Full Symmetry Groups and Similar Reductions of a (2+1)-Dimensional Resonant Davey–Stewartson System*

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Abstract *Applying the classical Lie symmetry method to the (2+1)-dimensional resonant Davey–Stewartson system introduced by Tang [X.Y. Tang et al., Chaos, Solitons and Fractals* **42** *(2007) 2707], a more general infinite dimensional Lie symmetry with Kac-Moody-Virasoro type Lie algebra is obtained, which involves four arbitrary functions of t. Alternatively, by a simple direct method, the full symmetry groups including Lie symmetry group and non-Lie symmetry group are gained straightly. In this way, the related Lie algebra can be easily found by a more simple limiting procedure. Lastly, via solving the characteristic equations, three types of the general similar reductions are derived.*

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1 Introduction

It is well known that the nonlinear Schrödinger (NLS) equation plays an important role in many areas of the physical sciences.[1−3] The standard (1+1)-dimensional NLS equation, reading

$$\mathrm{i}\psi_t = -\frac{1}{2}\psi_{xx} + \delta|\psi|^2\psi\,, \tag{1}$$

is a completely integral nonlinear equation and its exact solutions can be obtained by the inverse scattering method. Equation (1) has a (2+1)-dimensional extension named Davey–Stewartson (DS) equation,[4−9] i.e.

$$\mathrm{i}\psi_t + \frac{1}{2}(\psi_{xx}+\psi_{yy}) + \delta|\psi|^2\psi - Q\psi = 0\,, \tag{2}$$

$$Q_{xx} - Q_{yy} - 2\alpha(|\psi|^2)_{xx} = 0\,. \tag{3}$$

The DS equation is also an integral model and has two-dimensional dromion solution. Due to its nice analytical properties, it has been widely used in hydrodynamics, plasma physics, and nonlinear optics, etc.

Recently, a variant of the NLS equation named the resonant nonlinear Schrödinger (RNLS) equation aries in the context of the propagation of long magneto-acoustic waves in cold, collisionless plasma,[10] in the theory of Madelung fluids,[11] and in capillarity theory.[12] The RNLS equation reads

$$\mathrm{i}\psi_t + s^2\psi_{xx} - \alpha|\psi|^2\psi - \Big(\frac{2s^2|\psi|_{xx}}{|\psi|}\Big)\psi = 0\,. \tag{4}$$

Equation (4) has two nonlinear terms, including the conventional cubic nonlinearity in Eq. (1) and an additional nonlinear term involving the modulus of the wave envelope. Similar to the NLS equation, a natural (2+1)-dimensional extension of the RNLS equation was introduced[13] as

$$\mathrm{i}\psi_t + s^2\psi_{xx} + \psi_{yy} - 2s^2\Big(\frac{|\psi|_{xx}}{|\psi|} + \frac{s^2|\psi|_{yy}}{|\psi|}\Big)\psi - Q\psi + \alpha|\psi|^2\psi = 0\,, \tag{5}$$

$$Q_{xx} - s^2Q_{yy} - 2\alpha(|\psi|^2)_{xx} = 0\,. \tag{6}$$

Here, ψ denotes the complex wave envelope, and Q is a real quantity, which can be regarded as a forcing term in Eq. (5). $s^2 = 1$ and $s^2 = -1$ dictate the hyperbolic and elliptic nature, respectively of the governing Eq. (6) for this term, which is the same to the situation for the classical DS equation. The pair of Eqs. (5) and (6) is called the resonant Davey–Stewartson (RDS) system.

For the RDS system, Tang *et al.*[13−14] investigated it by the multi-linear variable separation approach and the Hirota bilinear method, and a class of exact solutions were obtained. In Ref. [14], it was also proved that the RDS system can pass the Painlevé test. Then by the classical Lie symmetry method, Gao and Tang[15] obtained an infinite-dimensional Lie algebra including one arbitrary function of t of the RDS system and gave four types of the two-dimensional reductions.

However, the symmetry obtained in Ref. [15] is not general and the symmetry group is absent. Hence, in this paper, we reinvestigate the symmetry property of the RDS system. Thanks to the famous first fundamental theorem

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of Lie, the symmetry group can be obtained via the corresponding Lie algebra. But in this way, for the Lie symmetry group, the final known expressions may be quite complicated and difficult for real applications. For this reason, Lou and Ma[16] introduced a simple direct method to solve this problem, by which one can straightforwardly get the symmetry group and the related Lie algebra can be obtained only by a more simple limiting procedure. Furthermore, in comparison with the classical Lie symmetry method, by the simple direct method, we can get not only the Lie symmetry group but also the non-Lie symmetry group.

The paper is arranged as follows: In Sec. 2, by the classical Lie symmetry approach, we get the general Lie symmetry including four arbitrary functions of t. This infinite dimensional Lie algebra possesses the Kac-Moody-Virasoro algebra structure like lots of (2+1)-dimensional integral models. In Sec. 3, we apply the simple direct method to get the transformation group. Both Lie symmetry group and non-Lie symmetry group are attained. In Sec. 4, by solving the characteristic equations, three types of the general similar reductions are presented. Section 5 is the conclusion.

2 Lie Point Symmetry by Classical Symmetry Method

Firstly, the similar decompositions

$$\psi = \mathrm{e}^{R-\mathrm{i}S}, \quad R = \frac{1}{2}\ln(-uv), \quad S = \frac{1}{2}\ln\Big(-\frac{u}{v}\Big), \tag{7}$$

which were used by Tang *et al.*,[13–15] are applied to transform the complex RDS system (5) and (6) into real equations, i.e.

$$u_t - s^2 u_{xx} - u_{yy} - uQ - \alpha u^2 v = 0\,, \tag{8}$$

$$v_t + s^2 v_{xx} + v_{yy} + vQ + \alpha u v^2 = 0\,, \tag{9}$$

$$Q_{xx} - s^2 Q_{yy} + 2\alpha(uv)_{xx} = 0\,. \tag{10}$$

Here, u and v are real quantities.

To Eqs. (8)–(10), by applying the classical Lie symmetry method, we consider the one-parameter group of infinitesimal transformations in (x,y,t,u,v,Q) given by

$$\begin{aligned} x^* &= x + \epsilon X(x,y,t,u,v,Q) + o(\epsilon^2)\,,\\ y^* &= y + \epsilon Y(x,y,t,u,v,Q) + o(\epsilon^2)\,,\\ t^* &= t + \epsilon T(x,y,t,u,v,Q) + o(\epsilon^2)\,,\\ u^* &= u + \epsilon \Phi_1(x,y,t,u,v,Q) + o(\epsilon^2)\,,\\ v^* &= v + \epsilon \Phi_2(x,y,t,u,v,Q) + o(\epsilon^2)\,,\\ Q^* &= Q + \epsilon \Phi_3(x,y,t,u,v,Q) + o(\epsilon^2)\,, \end{aligned} \tag{11}$$

where ϵ is group parameter. It is required that Eqs. (8)–(10) be invariant under the transformations (11). Accordingly, the Lie point symmetries have the form

$$\sigma_u = Xu_x + Yu_y + Tu_t - \Phi_1\,, \tag{12}$$

$$\sigma_v = Xv_x + Yv_y + Tv_t - \Phi_2\,, \tag{13}$$

$$\sigma_Q = XQ_x + YQ_y + TQ_t - \Phi_3\,. \tag{14}$$

To keep the invariance of Eqs. (8)–(10), σ_u, σ_v, and σ_Q should satisfy

$$\frac{\partial}{\partial t}\sigma_u - s^2\frac{\partial^2}{\partial x^2}\sigma_u - \frac{\partial^2}{\partial y^2}\sigma_u - \sigma_u(Q+2\alpha uv) - \alpha u^2\sigma_v - u\sigma_Q = 0\,, \tag{15}$$

$$\frac{\partial}{\partial t}\sigma_v - s^2\frac{\partial^2}{\partial x^2}\sigma_v - \frac{\partial^2}{\partial y^2}\sigma_v + \sigma_v(Q+2\alpha uv) + \alpha v^2\sigma_u + v\sigma_Q = 0\,, \tag{16}$$

$$\frac{\partial}{\partial x^2}\sigma_Q - s^2\frac{\partial^2}{\partial y^2}\sigma_Q + 2\alpha\frac{\partial^2}{\partial x^2}(v\sigma_u + u\sigma_v) = 0\,. \tag{17}$$

Taking Eqs. (12)–(14) into Eqs. (15)–(17), eliminating the terms u_{yy}, v_{yy}, Q_{xx} and their higher-order derivatives through Eqs. (8)–(10), then collecting together the coefficients of the dependent variables and their partial derivatives, and setting all of them to zero, this yields a system of overdetermined, linear equations for the infinitesimals X, Y, T, Φ_1, Φ_2, and Φ_3. By solving these equations, one can get

$$\begin{aligned} &X = \frac{1}{2}\dot{k}x + f, \quad Y = \frac{1}{2}\dot{k}y + g, \quad T = k, \quad \Phi_1 = \Big[-\frac{1}{8}\ddot{k}\Big(\frac{1}{s^2}x^2 + y^2\Big) - \frac{\dot{f}}{2s^2}x - \frac{1}{2}\dot{g}y + h\Big]u\,,\\ &\Phi_2 = \Big[\frac{1}{8}\ddot{k}\Big(\frac{1}{s^2}x^2 + y^2\Big) + \frac{\dot{f}}{2s^2}x + \frac{1}{2}\dot{g}y - h - \dot{k}\Big]v, \quad \Phi_3 = -\frac{1}{8}\dddot{k}\Big(\frac{1}{s^2}x^2 + y^2\Big) - \frac{\ddot{f}}{2s^2}x - \frac{1}{2}\ddot{g}y + \dot{h} + \frac{1}{2}\ddot{k} - \dot{k}Q\,, \end{aligned} \tag{18}$$

where $\{f,g,h,k\}$ are arbitrary functions of t and the dots indicate derivatives with respect to t. One can see that the results obtained in Ref. [15] are only special cases of (18) for choosing $k = (C_1/2)t^2 + C_2 t + C_3$, $h = C_4 t + C_5$, and $g = C_6 t + C_7$.

For Eqs. (8)–(10), the Lie point symmetries have the forms

$$\sigma \equiv \sigma_1(k) + \sigma_2(f) + \sigma_3(g) + \sigma_4(h) \equiv \begin{pmatrix} \frac{1}{2}\dot{k}(xu_x + yu_y) + ku_t + \frac{1}{8}\ddot{k}(\frac{1}{s^2}x^2 + y^2)u \\ \frac{1}{2}\dot{k}(xv_x + yv_y) + kv_t - \frac{1}{8}\ddot{k}(\frac{1}{s^2}x^2 + y^2)v + \dot{k}v \\ \frac{1}{2}\dot{k}(xQ_x + yQ_y) + kQ_t + \dot{k}Q + \frac{1}{8}\dddot{k}(\frac{1}{s^2}x^2 + y^2) - \frac{1}{2}\ddot{k} \end{pmatrix}$$

$$+ \begin{pmatrix} fu_x + \frac{1}{2s^2}\dot{f}xu \\ fv_x - \frac{1}{2s^2}\dot{f}xv \\ fQ_x + \frac{1}{2s^2}\ddot{f}x \end{pmatrix} + \begin{pmatrix} gu_y + \frac{1}{2}\dot{g}yu \\ gv_y - \frac{1}{2}\dot{g}yv \\ gQ_y + \frac{1}{2}\ddot{g}y \end{pmatrix} + \begin{pmatrix} -hu \\ hv \\ -\dot{h} \end{pmatrix}.$$

The nonzero commutators among $\sigma_1(k)$, $\sigma_2(f)$, $\sigma_3(g)$, and $\sigma_4(h)$ are

$$[\sigma_1(k_1),\sigma_1(k_2)]=\sigma_1(k_1\dot{k}_2-\dot{k}_1k_2),\quad [\sigma_2(f_1),\sigma_2(f_2)]=-\frac{1}{2s^2}\sigma_4(f_1\dot{f}_2-f_2\dot{f}_1)\,,\quad [\sigma_3(g_1),\sigma_3(g_2)]=-\frac{1}{2}\sigma_4(g_1\dot{g}_2-g_2\dot{g}_1),$$

$$[\sigma_1(k),\sigma_2(f)]=\sigma_2\Big(k\dot{f}-\frac{1}{2}f\dot{k}\Big)\,,\quad [\sigma_1(k),\sigma_3(g)]=\sigma_3\Big(k\dot{g}-\frac{1}{2}g\dot{k}\Big),\quad [\sigma_1(k),\sigma_4(h)]=\sigma_4(k\dot{h})\,.$$

Hence, $\sigma_1-\sigma_4$ constitute an infinite-dimensional closed Kac-Moody-Virasoro type Lie symmetry algebra.

To further obtain the related Lie symmetry group $G:(x,y,t,u,v,Q)\to(x',y',t',u',v',Q')$, one can adopt the first fundamental theorem of Lie as a powerful method. That is to say, we need to solve the following system of ordinary differential equations with initial value problem

$$\frac{\mathrm{d}x'}{X'}=\frac{\mathrm{d}y'}{Y'}=\frac{\mathrm{d}t'}{T'}=\frac{\mathrm{d}u'}{\Phi_1'}=\frac{\mathrm{d}v'}{\Phi_2'}=\frac{\mathrm{d}Q'}{\Phi_3'}=\mathrm{d}\epsilon,\quad (x',y',t',u',v',Q')|_{\epsilon=0}=(x,y,t,u,v,Q)\,. \tag{19}$$

Here $\{X',Y',T',\Phi_1',\Phi_2',\Phi_3'\}$ are obtained by insteading $\{x,y,t,u,v,Q\}$ in $\{X,Y,T,\Phi_1,\Phi_2,\Phi_3\}$ by $\{x',y',t',u',v',Q'\}$. However, for the existence of arbitrary functions of t, it is not easy to solve Eqs. (19) and even Eqs. (19) were solved, the final expressions may be quite complicated. So, in the following section, we will apply another method−a simple direct method to compute the symmetry group of Eqs. (8)–(10) straightly.

3 Full Symmetry Group by a Simple Direct Method

In Ref. [17], Clarkson and Kruskal (CK) introduced a direct method to derive symmetry reductions of a nonlinear system without using any group theory. For many types of nonlinear systems the method can be used to find all possible similarity reductions. Then Lou and Ma were inspired to modify the C-K direct method to find the generalized Lie and non-Lie symmetry groups of both integral and non-integral nonlinear differential equations. Here, we take this simple direct method to investigate Eqs. (8)–(10).

After finishing some quite tedious calculations, one can take the simplified symmetry transformation ansatz as

$$u=\alpha_1+\beta_1U(\xi,\eta,\tau),\quad v=\alpha_2+\beta_2V(\xi,\eta,\tau),\quad Q=\alpha_3+\beta_3q(\xi,\eta,\tau)\,, \tag{20}$$

where α_1, α_2, α_3, β_1, β_2, β_3, and ξ,η,τ are functions of $\{x,y,t\}$. It is required that $U(\xi,\eta,\tau)\equiv U, V(\xi,\eta,\tau)\equiv V$ and $q(\xi,\eta,\tau)\equiv q$ satisfy the same Eqs. (8)–(10) but with new independent variables ξ,η,τ.

Substituting (20) into Eqs. (8)–(10), eliminating $U_\tau, V_\tau, q_{\xi\xi}$ and their higher-order derivatives, then setting the coefficients of the polynomials of U, V, q and their derivatives to be zero, by solving these equations we can get the results

$$\xi=\delta_2\tau_{0t}^{1/2}x+\xi_{0t},\quad \eta=\delta_1\tau_{0t}^{1/2}y+\eta_{0t},\quad \tau=\tau_0,\quad \alpha_1=\alpha_2=0\,, \tag{21}$$

$$\alpha_3=\frac{2\tau_{0ttt}\tau_{0t}-3\tau_{0tt}^2}{16\tau_{0t}^2}\Big(\frac{x^2}{s^2}+y^2\Big)-\frac{\delta_2(\xi_{0t}\tau_{0tt}-\xi_{0tt}\tau_{0t})}{2s^2\tau_{0t}^{3/2}}x -\frac{\delta_1(\eta_{0t}\tau_{0tt}-\eta_{0tt}\tau_{0t})}{2\tau_{0t}^{3/2}}-\frac{1}{4\tau_{0t}}\Big(\frac{1}{s^2}\xi_{0t}^2+\eta_{0t}^2\Big)+\frac{f_t}{f}-\frac{\tau_{0tt}}{2\tau_{0t}}\,, \tag{22}$$

$$\beta_1=\beta_0\exp\Big(\frac{\tau_{0tt}}{8\tau_{0t}}\Big(\frac{x^2}{s^2}+y^2\Big)+\frac{1}{2\tau_{0t}^{1/2}}\Big(\frac{\delta_2\xi_{0t}}{s^2}x+\delta_1\eta_{0t}y\Big)\Big)\,, \tag{23}$$

$$\beta_2=\frac{\tau_{0t}}{\beta_0}\exp\Big(-\frac{\tau_{0tt}}{8\tau_{0t}}\Big(\frac{x^2}{s^2}+y^2\Big)-\frac{1}{2\tau_{0t}^{1/2}}\Big(\frac{\delta_2\xi_{0t}}{s^2}x+\delta_1\eta_{0t}y\Big)\Big)\,, \tag{24}$$

$$\beta_3=\tau_{0t}\,, \tag{25}$$

where $\xi_0\equiv\xi_0(t)$, $\eta_0\equiv\eta_0(t)$, $\tau_0\equiv\tau_0(t)$, and $\beta_0\equiv\beta_0(t)$ are arbitrary functions of t and

$$\delta_1^2=1,\quad \delta_2^2=1\,. \tag{26}$$

In summary, we can arrive at the following final transformation group theorem of Eqs. (8)–(10):

Theorem If $\{U(x,y,t),V(x,y,t),q(x,y,t),\}$ is a solution of Eqs. (8)–(10), so is $\{u,v,Q\}$ with

$$u=\beta_0\exp\Big(\frac{\tau_{0tt}}{8\tau_{0t}}\Big(\frac{x^2}{s^2}+y^2\Big)+\frac{1}{2\tau_{0t}^{1/2}}\Big(\frac{\delta_2\xi_{0t}}{s^2}x+\delta_1\eta_{0t}y\Big)\Big)U(\xi,\eta,\tau)\,, \tag{27}$$

$$v=\frac{\tau_{0t}}{\beta_0}\exp\Big(-\frac{\tau_{0tt}}{8\tau_{0t}}\Big(\frac{x^2}{s^2}+y^2\Big)-\frac{1}{2\tau_{0t}^{1/2}}\Big(\frac{\delta_2\xi_{0t}}{s^2}x+\delta_1\eta_{0t}y\Big)\Big)V(\xi,\eta,\tau)\,, \tag{28}$$

$$Q=\frac{2\tau_{0ttt}\tau_{0t}-3\tau_{0tt}^2}{16\tau_{0t}^2}\Big(\frac{x^2}{s^2}+y^2\Big)-\frac{\delta_2(\xi_{0t}\tau_{0tt}-\xi_{0tt}\tau_{0t})}{2s^2\tau_{0t}^{3/2}}x-\frac{\delta_1(\eta_{0t}\tau_{0tt}-\eta_{0tt}\tau_{0t})}{2\tau_{0t}^{3/2}}$$

$$-\frac{1}{4\tau_{0t}}\Big(\frac{1}{s^2}\xi_{0t}^2+\eta_{0t}^2\Big)+\frac{\beta_{0t}}{\beta_0}-\frac{\tau_{0tt}}{2\tau_{0t}}+\tau_{0t}q(\xi,\eta,\tau)\,, \tag{29}$$

with (21), where ξ_0, η_0, τ_0, and β_0 are arbitrary functions of t and the discrete value of the constants δ_1 and δ_2 are given by (26).

From above symmetry group theorem, one can see that for Eqs. (8)–(10), the full symmetry group $\mathcal{G}$ is divided into four sectors which correspond to

$$\delta_1=1,\quad \delta_2=1,\quad \delta_1=1,\quad \delta_2=-1,\quad \delta_1=-1,\quad \delta_2=1,\quad \delta_1=-1,\quad \delta_2=-1$$

of the theorem, respectively. In other words, the full symmetry group $\mathcal{G}$ can be considered as a product of the discrete group $\mathcal{C}_4$ and the usual Lie point symmetry group $\mathcal{S}$ (theorem with $\delta_1=\delta_2=1$)

$$\mathcal{G}=\mathcal{C}_4\otimes\mathcal{S},\qquad \mathcal{C}_4\equiv\{I,\sigma^x,\sigma^y,\sigma^x\sigma^y\}\,,$$

where I is the identity transformation, σ^x denotes the reflection of x, i.e. $\{x\to -x\}$, while σ^y denotes the reflection of y, i.e. $\{y\to -y\}$.

Furthermore, via the known transformation group, the Lie point symmetries and the related Lie symmetry algebra can be obtained straightforwardly by a more simple limiting procedure. In fact, if we set

$$\tau(t)=t+\epsilon k(t),\quad \xi_0(t)=\epsilon f(t),\quad \eta_0(t)=\epsilon g(t),\quad \beta_0(t)=1-\epsilon h(t),\quad \delta_1=\delta_2=1\,,$$

with an infinitesimal parameter ϵ, then (27)$-$(29) can be written as

$$\begin{aligned}
&u=U+\epsilon\sigma(U),\quad v=V+\epsilon\sigma(V)\quad Q=q+\epsilon\sigma(q)\,,\\
&\sigma(U)=\Big(\frac{1}{2}k_t(t)x+f(t)\Big)U_x+\Big(\frac{1}{2}k_t(t)y+g(t)\Big)U_y+k(t)U_t\\
&\qquad\quad+\Big(\frac{1}{8}k_{tt}(t)\Big(\frac{x^2}{s^2}+y^2\Big)+\frac{f_t(t)}{2s^2}x+\frac{1}{2}g_t(t)y-h(t)\Big)U\,,\\
&\sigma(V)=\Big(\frac{1}{2}k_t(t)x+f(t)\Big)V_x+\Big(\frac{1}{2}k_t(t)y+g(t)\Big)V_y+k(t)V_t\\
&\qquad\quad-\Big(\frac{1}{8}k_{tt}(t)\Big(\frac{x^2}{s^2}+y^2\Big)+\frac{f_t(t)}{2s^2}x+\frac{1}{2}g_t(t)y-h(t)-k_t(t)\Big)V\,,\\
&\sigma(q)=\Big(\frac{1}{2}k_t(t)x+f(t)\Big)q_x+\Big(\frac{1}{2}k_t(t)y+g(t)\Big)q_y+k(t)q_t+k_t(t)q-\frac{k_{ttt}(t)}{8}\Big(\frac{x^2}{s^2}+y^2\Big)\\
&\qquad\quad+\frac{f_{tt}(t)}{2s^2}x+\frac{g_{tt}(t)}{2}y-\frac{1}{2}k_{tt}(t)-h_t(t)\,.
\end{aligned}$$

The equivalent vector expression of the above symmetry is

$$\begin{aligned}
\underline{\mathrm{V}}=&\Big(\frac{1}{2}k_t(t)x+f(t)\Big)\frac{\partial}{\partial x}+\Big(\frac{1}{2}k_t(t)y+g(t)\Big)\frac{\partial}{\partial y}+k(t)\frac{\partial}{\partial t}-\Big(\frac{1}{8}k_{tt}(t)\Big(\frac{x^2}{s^2}+y^2\Big)\\
&+\frac{f_t(t)}{2s^2}x+\frac{1}{2}g_t(t)y-h(t)\Big)\frac{\partial}{\partial U}+\Big(\frac{1}{8}k_{tt}(t)\Big(\frac{x^2}{s^2}+y^2\Big)+\frac{f_t(t)}{2s^2}x+\frac{1}{2}g_t(t)y-h(t)-k_t(t)\Big)\frac{\partial}{\partial V}\\
&-\Big(k_t(t)q-\frac{k_{ttt}(t)}{8}\Big(\frac{x^2}{s^2}+y^2\Big)+\frac{f_{tt}(t)}{2s^2}x+\frac{g_{tt}(t)}{2}y-\frac{1}{2}k_{tt}(t)-h_t(t)\Big)\frac{\partial}{\partial q}\,,
\end{aligned}$$

which is exactly the same with that obtained by the classical Lie symmetry approach in Sec. 2.

4 Similar Reductions

We know that the study of the symmetries in soliton theory is very important. The reason lies in that not only one can obtain new solutions from old ones but also by symmetry we can reduce the dimensions of a partial differential equation. Via any subgroup of the symmetry group, the original equation can be reduced to some equations with fewer independent variables by solving the characteristic equation.[18–22] Here by solving the following characteristic equations

$$\frac{\mathrm{d}x}{X}=\frac{\mathrm{d}y}{Y}=\frac{dt}{T}=\frac{\mathrm{d}u}{\Phi_1}=\frac{\mathrm{d}v}{\Phi_2}=\frac{\mathrm{d}Q}{\Phi_3}\,, \tag{30}$$

with $\{X,Y,T,\Phi_1,\Phi_2,\Phi_3\}$ in (18), we can obtain three types of similarity reductions of Eqs. (8)–(10).

4.1 *The First Type of Similarity Reductions*

For $k(t)\neq 0$ in (18), we get the most general similarity reductions of Eqs. (8)–(10) by solving Eq. (30),

$$u=U(\xi,\eta)\,\mathrm{e}^{F_1(x,y,t)},\quad v=\frac{V(\xi,\eta)}{k}\,\mathrm{e}^{-F_1(x,y,t)},\quad Q=\frac{q(\xi,\eta)+F_2(x,y,t)}{k}\,, \tag{31}$$

where

$$F_1(x,y,t)=-\frac{k_t}{8}\Big(\frac{\xi^2}{s^2}+\eta^2\Big)-\frac{1}{4s^2}\Big(2\int\frac{f_t}{\sqrt{k}}\mathrm{d}t+\int ak_{tt}\mathrm{d}t\Big)\xi-\frac{1}{4}\Big(2\int\frac{g_t}{\sqrt{k}}\mathrm{d}t+\int bk_{tt}\mathrm{d}t\Big)\eta$$
$$-\frac{1}{2s^2}\int\frac{af_t+s^2bg_t}{\sqrt{k}}\mathrm{d}t-\frac{1}{8s^2}\int k_{tt}(a^2+s^2b^2)\mathrm{d}t+\int\frac{h}{k}\mathrm{d}t\,,$$
$$F_2(x,y,t)=\frac{1}{16}(k_t^2-2kk_{tt})\Big(\frac{\xi^2}{s^2}+\eta^2\Big)-\frac{1}{4s^2}\Big(2\int\sqrt{k}f_{tt}\mathrm{d}t+\int akk_{ttt}\mathrm{d}t\Big)\xi-\frac{1}{4}\Big(2\int\sqrt{k}g_{tt}\mathrm{d}t$$
$$+\int bkk_{ttt}\mathrm{d}t\Big)\eta-\frac{1}{2s^2}\int\sqrt{k}(af_{tt}+s^2bg_{tt})\mathrm{d}t-\frac{1}{8s^2}\int kk_{ttt}(a^2+s^2b^2)\mathrm{d}t+\frac{1}{2}k_t+h\,,$$

with

$$a\equiv a(t)=\int\frac{f}{k^{3/2}}\mathrm{d}t,\quad b\equiv b(t)=\int\frac{g}{k^{3/2}}\mathrm{d}t,\quad k\equiv k(t),\quad f\equiv f(t),\quad g\equiv g(t),\quad h\equiv h(t)\,.$$

In the solution (31), $U(\xi,\eta)$, $V(\xi,\eta)$, and $q(\xi,\eta)$ are similarity reduced functions with respect to the similarity variables

$$\xi=\frac{x}{\sqrt{k}}-a,\quad \eta=\frac{y}{\sqrt{k}}-b\,,$$

and satisfy the first type of similarity reduced equations

$$U_{\xi\xi}+s^2U_{\eta\eta}+s^2U(\alpha UV+q)=0,\quad V_{\xi\xi}+s^2V_{\eta\eta}+s^2V(\alpha UV+q)=0,\quad 2\alpha s^2(UV)_{\xi\xi}+s^2q_{\xi\xi}-q_{\eta\eta}=0\,.$$

4.2 *The Second Type of Similarity Reductions*

For $k(t)=0, f(t)\neq0$, the second type of similarity solutions can be obtained, i.e.

$$u=U(z,t)\,\mathrm{e}^{F_3(x,y,t)},\quad v=V(z,t)\,\mathrm{e}^{-F_3(x,y,t)},\quad Q=q(z,t)+\frac{s^2gg_{tt}-ff_{tt}}{4s^2f^2}x^2-\frac{yg_{tt}-2h_t}{2f}x\,,$$

where

$$F_3(x,y,t)=\frac{s^2gg_t-ff_t}{4s^2f^2}x^2-\frac{yg_t-2h}{2f}x\,.$$

The similarity functions $U(z,t)$, $V(z,t)$, and $q(z,t)$ with respect to the similarity variables $z=-(g/f)x+y$ and t satisfy the second type of similarity reduced equations

$$4(f^2+s^2g^2)U_{zz}+4s^2g(g_tz-2h)U_z-4f^2U_t+4\alpha f^2U^2V+4f^2qU$$
$$+(s^2g_t^2z^2-4s^2hg_tz+2s^2gg_t-2ff_t+4s^2h^2)U=0\,,$$
$$4(f^2+s^2g^2)V_{zz}-4s^2g(g_tz-2h)V_z+4f^2V_t+4\alpha f^2V^2U+4f^2qV$$
$$+(s^2g_t^2z^2-4s^2hg_tz-2s^2gg_t+2ff_t+4s^2h^2)V=0\,,$$
$$4\alpha s^2g^2(UV)_{zz}+2(s^2g^2-f^2)q_{zz}-ff_{tt}+s^2gg_{tt}=0\,.$$

4.3 *The Third Type of Similarity Reductions*

For $k=f=0$, $g\neq0$, we have the following similarity solutions

$$u=U(x,t)\exp\Big[-\frac{g_t}{4g}y^2+\frac{h}{g}y\Big],\quad v=V(x,t)\exp\Big[\frac{g_t}{4g}y^2-\frac{h}{g}y\Big],\quad Q=q(x,t)-\frac{g_{tt}}{4g}y^2+\frac{h_t}{g}y\,.$$

The similarity functions $U(x,t)$, $V(x,t)$, and $q(x,t)$ satisfy the third type of similarity reduced equations

$$2s^2U_{xx}-2U_t+2\alpha U^2V+2qU+\Big(\frac{2h^2}{g^2}-\frac{g_t}{g}\Big)U=0\,,$$
$$2s^2V_{xx}+2V_t+2\alpha V^2U+2qV+\Big(\frac{2h^2}{g^2}+\frac{g_t}{g}\Big)V=0,\quad 4\alpha(UV)_{xx}+2q_{xx}+s^2\frac{g_{tt}}{g}=0\,.$$

5 Conclusion

In summary, a natural (2+1)-dimensional extension of the resonant nonlinear Schrödinger equation, termed the resonant Davey–Stewartson (RDS) system proposed by Tang *et al.*, has been investigated. A decomposition relation is introduced to transform the complex RDS system into a coupled real (2+1)-dimensional system with three equations. For this coupled system, Gao and Tang obtained an infinite-dimensional Lie algebra of symmetries including only one arbitrary function of t. Here, we find the more general symmetry for there being four arbitrary functions of t in it, so it can cover the results that obtained by Gao and Tang. Like most of (2+1)-dimensional integrable systems, we can see that its Lie algebra possesses Kac-Moody-Virasoro type algebra structure. Unluckily, owing to the arbitrary functions in the symmetry, it is difficult to obtain the corresponding Lie symmetry group via the Lie algebra. Hence, alternatively, we adopt a simple direct method introduced by Lou and Ma to the (2+1)-dimensional coupled system and get the symmetry group theorem straightly. In this way, in addition to the local Lie point group, the discrete group is also gained. Then by a more simple limiting procedure, the related Lie algebra is found via the Lie group directly. Lastly, three types of the general similar reductions are given out. Via the reduced equations, many more group invariant solutions are certainly possible. Here, we do not investigate them.

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