k)Y^\dagger,$$

where $\overline{\mathcal{D}}(k)$ is $N\times N$ block Toeplitz matrix which has been given before, Y is a $3\times 2N$ matrix:

$$Y = \left(|y_1\rangle, \ldots, \frac{|y_1\rangle^{(N-1)}}{(N-1)!}, |y_2\rangle, \ldots, \frac{|y_2\rangle^{(N-1)}}{(N-1)!}\right),$$

$$|y_1\rangle^{(j)} = \lim_{\epsilon\to 0}\frac{d^j}{d\epsilon^j}|y_1(k_1+\epsilon)\rangle,\ |y_2\rangle^{(j)} = \lim_{\varepsilon\to 0}\frac{d^j}{d\varepsilon^j}|y_2(k_2+\varepsilon)\rangle,\ k_2 = -\overline{k}_1$$

and M is $2N\times 2N$ matrix:

$$M = \begin{pmatrix} M^{[11]} & M^{[12]} \\ M^{[21]} & M^{[22]} \end{pmatrix},\ M^{[ij]} = \left(M_{l,m}^{[i,j]}\right)_{N\times N},$$

with

$$M_{l,m}^{[i,j]} = \lim_{\epsilon,\overline{\epsilon}\to 0}\frac{1}{(l-1)!(m-1)!}\frac{\partial^{m-1}}{\partial\epsilon^{m-1}}\frac{\partial^{l-1}}{\partial(\overline{\epsilon})^{l-1}}\left[\frac{\langle y_i|y_j\rangle}{k_j-\overline{k}_i+\epsilon-\overline{\epsilon}}\right].$$

Theorem 2 can be proved via directly calculation as in Ref. [23].

Therefore, if $\Phi^{[N]} = T_N\Phi$, then $\Phi^{[N]}$ indeed solves spectral problem (8), i.e.:

$$(T_N\Phi)_x = \left(-ik\sigma_3 + Q^{[N]}\right)T_N\Phi.$$

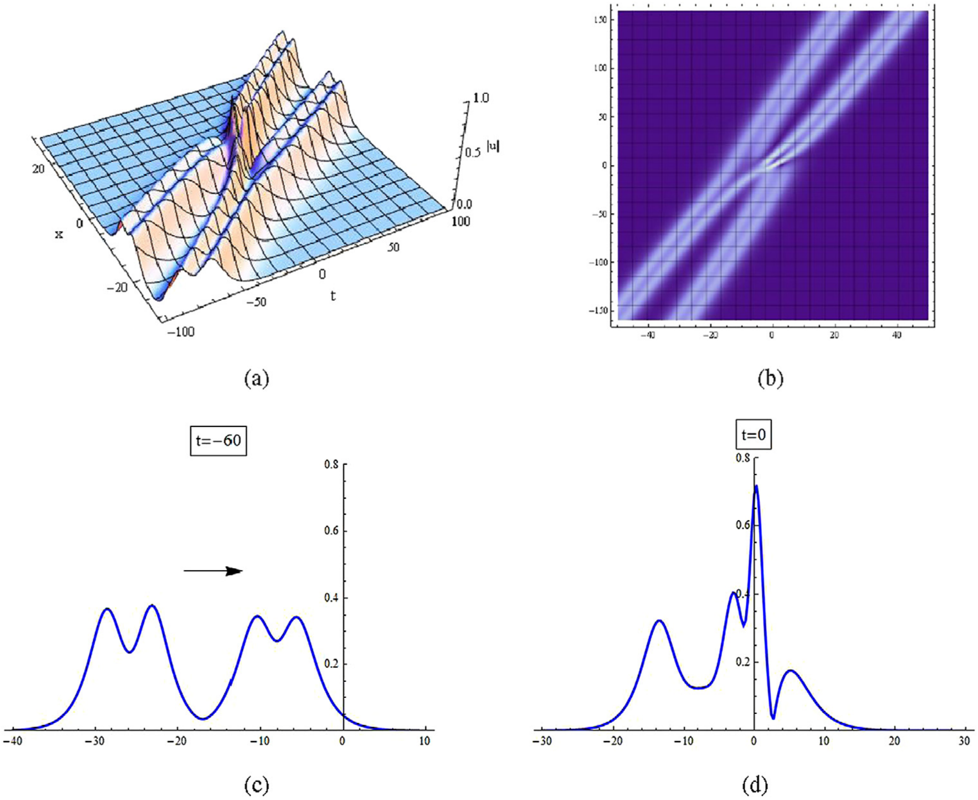

Fig. 7. (a). 3-D plot for the double-humped high-order soliton solution evolution; (b).The density plot for (a); (c)–(d).Transverse plot of (a) at moment $t=-60,0$, respectively. Where, $\xi_1=\frac{1}{24}$, $\eta_1=\frac{1}{4}$; $p_1=1$.

Substituting T_N into the above relation and letting spectral k go to infinity, we have the relation:

$$Q^{[N]}=Q-i[\sigma_3,\left(|y_1\rangle,\ldots,\frac{|y_2\rangle^{(N-1)}}{(N-1)!}\right)M^{-1}\begin{pmatrix}\langle y_1|\\ \vdots\\ \frac{\langle y_2|^{(N-1)}}{(N-1)!}\end{pmatrix}]. \tag{58}$$

Moreover, the transformations between the potential functions are:

$$Q^{[N]}_{j,l}=Q^{[0]}_{j,l}+2i\left(\frac{\det(A_{j,l})}{\det(M)}\right),\quad A_{j,l}=\begin{bmatrix}M & Y[l]^{\dagger}\\ Y[j] & 0\end{bmatrix},\ 1\le j,l\le 3. \tag{59}$$

Here the subscript $_{j,l}$ denotes the jth row and lth column element of matrix A, and $Y[l]$ represents the jth row of matrix Y.

Hence, formula (59) with a zero seed leads to the high-order N-soliton solution formula. Explicitly, taking $N=2$, $Q^{[0]}_{1,3}=0$ in (59) and considering the special case when $\xi_1=0$, we can obtain the high-order one-soliton or the 1-st order algebra soliton $u^{[1]}$ with the expression:

$$u_1(x,t)=\frac{8p_1|p_1|^2k_1\mathcal{F}_1(x,t,k_1)e^{2\eta_1x-8\eta_1^3t}+16p_1k_1\mathcal{F}_2(x,t,k_1)e^{-2\eta_1x+8\eta_1^3t}}{|p_1|^4e^{4\eta_1x-16\eta_1^3t}+4e^{-4\eta_1x+16\eta_1^3t}+4|p_1|^2\mathcal{G}_1(x,t,k_1)}, \tag{60}$$

where the rational polynomials $\mathcal{F}_1,\ \mathcal{F}_2,\ \mathcal{G}_1$ are defined as,

$$\mathcal{F}_1=1+f_1(x,t,k_1),\ \mathcal{F}_2=1-f_1(x,t,k_1),$$

$$\mathcal{G}_1 = 1 + 2f_1^2(x,t,k_1),\ \ f_1 = -2\eta_1 x + 24\eta_1^3 t.$$

First of all, we put these rational functions onto the exponential term, therefore, the above solution becomes:

(i) When $f_1(x,t,k_1) = 0$:

$$u_1(x,t) = \frac{2\sqrt{2}p_1k_1}{|p_1|}Sech(16\eta_1^3 t + \delta),\ \ \delta = \log(|p_1|/\sqrt{2}). \tag{61}$$

(ii) When $f_1(x,t,k_1) \neq 0,\ u_1(x,t)$ is:

$$C_{p_0}\frac{e^{2\eta_1(x-4\eta_1^2t)+\log[f_1]+\log\frac{|p_1|}{\sqrt{2}}} - e^{-2\eta_1(x-4\eta_1^2t)+\log[f_1]-\log\frac{|p_1|}{\sqrt{2}}} + e^{2\eta_1(x-4\eta_1^2t)+\log\frac{|p_1|}{\sqrt{2}}} + e^{2\eta_1(-x+4\eta_1^2t)-\log\frac{|p_1|}{\sqrt{2}}}}{e^{4\eta_1(x-4\eta_1^2t)+2\log|p_1|-\log 2} + e^{4\eta_1(-x+4\eta_1^2t)-2\log|p_1|+\log 2} + e^{\log[\mathcal{G}_1(x,t,k_1)]+\log 2}}, \tag{62}$$

where the constant C_{p_0} is $\frac{4\sqrt{2}k_1p_1}{|p_1|}$. The maximum value for solution $u_1(x,t)$ is $\frac{2\sqrt{2}p_1k_1}{|p_1|}$, which is attained in (62) with the extreme point is located at $x_0 = \frac{-3\delta}{4\eta_1},\ t_0 = \frac{-\delta}{16\eta_1^3}$.

Different from two-soliton solution. It is found that the above high-order one-soliton (62) has two paralleled center trajectories:

$$\phi_1 = 2\eta_1(x - 4\eta_1^2 t) + \log[f_1] + \log[p_1/\sqrt{2}] = 0,$$
$$\phi_2 = 2\eta_1(-x + 4\eta_1^2 t) + \log[f_1] - \log[p_1/\sqrt{2}] = 0,$$

which can be regarded as the special case of the regular two-soliton solution. In this case, two solitons are moving along the paralleled center trajectories in the same velocity, which is $-4\eta_1^2$. This kind of soliton, as it has been mentioned in [21], can describe a weak bound state of solitons. And it may appear in the study of train propagation of solitons with nearly equal velocities and amplitudes. In Fig. 6, we show the propagation of single-humped high-order soliton. Fig. 7 is the propagation of double-humped high-order soliton.

In the following, to derive the long time asymptotic estimation for the high-order one-soliton. Firstly, we do the simple variable substitutions:

$$x - 4\eta_1^2 t \to y,\ \ x + 4\eta_1^2 t \to z, \tag{63}$$

then $u_1(x,t)$ becomes,

$$u_1(y,z) = C_{p_0}\frac{e^{6\eta_1 y+\log(1-4\eta_1 y+2\eta_1 z)+\log|p_1|-\frac{1}{2}\log 2} + e^{2\eta_1 y+\log(1+4\eta_1 y-2\eta_1 z)-\log|p_1|+\frac{1}{2}\log 2}}{e^{8\eta_1 y+2\log|p_1|-\log 2} + e^{-2\log|p_1|+\log 2} + e^{\log[1+2(4\eta_1 y-2\eta_1 z)^2]+4\eta_1 y+\log 2}}. \tag{64}$$

In accordance with the results of the inverse scattering transformation, we need $\eta_1 > 0$. With simple calculation, it is found that $u_1(y,z)$ possesses the following asymptotic estimation:

(i) If $y \gg 0$, $x \gg 4\eta_1^2 t$, then $u_1(y,z) \to 0$, as $y \to +\infty$ for all z;
(ii) If $y \ll 0$, $x \ll 4\eta_1^2 t$, then $u_1(y,z) \to 0$, as $y \to -\infty$ for all z;
(iii) If $y \sim 0$, $x \sim 4\eta_1^2 t$, then $u_1(y,z) \sim \mathcal{O}(\frac{1}{z})$, as $z \to \pm\infty$ (or, as $t \to \pm\infty$);
(iv) If $f_1 > 0$, i.e. $z > 2y$ and $2\eta_1 y \sim \pm \ln z$, then $u_1(y,z) \sim \mathcal{O}_1(1)$, as $z \to +\infty$ (or, as $t \to +\infty$), where $\mathcal{O}_1(1)$ is seen as a constant;
(v) If $f_1 < 0$, i.e. $z < 2y$ and $2\eta_1 y \sim \pm \ln z$, then $u_1(y,z) \sim \mathcal{O}_2(1)$, as $z \to -\infty$ (or, as $t \to -\infty$), where $\mathcal{O}_2(1)$ is another constant;

It is noted that we only consider the elementary high order zeros of the Riemann–Hilbert problem, that is, the algebraic multiplicity of the zeros is arbitrary but the geometric multiplicity is one. However, in general case, the high-order zeros with arbitrary geometric multiplicity which can lead to more general soliton solutions should be considered in the further work.

5. Conclusion and discussion

In conclusion, the inverse scattering method is implemented to Sasa–Satsuma equation with a vanishing boundary condition, and the soliton matrices are constructed by studying the corresponding Riemann–Hilbert problem. By means of the regularization of the RHP with finite simple zeros, we obtain the general N-soliton formula for S–S equation, which was firstly derived in [24]. Furthermore, the high-order soliton matrices are also obtained by considering the multiple zeros of the RHP. It is interesting that pairs of zeros are considered in the process of the regularization for the high-order zeros, which is different from the situation in NLS equation, 3-wave system or the Manakov equation. Besides, the explicit form of DT and generalized DT for S–S equation was constructed which can be applied to generate interesting solutions. Our analysis mainly focuses on the two-solitons collision dynamics, asymptotic behavior and the long time asymptotic estimations for the high-order one-soliton solution.

In discussion, we note that a new integrable reverse space–time nonlocal Sasa–Satsuma equation is recently introduced and investigated in [31]:

$$u_t(x,t)+u_{xxx}(x,t)+\varepsilon\left\{6[\overline{u}(-x,-t)u(x,t)]u_x(x,t)+3u(x,t)[\overline{u}(-x,-t)u(x,t)]_x\right\}=0,\quad \varepsilon=\pm 1. \tag{65}$$

Via using binary Darboux transformation method, the periodic solutions with some localized solutions are constructed for this nonlocal equation [31], such as dark soliton, W-shaped soliton, M-shaped soliton and breather soliton. These solutions are generated from either zero or non-zero seed solution, and they can be nonsingular with certain parameters reductions, even though it is not easy to find the condition of non-singularity for each of them for this nonlocal Sasa–Satsuma equation. Comparatively speaking, for the general soliton as well as the high-order soliton we have derived for this local Sasa–Satsuma equation, their non-singularity has been naturally ensured within the Riemann–Hilbert formulation, which stems from the symmetry properties for the Jost solution, because one can utilize the involution property (17) to show that det(M) is nonsingular [23].

Recently, a detailed study of the inverse scattering theory for the integrable nonlocal NLS equation is presented using a new left–right Riemann–Hilbert problem and the Cauchy problem is formulated [32,33]. Furthermore, it was found in [32] that the symmetries of the eigenfunctions of the associated scattering problem are such that the eigenfunctions defined in the upper and lower half planes are not related. This is in sharp contrast to the classical local NLS equation [32]. Therefore, this would be an interesting topic for this integrable nonlocal Sasa–Satsuma equation. The corresponding inverse scattering transform and Riemann–Hilbert problem can be also formulated in this way. In that case, the important symmetry properties of the eigenfunctions and scattering data could be quite different from that in the local case. Actually, if we pose the reverse space–time nonlocal reduction on the potential matrix Q in the Lax-pair, which becomes:

$$Q=\begin{pmatrix} 0 & 0 & u(x,t)\\ 0 & 0 & \overline{u}(-x,-t)\\ -\overline{u}(-x,-t) & -u(x,t) & 0\end{pmatrix}.$$

Then it satisfies the following two new symmetry properties:

$$Q^{\dagger}(-x,-t)=\sigma_1 Q(x,t)\sigma_1,\quad \sigma_1=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix},$$

$$\overline{Q}(-x,-t)=-\sigma_2 Q(x,t)\sigma_2,\quad \sigma_2=\begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}.$$

In this case, the Jost solution is found to possess the following new symmetry property:

$$J_{\pm}^{\dagger}(-x,-t,-\overline{k})=\sigma_1 J_{\pm}^{-1}(x,k)\sigma_1,\quad J_{\pm}(x,k)=\sigma_2\overline{J_{\pm}}(-x,-t,\overline{k})\sigma_2.$$

Since the analytical and symmetry properties of the Jost solutions as well as the scattering data play a fundamental role in the RH formulation for the scattering problem, and also in the formulation of dressing matrices. According to the form of binary DT constructed in [31] for Eq. (65), we can see the patterns of Riemann–Hilbert zeros could possibly have certain locations on the complex k-plane: this kind of distribution of zeros is clearly different from what we have shown for the local case, and it might lead to new type of solutions associated with Cauchy problem for the nonlocal Sasa–Satsuma equation. In addition, it still remains to be seen whether there exist new symmetry properties which could give rise to unrelated eigenfunctions [32].

Acknowledgments

We would like to show our deepest gratitude to Prof. Liming Ling and Prof. Yu qi Li for their enthusiastic discussions and helpful suggestions. This work is supported by the Global Change Research Program of China (No. 2015CB953904), National Natural Science Foundation of China (No. 11275072, No. 11675054 and No. 11435005), and Shanghai Collaborative Innovation Center of Trustworthy Software for Internet of Things (No. ZF1213), Research Fund for the Doctoral Program of Higher Education of China (No. 20120076110024), The Network Information Physics Calculation of basic research innovation research group of China (Grant No. 61321064), and Shanghai Minhang District talents of high level scientific research project.

References

[1] D.J. Benneyand, A.C. Newell, The propagation of nonlinear wave envelopes, J. Math. Phys. 46 (1967) 133–139.
[2] V.E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, Sov. Phys. J. Appl. Mech. Tech. 4 (1968) 190–194.
[3] A. Hasegawaand, F. Tappert, Transmission of stationary nonlinear optical pulses in dispersive dielectric fibres I. Anomalous dispersion, Appl. Phys. Lett. 23 (1973) 142–144.
[4] A. Hasegawaand, F. Tappert, Transmission of stationary nonlinear optical pulses in dispersive dielectric fibres II. Normal dispersion, Appl. Phys. Lett. 23 (1973) 171–172.
[5] V.E. Zakharov, Collapse of langmuir waves, Sov. Phys.—JETP 35 (1972) 908–914.
[6] Y. Kodama, Optical solitons in a monomode fiber, J. Stat. Phys. 39 (1985) 597–614.
[7] Y. Kodamaand, A. Hasegawa, Nonlinear pulse propagation in a monomode dielectric guide, IEEE J. Quantum Electron. QE–23 (1987) 510–524.
[8] D.J. Kaupand, A.C. Newell, An exact solution for a derivative nonlinear Schrödinger equation, J. Math. Phys. 19 (1978) 798–801.
[9] H.H. Chen, Y.C. Lee, C.S. Liu, Integrability of nonlinear Hamiltonian systems by inverse scattering method, Phys. Scr. 20 (1979) 490–492.
[10] R. Hirota, Exact envelope-soliton solutions of a nonlinear wave equation, J. Math. Phys. 14 (1973) 805–809.
[11] N. Sasa, J. Satsuma, New type of soliton solutions for a higher-order nonlinear Schrödinger equation, J. Phys. Soc. Japan 60 (1991) 409–417.
[12] J.k. Yang, D.J. Kaup, Squared eigenfunctions for the Sasa-Satsuma equation, J. Math. Phys. 50 (2009) 023504.
[13] J. Xu, E. Fan, The unified transform method for the Sasa-Satsuma equation on the half-line, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 469 (2013) 20130068.
[14] C. Gilson, J. Hietarinta, J. Nimmo, Y. Ohta, Sasa-Satsuma higher-order nonlinear Schrödinger equation and its bilinearization and multisoliton solutions, Phys. Rev. E 68 (2003) 016614.
[15] T. Xu, D. Wang, M. Li, H. Liang, Soliton and breather solutions of the Sasa-Satsuma equation via the Darboux transformation, Phys. Scr. 89 (2014) 075207.
[16] U. Bandelow, N. Akhmediev, Sasa-Satsuma equation: Soliton on a background and its limiting cases, Phys. Rev. E 86 (2012) 026606.
[17] J.M. Soto-Crespo, N. Devine, N.P. Hoffmann, N. Akhmediev, Rogue waves of the Sasa-Satsuma equation in a chaotic wave field, Phys. Rev. E 90 (2014) 032902.
[18] G. Mu, Z. Qi, Dynamic patterns of high-order rogue waves for Sasa-Satsuma equation, Nonlinear Anal. RWA 31 (2016) 179–209.
[19] L. Ling, The algebraic representation for high order solution of Sasa-Satsuma equation, Discrete Contin. Dyn. Syst. Ser. S 9 (2017) 1975–2010.
[20] L. Gagnon, N. Stièvenart, N-soliton interaction in optical fibers: The multiple-pole case, Opt. Lett. 19 (1994) 619–621.
[21] V.S. Shchesnovichand, J.K. Yang, Higher-Order solitons in the N-wave system, Stud. Appl. Math. 110 (2003) 297–332.
[22] V.S. Shchesnovichand, J.K. Yang, General soliton matrices in the Riemann-Hilbert problem for integrable nonlinear equations, J. Math. Phys. 44 (2003) 4604–4639.

[23] D.F. Bian, B.L. Guo, L.M. Ling, High-order soliton solution of Landau-Lifshitz equation, Stud. Appl. Math. 134 (2015) 181–214.
[24] J.K. Yang, Nonlinear Waves in Integrable and Nonintegrable Systems, SIAM, Philadelphia, 2010.
[25] S. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zakharov, Theory of Solitons the Inverse Scattering Method, Plenum, New York, 1984.
[26] L.D. Faddeev, L.A. Takhtajan, Hamiltonian Methods in the Theory of Solitons, Springer-Verlag, Berlin-Heidelberg-NewYork, 1987.
[27] M.J. Ablowitz, P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, 1991.
[28] M.J. Ablowitz, A.S. Fokas, Complex Variables: Introduction and Applications, Cambridge University Press, Cambridge, 2003.
[29] A.S. Fokas, V.E. Zakharov, The dressing method and nonlocal Riemann-Hilbert problems, J. Nonlinear Sci. 2 (1992) 109–134.
[30] T. Xu, B. Tian, Bright N-soliton solutions innterms of the triple Wronskia for the coupled nonlinear Schödinger equations in optical fibers, J. Phys. A 43 (2010) 245205.
[31] C. Song, D. Xiao, Z. Zhu, Reverse space-time nonlocal Sasa-Satsuma equation and its solutions, J. Phys. Soc. Japan 86 (2017) 054001.
[32] M.J. Ablowitz, Z.H. Musslimani, Integrable nonlocal nonlinear Schrödinger equation, Phys. Rev. Lett. 110 (2013) 064105.
[33] M.J. Ablowitz, Z.H. Musslimani, Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation, Nonlinearity 29 (2016) 915–946.