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Prolongation structure of the variable coefficient KdV equation*

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The prolongation structure methodologies of Wahlquist–Estabrook [Wahlquist H D and Estabrook F B 1975 *J. Math. Phys.* **16** 1] for nonlinear differential equations are applied to a variable-coefficient KdV equation. Based on the obtained prolongation structure, a Lie algebra with five parameters is constructed. Under certain conditions, a Lie algebra representation and three kinds of Lax pairs for the variable- coefficient KdV equation are derived.

Keywords: prolongation structure, variable-coefficient KdV equation, Lax pairs
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1. Introduction

In 1971, Wahlquist and Estabrook (WE) proposed a prolongation method for nonlinear evolution equations and applied it to the KdV equation in their famous article,[1] in which the concept of pseudopotentials was introduced and the Bäcklund transformation of the KdV equation was obtained. Their preeminent work provides a general theoretical method for determining whether an equation can be associated with a Lax pair, in which a nonlinear evolution equation is formulated in terms of the ideals of Cartan's exterior differential forms.[2] It also showed that the prolongation structure can be determined by a set of Lie algebra relations. In 1976, Hermann[3] interpreted the prolongation structure as a connection. In 1984, Dodd and Fordy[4,5] studied the prolongation structure of quasi-polynomial and complex quasi-polynomial evolution equations, and provided a method for obtaining the matrix representations of the Lie algebra. They applied this method to a series of nonlinear differential equations and obtained their Lax pairs.

During the past three decades, prolongation structure has been investigated from a geometric point of view and it has been shown that it is related to a variety of the solution-method techniques for nonlinear differential equations, such as Lax pairs, the Bäcklund transformation, conservation law, the Miura transformation, Hamilton structure, the inverse scattering problem and so on.[6,7]

The study of variable-coefficient nonlinear partial differential equations in integral systems is today's challenge. In particular, ever since the variable coefficient KdV (vcKdV) equation was originally proposed by Grimshaw[8] in 1979, there have been many excellent works investigating vcKdV.[9−12] For example, in the special case of $f(t) = \alpha t^n$ and $g(t) = \beta t^m$, based on the Weiss–Tabor–Carnevale (WTC) method[13] and the classical Lie group method, the auto-Bäcklund transformation, the Painlevé property and the similarity reductions of the vcKdV equation were independently investigated by Nirmala *et al.*,[9]. The infinite conservation laws of the vcKdV equation and the variable-coefficient mKdV equation were obtained by Lou and Ruan.[10] Fan[11] obtained the Bäcklund transformation and the similarity reductions of the general vcKdV equation. Recently, authors have obtained an exact solution of vcKdV.[14−16]

Based on the prolongation structure methodologies of WE,[1] we study the prolongation structure of the vcKdV equation. The Lax pairs of vcKdV, in three cases, are constructed successfully. In the first case, we successfully reproduce the eight-parameter Lie algebra of KdV obtained by WE.[1] In the third case, we find that the constraint condition is the exact condition in which the vcKdV equation possesses the Painlevé property[17] The paper is organised as follows: In Sec-

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tion 2, we give the differential forms of the vcKdV equation. In Section 3, we obtain the prolongation structure of the vcKdV equation and give the Lie algebra relations associated with its structure. In Section 4, we give the Lax pairs and pseudopotentials of thevcKdV equation. Finally, we give a brief summary and discussion.

2. The differential forms of the vcKdV equation

We consider the following vcKdV equation:[8]

$$u_t + f(t)uu_x + g(t)u_{xxx} = 0, \tag{1}$$

in which the subscript indicates partial differentiation with regard to some variable. In order to express this equation in differential forms, we define the independent variables as

$$z = u_x, \quad p = z_x = u_{xx}. \tag{2}$$

The vcKdV equation can then be expressed in the following set of 2-forms

$$\begin{aligned}\alpha_1 &= \mathrm{d}u \wedge \mathrm{d}t - z\mathrm{d}x \wedge \mathrm{d}t,\\ \alpha_2 &= \mathrm{d}z \wedge \mathrm{d}t - p\mathrm{d}x \wedge \mathrm{d}t,\\ \alpha_3 &= -\mathrm{d}u \wedge \mathrm{d}x + g(t)\mathrm{d}p \wedge \mathrm{d}t + f(t)uz\mathrm{d}x \wedge \mathrm{d}t.\end{aligned} \tag{3}$$

We denote the ideal generated by the $\alpha_i (i = 1, 2, 3)$ in the exterior algebra with basis $\{\mathrm{d}x, \mathrm{d}t, \mathrm{d}u, \mathrm{d}z, \mathrm{d}p\}$ by $I = I(\alpha_1, \alpha_2, \alpha_3)$. In order to ensure complete equivalence between the forms (3) and the vcKdV equation (1), the ideal I must be 'closed', i.e., $\mathrm{d}I \subset I$. By direct calculation, we can easily find

$$\begin{aligned}\mathrm{d}\alpha_1 &= -\mathrm{d}x \wedge \mathrm{d}t \wedge \mathrm{d}z = \mathrm{d}x \wedge \alpha_2,\\ \mathrm{d}\alpha_2 &= -\mathrm{d}x \wedge \mathrm{d}t \wedge \mathrm{d}p = \frac{1}{g(t)}\mathrm{d}x \wedge \alpha_3,\\ \mathrm{d}\alpha_3 &= f(t)(z\mathrm{d}x \wedge \mathrm{d}t \wedge \mathrm{d}u + u\mathrm{d}x \wedge \mathrm{d}t \wedge \mathrm{d}z)\\ &= -f(t)\mathrm{d}x \wedge (z\alpha_1 + u\alpha_2).\end{aligned} \tag{4}$$

Therefore, we can apply Cartan's theory in Ref. [12].

3. Prolongation structure of the vcKdV equation

According to the prolongation method of WE,[1] we add new 1-forms

$$\omega = \mathrm{d}y + F(t, u, z, p, y)\mathrm{d}x + G(t, u, z, p, y)\mathrm{d}t, \tag{5}$$

in which y is a pseudopotential. In the case of the KdV equation, F and G are restricted to implicit functions of the independent variables x and t; however, here we demand that F and G be explicit functions of t. Then, the exterior derivatives of ω are required to lie in the 'augmented' ideal of forms $I' = \{I, \omega\}$

$$\mathrm{d}\omega = \sum_{i=1}^{3} f_i\alpha_i + \eta \wedge \omega. \tag{6}$$

From Eq. (6) we obtain the following system of overdetermined differential equations:

$$\begin{aligned}&F_{\mathrm{p}} = 0, \quad F_z = 0, \quad gF_u + G_{\mathrm{p}} = 0,\\ &uzfF_u + pG_z + zG_u - F_t + GF_y - FG_y = 0,\end{aligned} \tag{7}$$

where $GF_y - FG_y$ can be denoted by $[F, G]$, then, Eq. (7) gives rise to a Lie algebra for determination. For simplicity, we omit the concrete process, the reader can refer to Ref. [1] for details. With knowledge of Lie algebra, we find that F and G can be expressed as follows:

$$\begin{aligned}F &= \frac{1}{2}u^2X_3 + uX_2 + X_1,\\ G &= -gpuX_3 - gpX_2 + \frac{1}{2}gz^2X_3 - \frac{1}{2}gzu^2[X_2, X_3]\\ &\quad - gzu[X_1, X_3] - gz[X_1, X_2]\\ &\quad - \frac{1}{6}gu^3[X_3, [X_1, X_2]] - \frac{1}{3}fu^3X_3 - \frac{1}{2}fu^2X_2\\ &\quad - \frac{1}{2}gu^2[X_2, [X_1, X_2]]\\ &\quad - gu[X_1, [X_1, X_2]] + X_4.\end{aligned} \tag{8}$$

It is worth noting that in the prolongation structure of the KdV equation, $X[i] (i = 1, 2, 3, 4)$ depend only on prolongation variable y, but $X[i] (i = 1, 2, 3, 4)$ in Eq. (8) depend on y and t. Substituting Eq. (8) into Eq. (7) and introducing

$$[X_1, X_2] = X_5, \tag{9}$$

we obtain the following commutation relations:

$$\begin{aligned}&[X_3, [X_3, X_5]] = 0, \quad \frac{1}{4}[X_3, [X_2, X_5]] + \frac{1}{6}[X_2, [X_3, X_5]] = 0, \quad -\frac{\partial}{\partial t}X_1 - [X_1, X_4] = 0,\\ &\frac{1}{6}[X_1, [X_3, X_5]] + \frac{1}{2}[X_2, [X_2, X_5]] + \frac{1}{2}[X_3, [X_1, X_5]] = 0, \quad [X_1, X_3] = 0,\end{aligned}$$

$$\frac{1}{2}g[X_1,[X_2,X_5]]+\frac{1}{2}fX_5-\frac{1}{2}\frac{\partial}{\partial t}X_3+g[X_2,[X_1,X_5]]-\frac{1}{2}[X_3,X_4]=0,$$
$$-\frac{\partial}{\partial t}X_2+g[X_1,[X_1,X_5]]-[X_2,X_4]=0,\quad [X_2,X_3]=0. \tag{10}$$

In order to acquire the commutation relations of the functions that only depend on y, we adopt the classical method of separation of variables for X_i. By setting

$$X_i=T_iY_i, \tag{11}$$

where T_i only depends on t, and Y_i only depends on y, then commutation relations (10) and (9) are equivalent to

$$[Y_3,[Y_3,Y_5]]=0,\quad \frac{1}{4}[Y_3,[Y_2,Y_5]]+\frac{1}{6}[Y_2,[Y_3,Y_5]]=0,\quad -\frac{\partial}{\partial t}T_1Y_1-T_1T_4[Y_1,Y_4]=0,$$
$$\frac{1}{6}T_1T_3T_5[Y_1,[Y_3,Y_5]]+\frac{1}{2}T_2^2T_5[Y_2,[Y_2,Y_5]]+\frac{1}{2}T_1T_3T_5[Y_3,[Y_1,Y_5]]=0,$$
$$\frac{1}{2}gT_1T_2T_5[Y_1,[Y_2,Y_5]]+\frac{1}{2}fT_5Y_5-\frac{1}{2}\frac{\partial}{\partial t}T_3Y_3+gT_1T_2T_5[Y_2,[Y_1,Y_5]]-\frac{1}{2}T_3T_4[Y_3,Y_4]=0,$$
$$-\frac{\partial}{\partial t}T_2Y_2+gT_1^2T_5[Y_1,[Y_1,Y_5]]-T_2T_4[Y_2,Y_4]=0,\quad [Y_1,Y_3]=0,\quad [Y_2,Y_3]=0,\quad T_1T_2[Y_1,Y_2]=T_5Y_5. \tag{12}$$

In order to obtain the ordinary commutation relations of Y_i, we investigate whether the coefficients of Lie-brackets and the Y_i of every relation in Eq. (12) should be equal or proportional, to what or to each other; a solution of T_i and A_i is then obtained

$$T_1=T_1,\quad T_2=\frac{f}{gT_1},\quad T_3=\frac{f^2}{g^2T_1^3},\quad T_4=gT_1^3,\quad T_5=\frac{f}{g} \tag{13}$$

and the relations (12) turn into

$$[Y_3,[Y_3,Y_5]]=0,\quad \frac{1}{4}[Y_3,[Y_2,Y_5]]+\frac{1}{6}[Y_2,[Y_3,Y_5]]=0,\quad [Y_1,Y_2]=Y_5,$$
$$-\frac{\partial}{\partial t}T_1Y_1-T_1^4g[Y_1,Y_4]=0,\quad \frac{1}{6}[Y_1,[Y_3,Y_5]]+\frac{1}{2}[Y_2,[Y_2,Y_5]]+\frac{1}{2}[Y_3,[Y_1,Y_5]]=0,$$
$$-\frac{\partial}{\partial t}T_2Y_2+fT_1^2([Y_1,[Y_1,Y_5]]-[Y_2,Y_4])=0,\quad [Y_1,Y_3]=0,\quad [Y_2,Y_3]=0,$$
$$\frac{1}{2}\frac{f^2}{g}([Y_1,[Y_2,Y_5]]+Y_5+2[Y_2,[Y_1,Y_5]]-[Y_3,Y_4])-\frac{1}{2}\frac{\partial}{\partial t}T_3Y_3=0. \tag{14}$$

Moreover, we demand that

$$\frac{\partial}{\partial t}T_1=aT_1^4g,\quad \frac{\partial}{\partial t}T_2=bfT_1^2,\quad \frac{\partial}{\partial t}T_3=c\frac{f^2}{g}, \tag{15}$$

then, the ordinary commutation relations of Y_i can be derived

$$[Y_3,[Y_3,Y_5]]=0,\quad \frac{1}{4}[Y_3,[Y_2,Y_5]]+\frac{1}{6}[Y_2,[Y_3,Y_5]]=0,\quad -aY_1-[Y_1,Y_4]=0,\quad [Y_1,Y_2]=Y_5,$$
$$\frac{1}{6}[Y_1,[Y_3,Y_5]]+\frac{1}{2}[Y_2,[Y_2,Y_5]]+\frac{1}{2}[Y_3,[Y_1,Y_5]]=0,\quad -bY_2+([Y_1,[Y_1,Y_5]]-[Y_2,Y_4])=0,$$
$$\frac{1}{2}([Y_1,[Y_2,Y_5]]+Y_5+2[Y_2,[Y_1,Y_5]]-[Y_3,Y_4])-cY_3=0,\quad [Y_1,Y_3]=0,\quad [Y_2,Y_3]=0. \tag{16}$$

This structure can be regarded as an open Lie structure in which Y_i is a generator, and there may be infinite numbers of prolongation generators. However, for this Lie structure we force it to close by demanding that every Lie-bracket of two generators is the linear combination of all the generators. Therefore, based on the relation (16) and the Jacobi identity, together with Eqs. (15) and (13), we have the following three solutions:

case 1 $a=0, b=0, c=0$

In this case, the solution of T_1 and the constraint condition of f and g are

$$g=Cf,\quad T_1=C1, \tag{17}$$

in which C and $C1$ are constants. This implies that F is independent of t and that the closed five-parameter Lie algebra is

$$
\begin{aligned}
&[Y_1, Y_2] = Y_5, \quad [Y_1, Y_3] = 0, \quad [Y_1, Y_4] = 0, \quad [Y_1, Y_5] = \frac{1}{3}Y_1 + \lambda Y_2, \quad [Y_2, Y_3] = 0, \\
&[Y_2, Y_4] = \lambda Y_5, \quad [Y_2, Y_5] = -\frac{1}{3}Y_2, \quad [Y_3, Y_4] = 0, \quad [Y_3, Y_5] = 0, \quad [Y_4, Y_5] = -\frac{1}{3}\lambda Y_1 - \lambda^2 Y_2.
\end{aligned} \tag{18}
$$

If we set $f = 12, g = 1, Y_5 = -Y_7$ in Eqs. (16) and introduce the following relations

$$
[Y_1, Y_7] = Y_5, \quad [Y_2, Y_7] = Y_6, \quad [Y_3, Y_4] = Y_8, \tag{19}
$$

and force the Lie structure, we can successfully recover the eight-parameter Lie algebra of KdV obtained by WE in Ref. [1].

case 2 $a = 1, b = -3, c = -1$

In this case the solution of T_1 and the constraint condition of f and g are

$$
g = Cf, \quad T_1^3 = -\frac{1}{3\int g\,\mathrm{d}t + C1}, \tag{20}
$$

in which C and $C1$ are constants, and the closed five-parameter Lie algebra is

$$
\begin{aligned}
&[Y_1, Y_2] = Y_5, \quad [Y_1, Y_3] = 0, \quad [Y_1, Y_4] = -Y_1, \quad [Y_1, Y_5] = \frac{1}{3}Y_1, \quad [Y_2, Y_3] = 0, \\
&[Y_2, Y_4] = Y_2, \quad [Y_2, Y_5] = -\frac{1}{3}Y_2, \quad [Y_3, Y_4] = 3Y_3, \quad [Y_3, Y_5] = 0, \quad [Y_4, Y_5] = 0.
\end{aligned} \tag{21}
$$

case 3 $a = 2, b = 1, c = 0$

In this case, we obtain that T_1 is the solution of the following equation

$$
T_1^3 = -\frac{1}{3}\frac{C2f^2}{g^2} \tag{22}
$$

and the constraint condition of f and g is

$$
g(t) = (C1 + C2\int f(t)\mathrm{d}t)f(t), \tag{23}
$$

where $C1$ and $C2$ are integration constants. The closed five-parameter Lie algebra is

$$
\begin{aligned}
&[Y_1, Y_2] = Y_5, \quad [Y_1, Y_3] = 0, \quad [Y_1, Y_4] = -2Y_1, \quad [Y_1, Y_5] = 0, \quad [Y_2, Y_3] = 0, \\
&[Y_2, Y_4] = -Y_2, \quad [Y_2, Y_5] = 0, \quad [Y_3, Y_4] = Y_5, \quad [Y_3, Y_5] = 0, \quad [Y_4, Y_5] = 3Y_5.
\end{aligned} \tag{24}
$$

What is noteworthy is that constraint condition (23) is the exact condition in which the vcKdV equation possesses the Painlevé property.[17]

4. Lax pairs and Pfaffian forms of the vcKdV equation

If the Lie structure can be expressed in matrix forms, the Lax pairs of the vcKdV equation can be obtained. In the above situation, we give the matric representation.

case 1 $a = 0, b = 0, c = 0$

The matric representations of Eq. (18) are

$$
Y_1 = \begin{bmatrix} 0 & \frac{\lambda}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}, \quad Y_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad Y_4 = \begin{bmatrix} 0 & -\frac{\lambda^2}{2} \\ -\frac{\lambda}{2} & 0 \end{bmatrix}, \quad Y_5 = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & -\frac{1}{6} \end{bmatrix}, \tag{25}
$$

and the F and G are

$$
F = \begin{bmatrix} 0 & \lambda - \frac{f}{6g}u \\ 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} -\frac{1}{6}fz & \frac{1}{6}fp + \frac{1}{18}\frac{f^2}{g}u^2 + \frac{1}{3}\lambda fu - 4\lambda^2 g \\ -\frac{1}{3}fu - 4\lambda g & \frac{1}{6}fz \end{bmatrix}. \tag{26}
$$

Here, if we set $f=12, g=1$, then we can obtain the Lax pair of the KdV equation.[18]

case 2 $a=1, b=-3, c=-1$

The matric representations of Eq. (21) are

$$Y_1=\begin{bmatrix}1 & -\dfrac{1}{\lambda}\\ \lambda & -1\end{bmatrix},\quad Y_2=\begin{bmatrix}0 & 0\\ \dfrac{\lambda}{6} & 0\end{bmatrix},\quad Y_3=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},\quad Y_4=\begin{bmatrix}0 & -\dfrac{\lambda^2}{2}\\ \lambda & -1\end{bmatrix},\quad Y_5=\begin{bmatrix}-\dfrac{1}{6} & 0\\ -\dfrac{\lambda}{3} & \dfrac{1}{6}\end{bmatrix}, \tag{27}$$

and the F and G are

$$\begin{aligned}F&=\begin{bmatrix}T_1 & -\dfrac{T_1}{\lambda}\\ \dfrac{f}{6gT_1}\lambda u+\lambda T_1 & -T_1\end{bmatrix},\\ G&=\begin{bmatrix}\dfrac{1}{6}fz-\dfrac{1}{3}T_1fu & \dfrac{T_1f}{3\lambda}u\\ -\dfrac{1}{6T_1}\lambda fp+\dfrac{1}{3}f\lambda z-\dfrac{f^2\lambda}{18gT_1}u^2-\dfrac{1}{3}\lambda fT_1u+\lambda gT_1^3 & -\dfrac{1}{6}fz+\dfrac{1}{3}T_1fu-gT_1^3\end{bmatrix},\end{aligned} \tag{28}$$

where T_1 is the solution of Eq. (20).

case 3 $a=2, b=1, c=0$

The matric representations of Eq. (24) are

$$Y_1=\begin{bmatrix}\lambda & -\dfrac{\lambda^2}{2}\\ 2 & -\lambda\end{bmatrix},\quad Y_2=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},\quad Y_3=\begin{bmatrix}\dfrac{2}{3} & 0\\ 0 & \dfrac{2}{3}\end{bmatrix},\quad Y_4=\begin{bmatrix}-1 & \lambda\\ 0 & 1\end{bmatrix},\quad Y_5=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \tag{29}$$

and the F and G are

$$\begin{aligned}F&=\begin{bmatrix}-\dfrac{u^2}{C2}+\lambda T_1 & -\dfrac{1}{2}\lambda^2T_1\\ 2T_1 & -\dfrac{u^2}{C2}-\lambda T_1\end{bmatrix},\\ G&=\begin{bmatrix}\dfrac{1}{3}\dfrac{6g^2pu-3g^2z^2+2fgu^3+C2^2f^2}{C2g} & -\dfrac{1}{3}\dfrac{\lambda C2f^2}{g}\\ 0 & \dfrac{1}{3}\dfrac{6g^2pu-3g^2z^2+2fgu^3-C2^2f^2}{C2g}\end{bmatrix},\end{aligned} \tag{30}$$

where T_1 is the solution of Eq. (22).

We can also give another representation for this Lie algebra. If we define $\partial_{y_i}(i=1,\ldots,5)$ as the basis vector, where y_i is a set of coordinates in the space of prolongation variables. Then the generators under relations (24) can be expressed as follows:

$$\begin{aligned}Y_1&=3y_2\partial_{y_3}-y_2\partial_{y_5}+\partial_{y_1},\quad Y_2=\partial_{y_2},\quad Y_3=\partial_{y_3},\\ Y_4&=-2y_1\partial_{y_1}-y_2\partial_{y_2}-3y_3\partial_{y_3}+y_3\partial_{y_5}+\partial_{y_4},\quad Y_5=-3\partial_{y_3}+\partial_{y_5}.\end{aligned} \tag{31}$$

Using this result, we can obtain five Pfaffian forms:

$$\begin{aligned}\omega_1&=\mathrm{d}y_1+T_1\,\mathrm{d}x+\frac{2}{3}\frac{bf^2y_1}{g}\,\mathrm{d}t,\\ \omega_2&=\mathrm{d}y_2+\frac{fu}{T_1g}\,\mathrm{d}x+\frac{1}{6}\frac{f(2fbT_1y_2-3u^2f-6pg)}{T_1g}\,\mathrm{d}t,\\ \omega_3&=\mathrm{d}y_3+\frac{3}{2}\frac{(-u^2+2bT_1y_2)}{b}\,\mathrm{d}x+\frac{1}{2}\frac{2f^2b^2y_3+2fgu^3+6bfgz-3g^2z^2+6g^2up}{bg}\,\mathrm{d}t,\\ \omega_4&=\mathrm{d}y_4-\frac{1}{3}\frac{bf^2}{g}\,\mathrm{d}t,\quad \omega_5=\mathrm{d}y_5-T_1y_2\,\mathrm{d}x-\frac{1}{3}\frac{f(bfy_3+3gz)}{g}\,\mathrm{d}t.\end{aligned} \tag{32}$$

Remark In 1983, Weiss[19] introduced the concept of the singularity manifold from the Painlevé property of the partial differential equation. In 1988, Nucci[20] found that one can easily obtain both the Lax equation and the auto-Bäcklund transformation for the nonlinear evolution from the Riccati pseudopential. Soon afterwards, he showed how to obtain the singularity manifold equation from the Riccati pseudopential. However, the relationship of singularity and non-Riccati pseudopentials is not clear. If we can obtain the singularity manifold equation from the prolongation of vcKdV under the third condition given above, and from the Painlevé property of vcKdV,[9] we believe that they should be equivalent.

5. Summary and discussion

We have derived the Lax pairs of the vcKdV equation under three conditions. In particular, we obtained the Lax pair of the KdV equation from vcKdV under the first condition and the third condition where the vcKdV equation possesses the Painlevé property. We know that a nonlinear equation with a nontrivial prolongation structure may construct the Lax pairs; from which one can find infinitly many higher symmetries, recursion operators and bi-Hamiltonian structures. However, it is difficult to determine whether a nonlinear equation can be associated with a Lax pair and, in particular, to the variable coefficient equation. Our results show that the construction of the Lax pairs of the vcKdV equation is very different from the Lax pairs of the KdV equation. This shows that is more difficult to investigate the integrability of the variable coefficient equation than the constant coefficient equation. Further work is required on how to derive the explicit exact solutions using the vcKdV to obtain Lax pairs. A further challenge is how to find out the precise relationship between the Lax pairs of the the variable coefficient equation and the constant coefficient equation.

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