Binary Bell Polynomials Approach to Generalized Nizhnik–Novikov–Veselov Equation*

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Abstract *The elementary and systematic binary Bell polynomials method is applied to the generalized Nizhnik–Novikov–Veselov (GNNV) equation. The bilinear representation, bilinear Bäcklund transformation, Lax pair and infinite conservation laws of the GNNV equation are obtained directly, without too much trick like Hirota's bilinear method.*

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1 Introduction

With the development of the nonlinear science, soliton theory has played an important role in various domains of natural science. Moreover, the traditional mathematics theory is greatly advanced, so the investigation of integrable systems has caused great interest of physicists and mathematicians. It shows that for integrable systems, there must exist solitary solutions. However, there has not been a certain and uniform definition about the "integrability" of a nonlinear system so far. For example, there are Liouville integrable, inverse scattering transformation integrable, Lax integrable, symmetry integrable, Painlevé integrable, C integrable, and so on. Hence, when a nonlinear system is proved to be integrable, one needs to point that under what mean it is integrable. Then this may pave the way for constructing the exact solutions explicitly in future.

It is known that the bilinear method introduced by Hirota[1−2] has been proved greatly powerful for constructing multi-soliton solutions of nonlinear system. The first step of this method is looking for appropriate dependent variable transformation, which can translate the original nonlinear equation into bilinear forms. Obviously, the selection for this transformation is the key. A noteworthy feature of Hirota's bilinear method is that it allows the construction of Bäcklund transformation,[3−4] which may lead to the underlying Lax pairs,[5−6] infinite conservation laws,[7−9] etc. The calculation of this bilinear Bäcklund transformation relies on a particular skill in using suitable "exchange formulas" related to bilinear representation of the system. However, whether the adoption of the dependent variable transformation in the first step or the utilization of "exchange formulas" for the construction of bilinear Bäcklund transformation are not as direct as one would wish. Recently, Lembert, Gilson *et al.* proposed an alternative procedure[10−12] based on the Bell polynomials[13] to solve above questions. In this way, on one hand, one can naturally write the bilinear form of the original equation; on the other hand, without prior bilinearization of the system, it can enable one to obtain bilinear Bäcklund transformation for soliton systems in a lucid and systematic way and then the corresponding spectral formulation can be written out directly. Fan[14] further developed this method with new applications to construct infinite conservation laws through decoupling binary Bell polynomials into a Riccati type equation and divergence type equation.

In this paper, based on the binary Bell polynomials method, we investigate a generalized Nizhnik–Novikov–Veselov (GNNV) equation and the bilinear representation, bilinear Bäcklund transformation, Lax pair and infinite conservation laws of the GNNV equation are presented. The paper is arranged as follows. In Sec. 2, we briefly recall the main properties of the multi-dimensional binary Bell polynomials that will be used in this paper. In Sec. 3, we apply this simple method to the GNNV equation and show its integrability under the meaning of its bilinear form, bilinear Bäcklund transformation, Lax pair and infinite conservation laws. Section 4 is the conclusion.

2 Binary Bell Polynomials

The multi-dimensional Bell's exponential Y-polynomials[13] are defined as follows

$$\begin{aligned} Y_{n_1x_1,\ldots,n_lx_l}(f) &\equiv Y_{n_1,\ldots,n_l}(f_{r_1x_1,\ldots,r_lx_l}) \\ &= \mathrm{e}^{-f}\partial_{x_1}^{n_1}\ldots\partial_{x_l}^{n_l}\,\mathrm{e}^{f}\,, \end{aligned} \tag{1}$$

where $f\equiv f(x_1,\ldots,x_n)$ is a C^∞ function with n independent variables and we denote

$$f_{r_1x_1,\ldots,r_lx_l}=\partial_{x_1}^{r_1}\ldots\partial_{x_l}^{r_l}(f)\,.$$

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The most important multi-dimensional binary Bell polynomials,[10] which are related to the standard Hirota's bilinear operator are only a two-field generalizations of above exponential Bell polynomials:

$$\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v,w)=Y_{n_1x_1,\ldots,n_lx_l}(f)\Big|_{f_{r_1x_1,\ldots,r_lx_l}=\begin{cases}v_{r_1x_1,\ldots,r_lx_l}, & r_1+\cdots+r_l=\text{odd},\\ w_{r_1x_1,\ldots,r_lx_l}, & r_1+\cdots+r_l=\text{even}.\end{cases}} \tag{2}$$

To illustrate above generalized formulation, we take $f=f(x,t)$ for example. According to formulation (1), one can easily get

$$\begin{aligned}&Y_x(f)=f_x,\quad Y_{2x}(f)=f_{2x}+f_x^2\,,\\&Y_{x,t}(f)=f_{xt}+f_xf_t\,,\\&Y_{3x}(f)=f_{3x}+3f_xf_{2x}+f_x^3,\ldots\end{aligned} \tag{3}$$

The corresponding binary Bell polynomials are

$$\begin{aligned}&\mathcal{Y}_x(v)=v_x,\quad \mathcal{Y}_{2x}(v,w)=w_{2x}+v_x^2\,,\\&\mathcal{Y}_{x,t}(v,w)=w_{xt}+v_xv_t\,,\\&\mathcal{Y}_{3x}(v,w)=v_{3x}+3v_xw_{2x}+v_x^3,\ldots\end{aligned} \tag{4}$$

On one hand, the link between $\mathcal{Y}$-polynomials (2) and the standard Hirota expressions is given by the follow identity

$$\begin{aligned}&\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}\Big(v=\ln\frac{F}{G},w=\ln FG\Big)\\&=(FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G\,,\end{aligned} \tag{5}$$

in which $n_1+n_2+\cdots+n_l\geq 1$ and

$$\begin{aligned}&D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G\\&\equiv(\partial_{x_1}-\partial_{x_1'})^{n_1}\ldots(\partial_{x_l}-\partial_{x_l'})^{n_l}\\&\times\ F(x_1,\ldots,x_l)G(x_1',\ldots,x_l')|_{x_1'=x_1,\ldots,x_l'=x_l}\,.\end{aligned} \tag{6}$$

On the other hand, the $\mathcal{Y}$-polynomials (2) can be "linearizabled" by the following formula

$$\begin{aligned}&\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v=v,w=v+q)|_{v=\ln\psi}\\&=\psi^{-1}\sum_{s_1=0}^{n_1}\cdots\sum_{s_l=0}^{n_l}\binom{n_1}{s_1}\cdots\binom{n_l}{s_l}\\&\times\mathcal{Y}_{s_1x_1,\ldots,s_lx_l}(0,q)\cdot\psi_{(n_1-s_1)x_1,\ldots,(n_l-s_l)x_l}\,,\end{aligned} \tag{7}$$

which is the basis for constructing Lax pair. Specially, it is noted that

$$P_{s_1x_1,\ldots,s_lx_l}(q)=\mathcal{Y}_{s_1x_1,\ldots,s_lx_l}(0,q)\,, \tag{8}$$

where $s_1+\cdots+s_l=\text{even}$ and the odd parts included are zero for $v=0$. Formula (8) restricts the Bell recipe to even part partitions:

$$\begin{aligned}&P_0(q)=1,\quad P_{2x}(q)=q_{2x},\quad P_{x,t}(q)=q_{xt}\,,\\&P_{4x}(q)=q_{4x}+3q_{2x}^2,\ldots\end{aligned} \tag{9}$$

According to formula (5) and definition (8), it is clear that

$$\begin{aligned}&P_{s_1x_1,\ldots,s_lx_l}(q=2\ln F)=F^{-2}D_{x_1}^{s_1}\ldots D_{x_l}^{s_l}F\cdot F\\&(s_1+\ldots+s_l=\text{even})\,.\end{aligned} \tag{10}$$

The binary Bell polynomials have been found to play an important role in the characterization of bilinearizable equations and provide a shortest way between bilinear Bäcklund transformation and Lax pair. About the more detailed applications of binary Bell polynomials, one can consult Refs. [10–12].

3 Integrability of GNNV Equation

In this section, we will try to apply the binary Bell polynomials method to the generalized Nizhnik–Novikov–Veselov (GNNV) equation.[15–16] The GNNV equation reads

$$\begin{aligned}&u_t+au_{xxx}+bu_{yyy}+3a(uv)_x\\&\quad+3b(uw)_y+cu_x+du_y=0\,,\\&u_x-v_y=0,\quad u_y-w_x=0\,,\end{aligned} \tag{11}$$

which is an isotropic Lax integrable extension of the (1+1)-dimensional KdV equation and a,b,c,d are free constants. Boilti *et al.*[17] have solved the GNNV equation by means of the inverse scattering transformation method. Moreover, the multidromion solutions and doubly periodic wave solutions for the GNNV equation were obtained by Radha *et al.*[18] and Peng,[19] respectively. When $c=d=0$, Eq. (11) would become the well-known Nizhnik–Novikov–Veselov (NNV) equation.[20–25] Here, we will adopt an alternative way, i.e. binary Bell polynomials, to investigate Eq. (11).

3.1 *Bilinear Representation*

Firstly, we introduce a potential field q by setting

$$u=mq_{xy}, \tag{12}$$

with m being a constant to be determined later. Substituting the ansatz (12) into the last two of Eq. (11), it leads to

$$v=mq_{xx}+v_0(x,t),\quad w=mq_{yy}+w_0(y,t)\,, \tag{13}$$

where $v_0\equiv v_0(x,t)$ and $w_0\equiv w_0(y,t)$ are arbitrary functions of indicated variables.

Due to (12) and (13), after being integrated once with respect to x, Eq. (11) becomes

$$\begin{aligned}E(q)=&\,q_{yt}+a(q_{3x,y}+3mq_{xx}q_{xy})\\&+b\partial_x^{-1}\partial_y(q_{x,3y}+3mq_{xy}q_{yy})+(3av_0+c)q_{xy}\\&+\partial_x^{-1}\partial_y(3bw_0+d)q_{xy}=0\,,\end{aligned} \tag{14}$$

with $\partial_x^{-1}=\int\cdot\,\mathrm{d}x$.

By selecting $m=1$, one can see that Eq. (14) can be cast into a combination form of P-polynomials

$$\begin{aligned}&P_{yt}(q)+aP_{3x,y}(q)+b\partial_x^{-1}\partial_yP_{x,3y}(q)+(3av_0+c)P_{xy}(q)\\&+\partial_x^{-1}\partial_y(3bw_0+d)P_{xy}(q)=0\,.\end{aligned} \tag{15}$$

Then we may decouple (15) into a pair of P-polynomials, by introducing an auxiliary z and an auxiliary P-condition

$$bP_{x,3y}(q) + (3bw_0 + d)P_{xy}(q) = P_{xz}(q)\,. \tag{16}$$

This decoupling leads to a P-representation, which comprises (16) and the following (3+1)-dimensional condition

$$P_{yt}(q) + aP_{3x,y}(q) + (3av_0 + c)P_{xy}(q) + P_{yz}(q) = 0\,. \tag{17}$$

Because of the property (10), Eqs. (16) and (17) give the bilinear representation of Eq. (11) as follows:

$$\begin{aligned}&(bD_xD_y^3 + (3bw_0 + d)D_xD_y - D_xD_z)F\cdot F = 0\,,\\&(D_yD_t + aD_x^3D_y + (3av_0 + c)D_xD_y\\&\quad + D_yD_z)F\cdot F = 0\,,\end{aligned} \tag{18}$$

under the change of dependent variable $u = q_{xy} = 2(\ln F)_{xy}$.

Starting from the bilinear form (18), one can easily obtain the multi-soliton solutions of Eq. (11) by regular perturbation method.

Remark 1 Other than Hirota's bilinear method, the binary Bell polynomials method can transform Eq. (14) into bilinear form straightforwardly without the clever guesswork of τ-function.

3.2 *Bilinear Bäcklund Transformation and Lax Pair*

It is known that so-called Bäcklund transformation is the relational expression between one solution and another of the soliton equations. Next, with this binary Bell polynomials method, we will search for the Bäcklund transformation in the bilinear form and the corresponding spectral formulation of Eq. (11).

Let q' and q be two different solutions of Eq. (14), respectively. Substituting them into Eq. (14) with $m = 1$, it leads to

$$\begin{aligned}&E(q') - E(q)\\&= (q'-q)_{yt} + a[(q'-q)_{3x,y} + 3q'_{xx}q'_{xy} - 3q_{xx}q_{xy}]\\&\quad + b\partial_x^{-1}\partial_y[(q'-q)_{x,3y} + 3q'_{xy}q'_{yy} - 3q_{xy}q_{yy}]\\&\quad + (3av_0 + c)(q'-q)_{xy}\\&\quad + \partial_x^{-1}\partial_y(3bw_0 + d)(q'-q)_{xy}\,.\end{aligned} \tag{19}$$

To connect Eq. (19) with the binary Bell polynomials, one can introduce two more new variables, i.e.

$$v = \frac{q'-q}{2}\,,\quad w = \frac{q'+q}{2}\,, \tag{20}$$

and rewrite the two-field condition (19) into the form

$$\begin{aligned}&\frac{1}{2}(E(q') - E(q))\\&= v_{yt} + a(v_{3x,y} + 3v_{xx}w_{xy} + 3w_{xx}v_{xy})\\&\quad + b\partial_x^{-1}\partial_y(v_{x,3y} + 3v_{xy}w_{yy} - 3w_{xy}v_{yy})\\&\quad + (3av_0 + c)v_{xy} + \partial_x^{-1}\partial_y(3bw_0 + d)v_{xy}\\&= v_{yt} + a[\partial_y\mathcal{Y}_{3x}(v,w) + 3v_{xx}w_{xy} - 3w_{2x,y}v_x - 3v_x^2v_{xy}]\\&\quad + b\partial_x^{-1}\partial_y[\partial_x\mathcal{Y}_{3y}(v,w)\\&\quad + 3w_{xy}v_{yy} - 3w_{2y,x}v_y - 3v_y^2v_{xy}]\\&\quad + (3av_0 + c)v_{xy} + \partial_y[(3bw_0 + d)v_y]\,.\end{aligned} \tag{21}$$

Then we need to add a constraint to express Eq. (21) as the y-derivative of a combination of $\mathcal{Y}$-polynomials. The simplest possible choice of such constraint may be

$$\mathcal{Y}_{xy}(v,w) = w_{xy} + v_xv_y = 0. \tag{22}$$

Owing to the constraint (22), Eq. (21) becomes

$$\begin{aligned}&\partial_y[v_t + a\mathcal{Y}_{3x}(v,w) + b\mathcal{Y}_{3y}(v,w)\\&\quad + (3av_0 + c)v_x + (3bw_0 + d)v_y] = 0\,,\end{aligned} \tag{23}$$

or

$$\begin{aligned}&\mathcal{Y}_t(v,w) + a\mathcal{Y}_{3x}(v,w) + b\mathcal{Y}_{3y}(v,w) + (3av_0 + c)\mathcal{Y}_x(v,w)\\&\quad + (3bw_0 + d)\mathcal{Y}_y(v,w) - \beta(x,t) = 0,\end{aligned} \tag{24}$$

with $\beta \equiv \beta(x,t)$ being an arbitrary function of indicated variables.

By application of the identity (5), one can naturally get the bilinear Bäcklund transformation of Eq. (14) from Eqs. (22) and (24):

$$\begin{aligned}&D_xD_yF\cdot G = 0\,,\\&(D_t + aD_x^3 + bD_y^3 + (3av_0 + c)D_x\\&\quad + (3bw_0 + d)D_y - \beta)F\cdot G = 0\,.\end{aligned} \tag{25}$$

For obtaining the corresponding Lax pair of system (25), one needs to make the transformation $q = 2\ln G$ and $\psi = F/G$. According to the identity (7), by the change of $w = v + q$ and $v = \ln\psi$, there are

$$\begin{aligned}&\mathcal{Y}_t(v,w) = \psi_t/\psi,\quad \mathcal{Y}_x(v,w) = \psi_x/\psi\,,\\&\mathcal{Y}_y(v,w) = \psi_y/\psi,\quad \mathcal{Y}_{3x}(v,w) = (\psi_{3x} + 3q_{xx}\psi_x)/\psi\,,\\&\mathcal{Y}_{3y}(v,w) = (\psi_{3y} + 3q_{yy}\psi_y)/\psi\,,\\&\mathcal{Y}_{xy}(v,w) = (\psi_{xy} + q_{xy}\psi)/\psi\,.\end{aligned} \tag{26}$$

On account of the above expressions (26), the system (22) and (24) is then immediately linearized into a Lax pair as follows

$$\begin{aligned}&\psi_{xy} + q_{xy}\psi = 0\,,\\&\psi_t + a\psi_{3x} + b\psi_{3y} + (3aq_{xx} + 3av_0 + c)\psi_x\\&\quad + (3bq_{yy} + 3bw_0 + d)\psi_y - \beta_0\psi = 0\,,\end{aligned} \tag{27}$$

where $v_0 \equiv v_0(x,t)$, $w_0 \equiv w_0(y,t)$, and $\beta_0 = \beta_0(t)$.

One can easily check that the compatibility condition of system (27) is just the potential of Eq. (14).

Remark 2 Here, one can see that in the derivation of Bilinear Bäcklund transformation and Lax pair of Eq. (14), the intractable problem about the choice and application of "exchange formulas" is circumvented. With the help of $\mathcal{Y}$-polynomials, one can directly get the Lax system and Bilinear Bäcklund transformation from two "$\mathcal{Y}$-constraints".

3.3 *Infinite Conservation Laws*

As another application of the binary Bell polynomials, Fan[14] derived the infinite conservation laws of soliton equations through decoupling binary Bell polynomials into a Riccati type equation and a divergence type equation. In fact, the conservation laws have been hinted in the two-field constraint system (22) and (23), which can be rewritten in the conserved form

$$\begin{aligned} & w_{xy} + v_x v_y = 0\,, \\ & \partial_t(v_y) + \partial_x(a v_{2x,y} + c v_y) + \partial_y[a(3w_{2x}v_x + v_x^3) \\ & + b\mathcal{Y}_{3y}(v,w) + 3av_0 v_x + (3bw_0 + d)v_y] = 0\,. \end{aligned} \tag{28}$$

Instead of introducing $\eta = (q'_x - q_x)/2$ by Fan, a new potential function is proposed as

$$\eta = \frac{q'_y - q_y}{2}\,.$$

In this way, there are

$$v_y = \eta, \quad w_y = q_y + \eta\,. \tag{29}$$

Substituting (29) into (28), we can obtain

$$\eta_x + \eta\partial_y^{-1}\eta_x + q_{xy} = 0\,, \tag{30}$$

$$\begin{aligned} & \eta_t + \partial_x(a\eta_{xx} + c\eta) + \partial_y[3a(\partial_y^{-1}\eta_{xx} + q_{xx})\partial_y^{-1}\eta_x \\ & + a(\partial_y^{-1}\eta_x)^3 + b(\eta_{yy} + 3\eta\eta_y + 3q_{yy}\eta + \eta^3) \\ & + 3av_0\partial_y^{-1}\eta_x + (3bw_0 + d)\eta] = 0\,. \end{aligned} \tag{31}$$

One can see that Eq. (30) is not a Riccati-type equation, which is different from that in Ref. [14]. Similarly, we expand η in the series form

$$\eta = \epsilon + \sum_{n=1}^{\infty} I_n(q, q_x, q_y, \ldots)\epsilon^{-n}\,. \tag{32}$$

Firstly, the substitution of (32) into Eq. (30) would lead to

$$\begin{aligned} & \sum_{n=1}^{\infty} I_{n,x}\epsilon^{-n} + \Bigl(\sum_{n=1}^{\infty} \partial_y^{-1} I_{n,x}\epsilon^{-n}\Bigr) \\ & \times \Bigl(\epsilon + \sum_{n=1}^{\infty} I_n\epsilon^{-n}\Bigr) + q_{xy} = 0\,. \end{aligned} \tag{33}$$

Collecting the coefficients for power of ϵ and equating them with zero, we then get the recursion for I_n

$$I_1 = -q_{yy} - \Theta_1(y,t), \quad I_2 = q_{3y} + \Theta_2(y,t)\,, \tag{34}$$

$$\begin{aligned} I_{n+1} = & -I_{n,y} - \sum_{k=1}^{n} \partial_x^{-1}\partial_y(I_k\partial_y^{-1}I_{n-k,x}) \\ & - \Theta'_{n+1}(y,t)\,, \quad (n \geq 2)\,. \end{aligned} \tag{35}$$

For example, by formula (35), one can calculate

$$I_3 = -q_{4y} - \partial_x^{-1}\partial_y(q_{xy}q_{yy}) - \partial_y(\Theta_1 q_y) - \Theta_3\,, \tag{36}$$

$$I_4 = q_{5y} + 2\partial_x^{-1}\partial_y^2(q_{xy}q_{yy}) + 2\partial_y^2(\Theta_1 q_y) + \Theta_4, \ldots \tag{37}$$

Here, $\Theta_n \equiv \Theta_n(y,t)$ is the function of indicated variables and may need to be determined later. Actually, the second equation of system (28) embodies the explicit expression of conservation laws

$$I_{n,t} + F_{n,x} + G_{n,y} = 0, \quad (n = 1, 2, \ldots)\,. \tag{38}$$

To proceed, applying (32) to Eq. (31) and collecting the coefficients of each order of ϵ, we have

$$\begin{cases} \epsilon : 3b(q_{yy} + I_1 + w_0) = 0 & \Rightarrow I_1 = -q_{yy} - w_0\,; \\ \epsilon^0 : 3b(I_{1,y} + I_2) = 0 & \Rightarrow I_2 = q_{3y} + w_{0y}\,; \end{cases} \tag{39}$$

$$\epsilon^{-1} : \begin{cases} F_1 = aI_{1,xx} + cI_1 = -aq_{2x,2y} - c(q_{yy} + w_0)\,, \\ G_1 = 3aq_{xx}\partial_y^{-1}I_{1,x} + bI_{1,yy} + 3bI_{2,y} + 3bq_{yy}I_1 + 3bI_1^2 + 3bI_3 + 3av_0\partial_y^{-1}I_{1,x} + (3bw_0 + d)I_1 \\ \quad = -3aq_{xy}(q_{xx} + v_0) - b[q_{4y} + 3\partial_x^{-1}\partial_y(q_{xy}q_{yy}) + 3w_{0y}q_y + 3w_0q_{yy} - 2w_{0yy} + 3\Theta_3] - d(q_{yy} + w_0)\,, \end{cases}$$

$$\epsilon^{-2} : \begin{cases} F_2 = aI_{2,xx} + cI_2 = aq_{2x,3y} + c(q_{3y} + w_{0y})\,, \\ G_2 = 3a(\partial_y^{-1}I_{1,x})(\partial_y^{-1}I_{1,xx}) + 3aq_{xx}\partial_y^{-1}I_{2,x} + bI_{2,yy} + 3bI_1I_{1,y} + 3bI_{3,y} + 3bq_{yy}I_2 \\ \qquad + 6bI_1I_2 + 3bI_4 + 3av_0\partial_y^{-1}I_{2,x} + (3bw_0 + d)I_2 \\ \quad = 3a(q_{xy}q_{xxy} + q_{xx}q_{xyy} + v_0q_{xyy}) + b[q_{5y} + 3\partial_x^{-1}\partial_y^2 q_{xy}q_{yy} + 3\partial_y^2(w_0q_y) \\ \qquad + w_{0,3y} - 3\Theta_{3,y} + 3\Theta_4] + d(q_{3y} + w_{0y})\,. \end{cases}$$

When $n \geq 3$, there are

$$\epsilon^{-n} : \begin{cases} F_n = aI_{n,xx} + cI_n\,, \\ G_n = 3a\sum_{k=1}^{n}(\partial_y^{-1}I_{k,x})(\partial_y^{-1}I_{n-k,xx}) + 3aq_{xx}\partial_y^{-1}I_{n,x} + bI_{n,yy} + 3b\sum_{k=1}^{n} I_kI_{n-k,y} \\ \qquad + 3bI_{n+1,y} + a\sum_{i+j+k=n}(\partial_y^{-1}I_{i,x})(\partial_y^{-1}I_{j,x})(\partial_y^{-1}I_{k,x}) + 3bq_{yy}I_n + b\sum_{i+j+k=n} I_iI_jI_k \\ \qquad + 3b\sum_{k=1}^{n+1} I_kI_{n+1-k} + 3bI_{n+2} + 3av_0\partial_y^{-1}I_{n,x} + (3bw_0 + d)I_n\,. \end{cases}$$

Comparing (39) with (34), we see that it requires $\Theta_1 = w_0$ and $\Theta_2 = w_{0y}$. When $n \geq 3$, $\Theta_n \equiv \Theta_n(y, t)$ are arbitrary functions of indicated variables. One can directly vary that the first equation of conservation law equation (38) is exactly Eq. (14).

4 Conclusion

Recently, Lembert, Gilson *et al.* shows that by means of a direct and systematic approach based on the Bell polynomials, the major integrability feature of a significant set of soliton equations can be disclosed. Here, this method is applied to the generalized Nizhnik–Novikov–Veselov (GNNV) equation and its corresponding bilinear formula, bilinear Bäcklund transformation, Lax pair and infinite conservation laws are obtained directly and naturally. It is known that the Hirota bilinear method has been proved valid and powerful in soliton theory. Except for constructing N-soliton solution, based on Hirota's bilinear equations, other underlying integrability including Lax pair, conservation laws, etc. may be obtained. But, there is, no general rule for selection of the Hirota dependent variables, nor for the choice of some essential formulas especially the "exchange formulas". The binary Bell polynomials method has filled this gap. It leads a simple and direct way from combinatorial aspects of logarithmically linearizable nonlinear partial differential equations to the characterization of fundamental hierarchies of soliton equations. From the application of this method to the GNNV equation, one can see that it takes a great advantage of reformulating the Hirota operators in terms of $\mathcal{Y}$- and P-polynomials. The pivotal spectral formulations can be derived straight away from the original equation itself step by step, without relying on clever guesswork. The more difficult problem about the conservation laws of (2+1)-dimensional nonlinear partial equation has also been solved by this method.

Finally, note that the use of the binary Bell polynomials method to investigate kinds of integrability of nonlinear system is novel and explanatory. As yet, it only connects the Bell polynomials with Hirota's bilinear operator by logarithmic transformation. In order to extend this method to more soliton equations, the search for other transformations is worth to be investigated later.

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