Pseudopotentials, Lax Pairs and Bäcklund Transformations for Generalized Fifth-Order KdV Equation*

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Abstract *Based on the method developed by Nucci, the pseudopotentials, Lax pairs and the singularity manifold equations of the generalized fifth-order KdV equation are derived. By choosing different coefficient, the corresponding results and the Bäcklund transformations can be obtained on three conditioners which include Caudrey–Dodd–Gibbon–Sawada–Kotera equation, the Lax equation and the Kaup-kupershmidt equation.*

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1 Introduction

In 1971, the prolongation method for nonlinear evolution equations was proposed by Wahlquist and Estabrook (WE),[1] in which the concept of pseudopotentials was introduced. In 1980, Kaup[2] gave a semidirect method to obtain the prolongation structure of WE without using the differential forms. In 1988, Nucci[3] showed that if the equations satisfied by the pseudopotential are of the following Riccati type:

$$u_x = F_2(q)u^2 + F_1(q)u + F_0(q)\,,$$
$$u_t = G(u, q, q_x, q_{xx}, \ldots)\,, \tag{1}$$

in which G is a polynomial of second order in u, then one can obtain the Lax equations and the auto-Bäcklund transformations of the corresponding nonlinear evolution equation. In particular, if the F_2 is a constant k, the singularity manifold equations and the relation between the dependent variable of the Lax equations and the dependent variable of the singularity manifold equations can be obtained. In 1991, Lou[4] developed Nucci' method to investigate some variable coefficient nonlinear equations.

The generalized fifth-order KdV (gfKdV) equation is written as the following form:

$$q_t + aq^2q_x + bq_xq_{xx} + cqq_{xxx} + dq_{xxxxx} = 0\,, \tag{2}$$

where a, b, and c are arbitrary nonzero and real constants, and $u = u(x,t)$ is a smooth function. This equation is known as the general form of the fifth-order KdV equation, including the Caudrey–Dodd–Gibbon–Sawada–Kotera (CDG-SK) equation,[5−6] the Lax equation[7] and the Kaup–Kupershmidt (KK) equation.[8]

In this paper, the method of Nucci is applied to the gfKdV eqution, the pseudopotentials, Lax pairs, and the singularity manifold equations of the generalized fifth-order equation are derived. By choosing different coefficient only, the corresponding results and the Bäcklund transformations can be obtained on three conditioners, which include CDG-SK equation, the Lax equation and the KK equation. The paper is organized as the following: In Sec. 2, the pseudopotential equation of the gfKdV equation is derived. In Sec. 3, the Lax equation of gfKdV equation, the Lax equation in AKNS form, and singularity manifold equation are obtained respectively. In Sec. 4, the auto-Bäcklund transformations of the CDG-SK equation, Lax equation, and the KK equation are recovered and obtained. In the end, a brief summarization is given.

2 Pseudopotential of gfKdV

According to the method of Nucci,[3] we assume there exist a pseudopotential $u = u(x,t)$ of the following form:

$$u_x = ku^2 + F_1(q)u + F_0(q)\,,$$
$$u_t = G(u, q, q_x, q_{xx}, \ldots)\,, \tag{3}$$

such that the integrability condition $u_{xt} = u_{tx}$ of system (3) is the gfKdV equation (2). With the help of *Maple*, we obtain a simply solution:

$$u_x = ku^2 + \lambda u + \frac{(3c-b)q}{10dk} + \frac{\lambda^2}{4k}\,, \tag{4a}$$

$$u_t = \frac{(b-3c)k}{50d}(10dq_{xx} + (2b-c)q^2)u^2$$

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$$+\frac{(b-3c)k}{50d}(10dq_{xxx}+10\lambda dq_{xx}+(2b-c)(2qq_x+\lambda q^2))u$$
$$+\frac{b-3c}{1000d^2k}(100d^2q_{xxxx}+100\lambda d^2q_{xxx}$$
$$+50\lambda^2d^2q_{xx}+20d(2b-c)q_x(\lambda q+q_x)$$
$$+20d(b+2c)qq_{xx}-(2b-c)(2(b-3c)q^3-5\lambda^2dq^2))\,, \quad (4b)$$

under the constraint condition:

$$a=-\frac{(2b-c)(b-3c)}{10d}\,. \quad (5)$$

In fact, if we solve u^2 from Eq. (4a) and substitute into Eq. (4b), we can get the following equivalent system:

$$u_x=ku^2+\lambda u+\frac{(3c-b)q}{10dk}+\frac{\lambda^2}{4k}\,,$$
$$u_t=\frac{b-3c}{50d}\Big((10dq_{xx}+(2b-c)kq^2)u+\frac{10dq_{xxx}+10\lambda dq_{xx}+(2b-c)(2qq_x+\lambda q^2)}{2k}\Big)_x\,. \quad (6)$$

In order to simplicity, we take the spectral parameter $\lambda=0$ and the constant $k=1$ in the following calculation. The result reads

$$u_x=u^2+\frac{(3c-b)q}{10d}\,,$$
$$u_t=\frac{b-3c}{50d}((10dq_{xx}+(2b-c)q^2)u+5dq_{xxx}+(2b-c)qq_x)_x\,, \quad (7)$$

where u is the pseudopotential of the gfKdV equation.

3 Lax Equations and Singularity Manifold Equations of gfKdV Equation

The Lax equations of the gfKdV equation can be obtained easily by the transformation

$$u=-(\ln\psi)_x\,, \quad (8)$$

and the result reads

$$\psi_{xx}=\frac{(b-3c)q}{10d}\psi\,,$$
$$\psi_t=\frac{(b-3c)}{50d}((10dq_{xx}+(2b-c)q^2)\psi_x-(5dq_{xxx}+(2b-c)qq_x)\psi)\,. \quad (9)$$

Moreover, if we let the linearizing transformation

$$u=\frac{v_1}{v_2}\,, \quad (10)$$

instead of (8), we can get the Lax equations of the AKNS form:[10]

$$\begin{pmatrix}v_1\\v_2\end{pmatrix}_x=\begin{pmatrix}0 & -\frac{b-3c}{10d}q\\-1 & 0\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}\,,$$
$$\begin{pmatrix}v_1\\v_2\end{pmatrix}_t=\begin{pmatrix}A & B\\C & -A\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}\,, \quad (11)$$

where

$$A=\frac{(b-3c)}{50d}(5dq_{xxx}+(2b-c)qq_x)\,,$$
$$B=\frac{b-3c}{10}\Big(q_{xxxx}+\frac{(b+2c)}{5d}qq_{xx}+\frac{(2b-c)}{5d}q_x^2-\frac{(2b-c)(b-3c)}{50d^2}q^3\Big)\,,$$
$$C=-\frac{(b-3c)}{50d}(10dq_{xx}+(2b-c)q^2)\,.$$

Alternatively, if we introduce the transformation

$$u=\frac{1}{2}(\ln(\phi_x))_x\,, \quad (12)$$

we can get the Singularity manifold equation:

$$\frac{\phi_t}{\phi_x}=-\frac{1}{2}d\{\phi,x\}_{xx}+\frac{(2b-c)}{4(b-3c)}d\{\phi,x\}^2\,, \quad (13)$$

in which $\{\phi,x\}$ is the Schwarzian derivative i.e.

$$\{\phi,x\}=\Big(\frac{\phi_{xx}}{\phi_x}\Big)_x-\frac{1}{2}\Big(\frac{\phi_{xx}}{\phi_x}\Big)^2\,. \quad (14)$$

This Singularity manifold equation can also be obtained by the transformation

$$\phi=\frac{\psi_1}{\psi_2}\,, \quad (15)$$

through the Lax equation (9). It is obvious that the relation between the dependent variable ψ of the Lax equations and the dependent variable ϕ of the singularity manifold equations is

$$\phi_x=\psi^{-2}\,. \quad (16)$$

In the above section, we get the pseudopotential, Lax pairs and singularity manifold of the gfKdV equation under the constraint condition (5), especially, some famous integrable fifth-order evolution equation such as CDG-SK equation, Lax equation and KK equation are just the special case of the condition (5). Therefore, by choosing suitable parameters we can get the corresponding pseudopotential, Lax pairs and singularity manifold of the SK equation and KK equation appeared in Ref. [3]. In the same way, we can also get the pseudopotential, Lax pairs and singularity manifold of the Lax equation and some other fifth-order evolution.

4 Bäcklund Transformation of gfKdV Equation

In order to obtain the Bäcklund transformation of the gfKdV equation, we introduce another singularity manifold equation of the gfKdV equation. Another singularity equation can be obtained under the following three cases:

Case 1 $b=c=\alpha$

In this case, the gfKdV equation reads

$$q_t+\frac{1}{5d}\alpha^2q^2q_x+\alpha q_xq_{xx}+\alpha qq_{xxx}+dq_{xxxxx}\,, \quad (17)$$

by means of the transformation:

$$u=-(\ln(\theta_x))_x\,, \quad (18)$$

we can get

$$\frac{\theta_t}{\theta_x}=d\{\theta,x\}_{xx}+4d\{\theta,x\}^2\,, \quad (19)$$

$$q=\frac{-5\mathrm{d}\theta_{xxx}}{\alpha\theta_x}\,. \tag{20}$$

The Schwarzian derivative (14) is invariant under the Möbius transformation

$$\theta^*=\frac{a+b\theta}{c+\mathrm{d}\theta}\,. \tag{21}$$

Therefore, Eq. (19) is also invariant under Möbius transformation (21), that is to say

$$\theta^*=\theta^{-1} \tag{22}$$

is also a solution of Eq. (19), and the corresponding

$$q^*=\frac{-5\mathrm{d}\theta^*_{xxx}}{\alpha\theta^*_x}\,, \tag{23}$$

is another solution of Eq. (17). Then combining (22) and (25) by means of Eq. (24), we obtain the spatial part of an auto-Bäcklund transformation for Eq. (17):

$$900d^2(Q^*_{xx}-Q_{xx})+\alpha^2(Q^*-Q)^3 + 90\alpha d(Q^*-Q)(Q^*_x+Q_x)=0\,, \tag{24}$$

where

$$Q=\int q\,\mathrm{d}x\,,\quad Q^*=\int q^*\,\mathrm{d}x\,. \tag{25}$$

The temporal part can be obtained in the same method, for simplicity we omit it here.

In particular, when $d=1$, Eq. (17) is the CDG-SK equation:

$$q_t+\frac{1}{5}\alpha^2q^2q_x+\alpha q_xq_{xx}+\alpha qq_{xxx}+q_{xxxxx}\,, \tag{26}$$

when $\alpha=-15$, Eq. (26) is the Sawa–Kotera–Parker–Dye (SKPD) equation:[11]

$$q_t+45q^2q_x-15q_xq_{xx}-15qq_{xxx}+q_{xxxxx}\,, \tag{27}$$

and the KK equation list in Ref. [3] is just the special case $\alpha=-5$, $d=-1$.

Case 2 $b=2c=2\alpha$

In this case, the gfKdV equation reads

$$q_t+\frac{3}{10d}\alpha^2q^2q_x+2\alpha q_xq_{xx}+\alpha qq_{xxx}+dq_{xxxxx}\,, \tag{28}$$

by means the transformation:

$$u=-\frac{1}{2}(\ln(\theta_x))_x\,, \tag{29}$$

we can get

$$\frac{\theta_t}{\theta_x}=\frac{1}{2}d\{\theta,x\}_{xx}+\frac{3}{4}d\{\theta,x\}^2\,, \tag{30}$$

$$q=-\frac{5d}{\alpha}\{\theta,x\}-\frac{5d}{\alpha}\frac{\theta^2_{xx}}{\theta^2_x}\,. \tag{31}$$

As in the case 1, it is straightforward to obtain another solution of Eq. (28):

$$q^*=-\frac{5d}{\alpha}\{\theta^*,x\}-\frac{5d}{\alpha}\frac{\theta^{*2}_{xx}}{\theta^{*2}_x}\,,\quad \theta^*=\theta^{-1}\,, \tag{32}$$

and the spatial part of an auto-Bäcklund transformation for Eq. (28):

$$800d^2(Q^*_{xx}-Q_{xx})(Q^*-Q)-400d^2(Q^*_x-Q_x)^2 +\alpha^2(Q^*-Q)^4+80\alpha d(Q^*-Q)^2(Q^*_x+Q_x)=0\,, \tag{33}$$

where Q and Q^* are the same as Eq. (25).

In particular, when $d=1$, Eq. (28) is the Lax equation,[6]

$$q_t+\frac{3}{10}\alpha^2q^2q_x+2\alpha q_xq_{xx}+\alpha qq_{xxx}+q_{xxxxx}\,. \tag{34}$$

Case 3 $b=(5/2)c=(5/2)\alpha$

In this case, the gfKdV equation reads

$$q_t+\frac{1}{5d}\alpha^2q^2q_x+\frac{5}{2}\alpha q_xq_{xx}+\alpha qq_{xxx}+dq_{xxxxx}\,, \tag{35}$$

by means of the transformation:

$$u=-\frac{1}{4}(\ln(\theta_x))_x\,, \tag{36}$$

we can get

$$\frac{\theta_t}{\theta_x}=\frac{1}{4}d\{\theta,x\}_{xx}+\frac{1}{16}d\{\theta,x\}^2\,, \tag{37}$$

$$q=-\frac{5d}{\alpha}\{\theta,x\}-\frac{15d}{4\alpha}\frac{\theta^2_{xx}}{\theta^2_x}\,. \tag{38}$$

As in the case 1, it is straightforward to obtain another solution of Eq. (35):

$$q^*=-\frac{5d}{\alpha}\{\theta^*,x\}-\frac{15d}{4\alpha}\frac{\theta^{*2}_{xx}}{\theta^2_x}\,,\quad \theta^*=\theta^{-1}\,, \tag{39}$$

and the spatial part of an auto-Bäcklund transformation for Eq. (35):

$$900d^2(Q^*_{xx}-Q_{xx})(Q^*-Q)-675d^2(Q^*_x-Q_x)^2 +\alpha^2(Q^*-Q)^4+90\alpha d(Q^*-Q)^2(Q^*_x+Q_x)=0\,, \tag{40}$$

where Q and Q^* are the same as Eq. (25).

In particular, when $d=1$, Eq. (35) is the KK equation,

$$q_t+\frac{1}{5}\alpha^2q^2q_x+\frac{5}{2}\alpha q_xq_{xx}+\alpha qq_{xxx}+q_{xxxxx}=0\,, \tag{41}$$

and when $\alpha=15$, Eq. (41) is the Kaup–Kupershmidt–Parker–Dye (SKPD) equation:[11]

$$q_t+45q^2q_x-\frac{75}{2}q_xq_{xx}-15qq_{xxx}+dq_{xxxxx}=0\,. \tag{42}$$

When $\alpha=-5$, $d=-1$, Eq. (35) is the KK equation in Ref. [3], and the auto-Bäcklund transformation (40) is the same as that of Ref. [3], but the pseudopotential, Lax pairs and singularity are different from that of Ref. [3]. However, if we take the transformation $u\to-u/2$ the pseudopotential, Lax pairs and singularity are same as the solution in Ref. [3].

5 Summary and Discussion

In this paper, the pseudopotentials, Lax pairs, singularity manifold equations, and auto-Bäcklund transformations for the gfKdV equation are diverted. In particulary, the corresponding properties of the CDG-SK equation, Lax equation, and KK equation can be obtained by choosing different parameters of the results obtained for gfKdV equation. Various other important properties such as conservation laws, Miura transformations, Hamilton structure, Lie symmetries and so on[12−15] are also have close connection with the pseudopotentials, which is our further investigation.

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