Prolongation Structure of the Equation Studied by Qiao*

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Abstract *The prolongation structure technique of Wahlquist and Estanbrook is improved and applied to a new equation proposed by Z.J. Qiao [J. Math. Phys.* **48** *(2007) 082701]. Two potentials and two pseudopotentials are obtained, from which a new type of inverse scattering problem, Lax equations, and infinite number of conservation laws are obtained.*

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1 Introduction

Recently, Qiao *et al.* in Refs. [1] and [2] proposed a new completely integrable equation:

$$m_t = \frac{1}{2}\Big(\frac{1}{m^2}\Big)_{xxx} - \frac{1}{2}\Big(\frac{1}{m^2}\Big)_x\,, \tag{1}$$

which may have smooth solitons. This equation was shown to have bi-Hamiltionian structure and Lax pair, which imply integrability of this equation, and two kinds of peaked solitons were given in their paper. By adopting the phase analysis method of planar dynamical systems and the theory of the singular traveling wave systems to the traveling wave solutions of the equation, Li *et al.*[3] gave many solitary wave solutions, periodic wave solutions, Kink/antiKink wave solutions, cusped solitary wave solutions and breaking loop solutions. At the same time, this equation has cuspon and W/M shape soliton solutions.[4−7] In Ref. [8], the equivalence of the Qiao equation and the modified CBS equation is proved. The Painlevé property, Bäcklund and Darboux transformations for modified CBS have been obtained in Ref. [9].

It is well known that completely integrable non-linear equations have common features, such as the existence of Lax pair, bi-Hamiltonian structures, recursion operators, and applicability of inverse scattering methods. One of the effective method to test integrability is the prolongation structure technique of Wahlquist and Estabrook (WE),[10] in which they showed that the prolongation structure of non-linear equation was an incomplete Lie algebra, then they took the force closure method to investigate this Lie algebra. Dodd and Fordy[11] made the method more algorithmic and put it in an algebraic, instead of differential geometric framework.

In this paper, we would not adopt the approach of force closure, but introduce the relations with the basis of Lie algebra sl(2,R) to study the Lie algebra construction of Eq. (1). We obtain the two potentials and two pseudopotentials. Moreover, we can recover the inverse scattering problem given by Qiao[1] from one of the pseudopotentials, and we can get the Lax equations and infinite conservation laws of the new equation from the other one. This paper is arranged as follows: In Sec. 2, the prolongation structure of the Qiao equation is given, which is an incomplete Lie algebra; In Sec. 3, the representations of the incomplete Lie algebra are given, and the potentials, pseudopotentials, and the inverse scattering problems associated with the Qiao equation are obtained; In Sec. 4, the lax equations and the infinite conservation laws of the Qiao equation are deserved; In Sec. 5, a brief summary is given.

2 Prolongation Structure of the Qiao Equation

By the transformation

$$m = u^{-2/3}\,, \tag{2}$$

Eq. (1) can be transformed to

$$u_t = -u^2 u_{xxx} + \frac{2}{9}u_x^3 - u u_x u_{xx} + u^2 u_x\,. \tag{3}$$

Introducing the new variables

$$z = u_x\,, \quad p = z_x = u_{xx}\,, \tag{4}$$

Eq. (3) can be associated with the following set of 2-forms:

$$\alpha_1 = \mathrm{d}u \wedge \mathrm{d}t - z\mathrm{d}x \wedge \mathrm{d}t\,, \tag{5}$$

$$\alpha_2 = \mathrm{d}z \wedge \mathrm{d}t - p\mathrm{d}x \wedge \mathrm{d}t\,, \tag{6}$$

$$\alpha_1 = \mathrm{d}x\wedge\mathrm{d}u + u^2\mathrm{d}p\wedge\mathrm{d}t + \Big(uzp - u^2 z - \frac{2}{9}z^3\Big)\mathrm{d}x\wedge\mathrm{d}t\,. \tag{7}$$

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By calculating, we can find that the set of 2-forms is a closed ideal and consequently by Cartans theory[12] is completely equivalent to Eq. (3).

Following the WE procedure, we introduce the 1-form

$$\Omega = \mathrm{d}y + F\,\mathrm{d}x + G\,\mathrm{d}t\,, \tag{8}$$

and demand the exterior derivatives of Ω in the ring of α_1, α_2, α_3

$$\mathrm{d}\Omega = f_1\alpha_1 + f_2\alpha_2 + f_3\alpha_3 + \eta\wedge\Omega\,. \tag{9}$$

Then we can get

$$F_z = 0\,,\quad F_p = 0\,,\quad u^2F_u + G_p = 0\,,$$
$$F_u uzp - F_u zu^2 - \frac{2}{9}F_u z^3 + G_z p + G_u z - [F,G] = 0\,, \tag{10}$$

in which

$$[F,G] = FG_y - GF_y\,. \tag{11}$$

By calculating, we find F and G are of the following forms:

$$F = X_1 + u^{2/3}X_2 + u^{-2/3}X_3\,, \tag{12}$$

$$\begin{aligned} G = &-\frac{1}{180}(-120u^{1/3}p + 90u^{4/3} - 40u^{-2/3}z^2)X_2 - \frac{1}{4}u^{8/3}[X_1,X_5] - \frac{1}{3}u^2[X_2,X_5] \\ &- \frac{1}{180}(-40u^{2/3}z + 120u^{5/3}p - 45u^{8/3})X_3 - X_4 - \frac{2}{3}zu^{5/3}X_5 - \frac{1}{5}u^{10/3}[X_3,X_5]\,, \end{aligned} \tag{13}$$

and the following Lie bracket relations:

$$\begin{aligned} &[X_1,X_2] = 0\,,\quad [X_1,X_3] = X_5\,,\quad [X_2,X_3] = 0\,,\quad [X_3,X_4] = 0\,,\\ &[X_1,X_4] + [X_3,[X_3,X_5]] = 0\,,\quad \frac{1}{2}[X_2,[X_1,X_5]] + \frac{1}{3}[X_1,[X_2,X_5]] = 0\,,\\ &[X_2,X_4] + [X_1,[X_3,X_5]] + \frac{1}{2}[X_3,[X_1,X_5]] = 0\,,\quad [X_2,[X_2,X_5]] = 0\,,\\ &[X_2,[X_3,X_5]] - \frac{1}{2}X_5 + \frac{1}{2}[X_1,[X_1,X_5]] + \frac{1}{3}[X_3,[X_2,X_5]] = 0\,. \end{aligned} \tag{14}$$

The Lie bracket relations (14) are the prolongation structure of Eq. (3). The prolongation structure is open. Therefore, its representation is not unique, and it may be an infinity dimensional algebra. In the next section, we will give some interesting representations of the open structure.

3 Representation of Incomplete Lie Algebra

It is well known that the standard Lie algebra relations for sl(2,R) are

$$[Y_0,Y_1] = Y_1\,,\quad [Y_0,Y_{-1}] = -Y_{-1}\,,$$
$$[Y_{-1},Y_1] = -2Y_0\,. \tag{15}$$

The one-dimensional representation of relations (15) are given by

$$Y_0 = y\frac{\partial}{\partial y}\,,\quad Y_1 = y^2\frac{\partial}{\partial y}\,,\quad Y_{-1} = -\frac{\partial}{\partial y}\,, \tag{16}$$

and the two-dimensional representation of the relations are given by

$$Y_0 = \frac{1}{2}\Big(y_2\frac{\partial}{\partial y_2} - y_1\frac{\partial}{\partial y_1}\Big)\,,$$
$$Y_1 = -y_2\frac{\partial}{\partial y_1}\,,\quad Y_{-1} = -y_1\frac{\partial}{\partial y_2}\,. \tag{17}$$

Here, we set

$$X_i = a_iY_{-1} + b_iY_0 + c_iY_1\,, \tag{18}$$

then substituting Eq. (18) into Eq. (14) and according to the relations (15), we can get the following four interesting solutions:

Case 1 $X_1 = X_3 = X_4 = X_5 = 0$, $X_2 = Y_{-1}$.

In this case, we can get

$$\Omega = \mathrm{d}y - u^{2/3}\mathrm{d}x - \Big(\frac{1}{4}u^{8/3} + \frac{2}{9}u^{2/3}u_x^2 - \frac{2}{3}u^{5/3}u_{xx}\Big)\mathrm{d}t\,, \tag{19}$$

and on the solution manifold of the prolonged ideal, we will have

$$y_x = u^{2/3}\,, \tag{20}$$
$$y_t = \frac{1}{4}u^{8/3} + \frac{2}{9}u^{2/3}u_x^2 - \frac{2}{3}u^{5/3}u_{xx}\,. \tag{21}$$

Therefore, if the equations satisfied by y are Eqs. (20)–(21), y is a potential and conservation law for Eq. (3).

Case 2 $X_1 = X_2 = X_4 = X_5 = 0$, $X_3 = Y_{-1}$.

In this case, we can get

$$\Omega = \mathrm{d}y - u^{-2/3}\mathrm{d}x - \Big(\frac{2}{3}u^{1/3}u_{xx} + \frac{2}{9}u^{-2/3}u_{xx} - \frac{1}{2}u^{4/3}\Big)\mathrm{d}t\,, \tag{22}$$

and on the solution manifold of the prolonged ideal, we will have

$$y_x = u^{-2/3}\,, \tag{23}$$
$$y_t = \frac{2}{3}u^{1/3}u_{xx} + \frac{2}{9}u^{-2/3}u_{xx} - \frac{1}{2}u^{4/3}\,. \tag{24}$$

Therefore, if the equations satisfied by y are Eqs. (23)–(24) is another potential and conservation law for Eq. (3).

Case 3 $X_1 = Y_0$, $X_2 = 0$, $X_3 = \lambda(-Y_{-1} + Y_1)$, $X_4 = 4\lambda^3(-Y_{-1} + Y_1)$, $X_5 = \lambda(Y_{-1} + Y_1)$.

In this case, according to the one-dimensional representation (16) of sl(2,R), we can get a pseudopotential of Eq. (3)

$$\Omega = \mathrm{d}y + (-\lambda u^{-2/3}y^2 - y - \lambda u^{-2/3})\mathrm{d}x + \Big(- \frac{2}{9}(3u^{1/3}(u_x + u_{xx}) + u^{-2/3}u_x^2 + 18\lambda^2)\lambda y^2$$

$$-4\lambda^2u^{2/3}y-\frac{2}{9}(3u^{1/3}(-u_x+u_{xx})+u^{-2/3}u_x^2+18\lambda^2)\lambda\Big)\,\mathrm{d}t\,. \tag{25}$$

Meanwhile, under the two-dimensional representation (17) of sl(2,R), we can obtain two Pfaffian forms

$$\Omega_1=\mathrm{d}y_1+\Big(-\frac{1}{2}y_1-\lambda u^{-2/3}y_2\Big)\,\mathrm{d}x+\Big(-2\lambda^2u^{-2/3}y_1-\frac{2}{9}(3u^{1/3}(z+p)+u^{-2/3}z^2+18\lambda^2)\lambda y_2\Big)\,\mathrm{d}t\,, \tag{26}$$

$$\Omega_2=\mathrm{d}y_2+\Big(\lambda u^{-2/3}y_1+\frac{1}{2}y_2\Big)\,\mathrm{d}x+\Big(\frac{2}{9}(3u^{1/3}(-z+p)+u^{-2/3}z^2+18\lambda^2)\lambda y_1+2\lambda^2u^{-2/3}y_2\Big)\,\mathrm{d}t\,. \tag{27}$$

On the solution manifold of the prolonged ideal we will get the inverse scattering problem associated to the Qiao equation (1)

$$\frac{\partial}{\partial x}y_1=-\frac{1}{2}y_1-\lambda u^{2/3}y_2\,, \tag{28}$$

$$\frac{\partial}{\partial x}y_2=\lambda u^{2/3}y_1+\frac{1}{2}y_2\,, \tag{29}$$

$$\frac{\partial}{\partial t}y_1=-2\lambda^2u^{-2/3}y_1-\frac{2}{9}(3u^{1/3}(u_x+u_{xx})+u^{-2/3}u_x^2+18\lambda^2)\lambda y_2\,, \tag{30}$$

$$\frac{\partial}{\partial t}y_2=-\frac{2}{9}(3u^{1/3}(u_x+u_{xx})+u^{2/3}u_x^2+18\lambda^2)\lambda y_1+2\lambda^2u^{-2/3}y_2\,. \tag{31}$$

Moreover, if we express Eqs. (28)–(31) in matrix forms, it is just an equivalent form of the Lax pair of Eq. (1) obtained by Qiao,[1] the formulations are omitted for the sake of concision, and we do not discuss this case further.

Case 4 $X_1=(1/2)(Y_{-1}+Y_1)$, $X_2=0$, $X_3=\lambda Y_0$, $X_4=-\lambda^3Y_0$, $X_5=(1/2)\lambda(Y_{-1}-Y_1)$.

In this case, according to the one-dimensional representation (16) of sl(2,R), we can get another pseudopotential of Eq. (3)

$$\begin{aligned}\Omega=\mathrm{d}y&+\Big(\frac{1}{2}y^2+\lambda u^{-2/3}y-\frac{1}{2}\Big)\,\mathrm{d}x\\&+\Big(\Big(-\frac{1}{2}\lambda^2u^{2/3}-\frac{1}{3}\lambda u^{1/3}z\Big)y^2\\&+\Big(\frac{2}{3}\lambda u^{1/3}p+\frac{2}{9}\lambda u^{-2/3}z^2-\lambda^3\Big)y\\&+\frac{1}{2}\lambda^2u^{2/3}-\frac{1}{3}\lambda u^{1/3}z\Big)\,\mathrm{d}t\,.\end{aligned} \tag{32}$$

On the solution manifold of the prolonged ideal, we will have

$$\frac{\partial}{\partial x}y=-\frac{1}{2}y^2-\lambda u^{-2/3}y+\frac{1}{2}\,, \tag{33}$$

$$\begin{aligned}\frac{\partial}{\partial t}y=&\Big(\frac{1}{2}\lambda^2u^{2/3}+\frac{1}{3}\lambda u^{1/3}u_x\Big)y^2\\&+\Big(-\frac{2}{3}\lambda u^{1/3}u_{xx}-\frac{2}{9}\lambda u^{-2/3}{u_x}^2+\lambda^3\Big)y\\&-\frac{1}{2}\lambda^2u^{2/3}+\frac{1}{3}\lambda u^{1/3}u_x\,.\end{aligned} \tag{34}$$

The first of these equations is a Riccati equation whose quadratic term coefficient is a constant, from which we can get the Lax equations and infinite number of conserved quantities, and the detail will be given in the next section.

Similar to Case 3, we can also obtain another two Pfaffian forms

$$\begin{aligned}\Omega_1=\mathrm{d}y_1&+\Big(-\frac{1}{2}\lambda u^{-2/3}y_1-\frac{1}{2}y_2\Big)\,\mathrm{d}x\\&+\Big(\Big(-\frac{1}{3}\lambda u^{1/3}p-\frac{1}{9}\lambda u^{-2/3}z^2+\frac{1}{2}\lambda^3\Big)y_1\\&+\Big(\frac{1}{2}\lambda^2u^{2/3}+\frac{1}{3}\lambda u^{1/3}z\Big)y_2\Big)\,\mathrm{d}t\,,\end{aligned} \tag{35}$$

$$\begin{aligned}\Omega_2=\mathrm{d}y_2&+\Big(-\frac{1}{2}y_1+\frac{1}{2}\lambda u^{-2/3}y_2\Big)\,\mathrm{d}x\\&+\Big(\Big(\frac{1}{2}\lambda^2u^{2/3}-\frac{1}{3}\lambda u^{1/3}z\Big)y_1\\&+\Big(\frac{1}{3}\lambda u^{1/3}p+\frac{1}{9}\lambda u^{-2/3}z^2-\frac{1}{2}\lambda^3\Big)y_2\Big)\,\mathrm{d}t\,,\end{aligned} \tag{36}$$

and on the solution manifold of the prolonged ideal we will get another new inverse scattering problem

$$\frac{\partial}{\partial x}y_1=\frac{1}{2}\lambda u^{-2/3}y_1+\frac{1}{2}y_2\,, \tag{37}$$

$$\frac{\partial}{\partial x}y_2=\frac{1}{2}y_1-\frac{1}{2}\lambda u^{-2/3}y_2\,, \tag{38}$$

$$\begin{aligned}\frac{\partial}{\partial t}y_1=&\Big(\frac{1}{3}\lambda u^{1/3}u_{xx}+\frac{1}{9}\lambda u^{-2/3}u_x^2-\frac{1}{2}\lambda^3\Big)y_1\\&+\Big(-\frac{1}{2}\lambda^2u^{2/3}-\frac{1}{3}\lambda u^{1/3}u_x\Big)y_2\,,\end{aligned} \tag{39}$$

$$\begin{aligned}\frac{\partial}{\partial t}y_2=&\Big(-\frac{1}{2}\lambda^2u^{2/3}+\frac{1}{3}\lambda u^{1/3}u_x\Big)y_1\\&+\Big(-\frac{1}{3}\lambda u^{1/3}u_{xx}-\frac{1}{9}\lambda u^{-2/3}u_x^2+\frac{1}{2}\lambda^3\Big)y_2\,.\end{aligned} \tag{40}$$

In fact Eqs. (37)–(40) can also be obtained from Eqs. (33)–(34) through the linearising transformation $y=y_2/y_1$. If Eqs. (37)–(40) are expressed in matric forms, we will get a new Lax pairs of Eq. (3):

$$\begin{pmatrix}y_1\\y_2\end{pmatrix}_x=\begin{pmatrix}\frac{1}{2}\lambda u^{-2/3} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\lambda u^{-2/3}\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}, \tag{41}$$

$$\begin{pmatrix}y_1\\y_2\end{pmatrix}_t=\begin{pmatrix}\frac{1}{3}\lambda u^{1/3}u_{xx}+\frac{1}{9}\lambda u^{-2/3}u_x^2-\frac{1}{2}\lambda^3 & -\frac{1}{2}\lambda^2u^{2/3}-\frac{1}{3}\lambda u^{1/3}u_x\\ -\frac{1}{2}\lambda^2u^{2/3}+\frac{1}{3}\lambda u^{1/3}u_x & -\frac{1}{3}\lambda u^{1/3}u_{xx}-\frac{1}{9}\lambda u^{-2/3}u_x^2+\frac{1}{2}\lambda^3\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}. \tag{42}$$

4 Conservation Laws of the Qiao Equation

If we solve y^2 from Eq. (33) and substitute it into Eq. (34), we can obtain

$$y_t = \Big(-\Big(\lambda^2 u^{2/3} + \frac{2}{3}u^{1/3}u_x\Big) y + \frac{1}{2}\lambda u^{4/3}\Big)_x . \tag{43}$$

Then by means of

$$y = 2(\ln\psi)_x , \tag{44}$$

we can obtain the Lax equations from Eqs. (33) and (43):

$$\psi_{xx} = -\lambda u^{-2/3}\psi_x + \frac{1}{4}\psi , \tag{45}$$

$$\psi_t = -\Big(\lambda^2 u^{2/3} + \frac{2}{3}\lambda u^{1/3}u_x\Big)\psi_x + \frac{1}{4}\lambda u^{4/3}\psi . \tag{46}$$

Moreover, we can obtain infinite number of conserved quantities from the Lax equtions (45) and (46). As we know

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x}\ln\psi = \frac{\partial}{\partial x}\frac{\partial}{\partial t}\ln\psi , \tag{47}$$

according to Eq. (46), we have

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\psi_x}{\psi} &= \frac{\partial}{\partial x}\frac{\psi_t}{\psi}\\ &= \frac{\partial}{\partial x}\frac{(\lambda^2 u^{2/3} + (2/3)\lambda u^{1/3}u_x)\psi_x + (1/4)\lambda u^{4/3}\psi}{\psi}\\ &= \frac{\partial}{\partial x}\Big(\frac{1}{4}\lambda u^{4/3} - \Big(\lambda^2 u^{2/3} + \frac{2}{3}\lambda u^{1/3}u_x\Big)\frac{\psi_x}{\psi}\Big) ,\end{aligned} \tag{48}$$

then setting $\psi_x/\psi = \Gamma$, and by means of Eq. (45) we have

$$\begin{aligned}\Gamma_x &= \frac{\psi_{xx}}{\psi} - \Big(\frac{\psi_x}{\psi}\Big)^2 = \frac{1}{4} - \lambda u^{-2/3}\frac{\psi_x}{\psi} - \Big(\frac{\psi_x}{\psi}\Big)^2\\ &= \frac{1}{4} - \lambda u^{-2/3}\Gamma - \Gamma^2 .\end{aligned} \tag{49}$$

Setting

$$\Gamma = \mu\Gamma_{-1} + \Gamma_0 + \frac{\Gamma_1}{\mu} + \frac{\Gamma_2}{\mu^2} + \cdots , \quad (\lambda = \mu) , \tag{50}$$

and substituting into Eq. (48), then comparing the coefficient of μ^i, we have

$$i = 2 : \Gamma_{-1}^2 + u^{-2/3}\Gamma_{-1} , \tag{51}$$

$$i = 1 : \Gamma_{-1,x} + 2\Gamma_{-1}\Gamma_0 + u^{-2/3}\Gamma_0 , \tag{52}$$

$$i = 0 : \Gamma_{0,x} - \frac{1}{4} + 2\Gamma_{-1}\Gamma_1 + \Gamma_0^2 + u^{-2/3}\Gamma_1 , \tag{53}$$

$$i = -1 : \Gamma_{1,x} + 2\Gamma_0\Gamma_1 + 2\Gamma_{-1}\Gamma_2 + u^{-2/3}\Gamma_2 , \tag{54}$$

$$i = -2 : \Gamma_{2,x} + 2\Gamma_{-1}\Gamma_3 + 2\Gamma_0\Gamma_2 + \Gamma_1^2 + u^{-2/3}\Gamma_3 , \tag{55}$$

$$\ldots$$

$$i = j : \Gamma_{j,x} + \sum_{l+k=j}\Gamma_l\Gamma_k + 2\Gamma_{-1}\Gamma_{j+1} , \tag{56}$$

from which we can obtain

$$\Gamma_{-1} = -u^{-2/3} , \tag{57}$$

$$\Gamma_0 = \frac{2}{3}u^{-1}u_x , \tag{58}$$

$$\Gamma_1 = \frac{2}{3}u^{-1/3}u_{xx} - \frac{2}{9}u^{-4/3}u_x^2 - \frac{1}{4}u^{2/3} , \tag{59}$$

$$\Gamma_2 = \frac{2}{3}u^{1/3}u_{xxx} + \frac{2}{9}u^{-2/3}u_x u_{xx} - \frac{1}{2}u^{-1/3}u_x , \tag{60}$$

$$\begin{aligned}\Gamma_3 = {}& \frac{2}{3}uu_{xxxx} + \frac{4}{3}u_x u_{xxx}\\ & - \frac{13}{18}u_x^2 - \frac{4}{27}u^{-1}u_{xx}u_x^2 - \frac{5}{6}uu_{xx}\\ & + \frac{2}{3}u_{xx}^2 + \frac{4}{81}u^{-2}u_x^4 + \frac{1}{16}u^2 ,\end{aligned} \tag{61}$$

$$\ldots$$

$$\Gamma_{j+1} = -\frac{1}{2\Gamma_{-1}}\Big[\Gamma_{j,x} + \sum_{l+k=j}\Gamma_l\Gamma_k\Big] . \tag{62}$$

Every Γ_n is a conservation density. The conserved quantity of the Γ_{-1} is just the potential we have obtained in Case 2.

When we make the transformation

$$u = m^{-3/2} , \tag{63}$$

all the above conclusion can be transformed into the corresponding conclusion of the Qiao equation (1).

5 Conclusions

In this letter, we obtain the prolongation structure of the equation proposed by Qiao *et al.*[1−2] and introduce the relations with the basis of Lie algebra sl(2,R) to study the Lie algebra construction of this prolongation structure. Meanwhile, two potentials and two pseudopotentials are obtained, from which some integrable features including inverse scattering problems, Lax equations and an infinite number of conservation quantities are obtained, from which the integrability of this equation is further proved.

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