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Binary Bell Polynomials, Bilinear Approach to Exact Periodic Wave Solutions of (2 + 1)-Dimensional Nonlinear Evolution Equations*

WANG Yun-Hu (王云虎)[1] and CHEN Yong (陈勇)[1,2,†]

[1]Software Engineering institute of East China Normal University Shanghai Key Laboratory of Trustworthy Computing, Shanghai 200062, China

[2]Nonlinear Science Center and Department of Mathematics, Ningbo University, Ningbo 315211, China

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Abstract *In the present letter, we get the appropriate bilinear forms of* $(2+1)$*-dimensional KdV equation, extended* $(2+1)$*-dimensional shallow water wave equation and* $(2+1)$*-dimensional Sawada–Kotera equation in a quick and natural manner, namely by appling the binary Bell polynomials. Then the Hirota direct method and Riemann theta function are combined to construct the periodic wave solutions of the three types nonlinear evolution equations. And the corresponding figures of the periodic wave solutions are given. Furthermore, the asymptotic properties of the periodic wave solutions indicate that the soliton solutions can be derived from the periodic wave solutions.*

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Key words: binary Bell polynomial, Riemann theta function, periodic wave solution, asymptotic property

1 Introduction

Owing to soliton theory are being applied to mathematics, physics, biology, astrophysics and other potential field, the studies on the nonlinear evolution equations (NNEEs) has arisen great attention during the past few decades.[1−15] As an important part in the soliton theory, seeking for analytic solutions of the NLEEs is always the hot research areas.[16−21] Starting from the analytic solutions, we may obtain the properties of corresponding NNEEs and connect them with practical physical phenomenons. Also, some methods have been proposed to deal with the problem, such as inverse scattering transformation,[22] Darboux transformation,[23] Hirota direct method.[24] Among these methods, the Hirota method is a powerful tool and direct approaches to construct exact solutions of nonlinear equations. The advantage of the Hirota method is that once the bilinear forms of nonlinear evolution equations are obtained, then not only the multi-soliton solutions, but also the bilinear BT, Lax pairs are constructed. It is clear that the key problem is transform the given NLEEs into corresponding bilinear forms. There has some methods to deal with the problem, such as rational transformation, logarithmic transformation, and double logarithmic transformation, but there is no universal method to find the needed transformation. Furthermore, the obtaining process of the bilinear forms for the given NNEEs are often cockamamie. Lately, Lembert and Gilson *et al.*[25−34] proposed a lucid and systematic approach to obtain the bilinear representations as well as its bilinear Bäcklund transformation (BT), Lax pairs of the NLEEs, namely by applying the Bell polynomials. Compare with the traditional methods, the efficiency of the Bell polynomials is obvious stand to reason.

Once the bilinear forms of the given NNEEs are given, we could construct the Wronskian solutions, Pfaffian solutions, and explicit periodic wave solutions by use of the Riemann theta functions.[35−38] Nakamura, Fan *et al.* have obtained periodic wave solutions of the KdV and KP equations by the bilinear approach.[37−39] The appeal and success of this approach lies in the fact that we obtain the periodic wave solutions in a direct approach without apply algebro-geometric theory. Besides, we could get corresponding soliton solutions via asymptotic analysis for the periodic wave solutions.

In this paper, $(2 + 1)$-dimensional KdV equation, extended $(2 + 1)$-dimensional shallow water wave equation and $(2 + 1)$-dimensional Sawada–Kotera equation will be dealt with to illustrate the efficiency of obtaining the bilinear forms by applying the Bell polynomials. Then, with the help of the Riemann theta functions and Hirota method, we obtain the periodic wave solutions of these three kinds of nonlinear equations. Furthermore, the periodic wave solutions are reduced to their soliton solutions via asymptotic analysis.

The paper is organized as follows. In Sec. 2, we give a brief introduction about the binary Bell polynomial. In Sec. 3, we give the bilinear form of $(2 + 1)$-dimensional KdV equation, extended $(2 + 1)$-dimensional shallow water wave equation and $(2+1)$-dimensional Sawada–Kotera equation by appling binary Bell polynomial. In Sec. 4, by using the Riemann theta functions and Hirota method,

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†E-mail: ychen@sei.ecnu.edu.cn

their periodic wave solutions and reductions are presented, respectively. Finally, some conclusions are given in Sec. 5.

2 Binary Bell Polynomial

Bell[29] proposed three kinds of exponent-form polynomials, the third type of Bell polynomials is the main tool used in this paper. For the convenience of the understanding for our description, we briefly introduce the necessary notations of the Bell polynomials. For details of Bell polynomial, refer to Lembert and Gilson *et al.* work.[25–34]

The multi-dimensional binary Bell polynomials that we use are defined as the following

$$\begin{aligned} &Y_{n_1x_1,\ldots,n_lx_l}(f) \equiv Y_{n_1,\ldots,n_l}(f_{r_1x_1,\ldots,r_lx_l}) = \mathrm{e}^{-f}\partial_{x_1}^{n_1}\cdots\partial_{x_l}^{n_l}\,\mathrm{e}^f\,,\\ &\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v,w) \equiv Y_{n_1,\ldots,n_l}(f)\Big|_{f_{r_1x_1,\ldots,r_lx_l}=\begin{cases} v_{r_1x_1,\ldots,r_lx_l}\,, r_1+\cdots+r_l \text{ is odd}\,,\\ w_{r_1x_1,\ldots,r_lx_l}\,, r_1+\cdots+r_l \text{ is even}\,,\end{cases}} \end{aligned} \tag{1}$$

with $f_{r_1x_1,\ldots,r_lx_l} = \partial_{x_1}^{n_1}\cdots\partial_{x_l}^{n_l}f$, $r_1 = 0,\ldots,n_1,\ldots,r_l = 0,\ldots,n_l$.

The first few lowest order binary Bell polynomials are

$$\begin{aligned} &\mathcal{Y}_x(v) = v_x\,,\quad \mathcal{Y}_{2x}(v,w) = w_{2x}+v_x^2\,,\\ &\mathcal{Y}_{x,y}(v,w) = w_{x,y}+v_xv_y,\\ &\mathcal{Y}_{3x}(v,w) = v_{3x}+3v_xw_{2x}+v_x^3,\ldots \end{aligned} \tag{2}$$

The link between $\mathcal{Y}$-polynomials and the standard Hirota expressions can be given by the identity

$$\begin{aligned} &\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v=\ln F/G, w=\ln FG)\\ &\quad = (FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G\,. \end{aligned} \tag{3}$$

In the particular case when $F=G$, the formula becomes

$$\begin{aligned} &F^{-2}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot F = \mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(0,q=2\ln F)\\ &\quad = \begin{cases} 0\,, & n_1+\cdots+n_l \text{ is odd}\,,\\ P_{n_1x_1,\ldots,n_lx_l}(q)\,, & n_1+\cdots+n_l \text{ is even}\,,\end{cases} \end{aligned} \tag{4}$$

in which the P-polynomials can be characterized by an equally recognizable even part partitional structure

$$\begin{aligned} &P_{2x}(q) = q_{2x}\,,\quad P_{x,t}(q) = q_{xt}\,,\quad P_{4x}(q) = q_{4x}+3q_{2x}^2\,,\\ &P_{3x,y}(q) = q_{3xy}+3q_{xy}q_{2x}\,,\quad \ldots \end{aligned} \tag{5}$$

3 Bilinear Form

In this section, we will give the bilinear forms for the $(2+1)$-dimensional KdV equation, extended $(2+1)$-dimensional shallow water wave equation and $(2+1)$-dimensional Sawada–Kotera equation by applying binary Bell polynomials.

3.1 $(2+1)$-*Dimensional KdV Equation*[40]

$$u_t + 3uu_y + u_{xxy} + 3u_x\int u_y\,\mathrm{d}x = 0\,. \tag{6}$$

Setting $u = q_{2x}$, substituting it into Eq. (6) and integrating respect to x yields

$$q_{x,t} + 3q_{x,y}q_{2x} + q_{3x,y} - \lambda = 0\,, \tag{7}$$

where λ is an integral constant.

Then, Eq. (6) can be written as follows

$$E(q) = P_{x,t}(q) + P_{3x,y}(q) - \lambda = 0\,. \tag{8}$$

Introducing a change of dependent variable

$$q = 2\ln F \Leftrightarrow u = q_{2x} = 2(\ln F)_{2x}\,, \tag{9}$$

we get the bilinear representation of the Eq. (6) in terms of the identity (3)

$$G(D_x,D_y,D_t) \equiv (D_xD_t + D_x^3D_y)F\cdot F - \lambda F^2 = 0\,. \tag{10}$$

3.2 *Extended* $(2+1)$-*Dimensional Shallow Water Wave Equation*[41]

$$u_{y,t} + u_{3x,y} - 3u_{2x}u_y - 3u_xu_{x,y} + \alpha u_{x,y} = 0\,, \tag{11}$$

where α is a constant.

Setting $u = -q_x$, substituting it into Eq. (11) and integrating with respect to x yields

$$q_{y,t} + q_{3x,y} + 3q_{2x}q_{x,y} + \alpha q_{x,y} - \gamma = 0\,, \tag{12}$$

where γ is an integral constant.

Then, Eq. (11) can be written as follows

$$P_{y,t}(q) + P_{3x,y}(q) + \alpha P_{x,y}(q) - \gamma = 0\,. \tag{13}$$

Introducing a change of dependent variable

$$q = 2\ln F \Leftrightarrow u = -q_x = -2(\ln F)_x\,, \tag{14}$$

we get the bilinear representation of the Eq. (11) as follows

$$\begin{aligned} G(D_x,D_y,D_t) &\equiv (D_yD_t + D_x^3D_y\\ &\quad + \alpha D_xD_y)F\cdot F - \gamma F^2 = 0\,. \end{aligned} \tag{15}$$

3.3 $(2+1)$-*Dimensional Sawada–Kotera Equation*[42]

$$\begin{aligned} &u_t - \Big(u_{4x} + 5uu_{2x} + \frac{5}{3}u^3 + 5u_{x,y}\Big)_x\\ &\quad +5\int u_{2y}\,\mathrm{d}x - 5uu_y - 5u_x\int u_y\,\mathrm{d}x = 0\,. \end{aligned} \tag{16}$$

Setting $u = 3q_{2x}$, substituting it into Eq. (16) and integrating with respect to x yields

$$\begin{aligned} &q_{xt} + 5q_{2y} - (q_{6x} + 15q_{2x}q_{4x} + 15q_{2x}^3)\\ &\quad - 5(q_{3xy} + 3q_{2x}q_{xy}) - \varsigma = 0\,, \end{aligned} \tag{17}$$

where ς is an integral constant.

Then, Eq. (16) can be written as follows

$$E(q) = P_{x,t}(q) + 5P_{2y}(q) - P_{6x}(q) - 5P_{3x,y}(q) - \varsigma = 0\,. \tag{18}$$

Introducing a change of dependent variable

$$q = 2\ln F \Leftrightarrow u = 3q_{2x} = 6(\ln F)_{2x}\,, \tag{19}$$

we get the bilinear representation of the Eq. (16) as follows

$$G(D_x,D_y,D_t) \equiv (D_xD_t + 5D_y^2 - D_x^6$$

$$-5D_x^3D_y)F\cdot F-\varsigma F^2=0\,. \tag{20}$$

From the above process for seeking the bilinear forms of three kinds of $(2+1)$-dimensional nonlinear equation, we could find that binary Bell polynomials provide us a direct and simple approach for constructing the bilinear forms for some nonlinear equations.

4 Periodic Wave Solutions

We notice that D-operators have good property when acting on exponential functions[43−44]

$$\begin{aligned}&D_x^mD_y^sD_t^n\,\mathrm{e}^{\xi 1}\cdot\mathrm{e}^{\xi 2}\\&=(\kappa_1-\kappa_2)^m(\iota_1-\iota_2)^s(\omega_1-\omega_2)^n\,\mathrm{e}^{\xi 1+\xi 2}\,,\end{aligned} \tag{21}$$

where $\xi_j=\kappa_jx+\iota_jy+\omega_jt+\xi_j^{(0)}$, $j=1,2$.

More generally, we have

$$\begin{aligned}&G(D_x,D_y,D_t)\,\mathrm{e}^{\xi 1}\cdot\mathrm{e}^{\xi 2}\\&=G(\kappa_1-\kappa_2,\iota_1-\iota_2,\omega_1-\omega_2)\,\mathrm{e}^{\xi 1+\xi 2}\,.\end{aligned} \tag{22}$$

In the following, we will use the Riemann theta function, Hirota direct method as well as property (22) to construct periodic wave solutions of Eqs. (6), (11), and (16).

4.1 $(2+1)$-*Dimensional KdV Equation*

We consider the Riemann theta function solution of Eq. (6)

$$F=\sum_{n=-\infty}^{\infty}\mathrm{e}^{2\pi\mathrm{i}n\xi+\pi\mathrm{i}n^2\tau}\,, \tag{23}$$

where $n\in\mathcal{Z}$, $\tau\in\mathcal{C}$, $\mathrm{Im}\tau>0$, and $\xi=\kappa x+\iota y+\omega t$, with κ, ι, and ω are constants to be determined.

Inserting Eq. (23) into Eq. (10), we have

$$\begin{aligned}GF\cdot F&=G(D_x,D_y,D_t)\sum_{n=-\infty}^{\infty}\mathrm{e}^{2\pi\mathrm{i}n\xi+\pi\mathrm{i}n^2\tau}\sum_{m=-\infty}^{\infty}\mathrm{e}^{2\pi\mathrm{i}m\xi+\pi\mathrm{i}m^2\tau}\\&=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}G(D_x,D_y,D_t)\,\mathrm{e}^{2\pi\mathrm{i}n\xi+\pi\mathrm{i}n^2\tau}\cdot\mathrm{e}^{2\pi\mathrm{i}m\xi+\pi\mathrm{i}m^2\tau}\\&=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}G(2\pi\mathrm{i}(n-m)\kappa,2\pi\mathrm{i}(n-m)\iota,2\pi\mathrm{i}(n-m)\omega)\,\mathrm{e}^{2\pi\mathrm{i}(n+m)\xi+\pi\mathrm{i}(n^2+m^2)\tau}\\&=\sum_{p=-\infty}^{\infty}\Big\{\sum_{n=-\infty}^{\infty}G[2\pi\mathrm{i}(2n-p)\kappa,2\pi\mathrm{i}(2n-p)\iota,2\pi\mathrm{i}(2n-p)\omega]\,\mathrm{e}^{\pi\mathrm{i}(n^2+(p-n)^2)\tau}\Big\}\mathrm{e}^{2\pi\mathrm{i}p\xi}=\sum_{p=-\infty}^{\infty}\bar{G}(p)\,\mathrm{e}^{2\pi\mathrm{i}p\xi}\,.\end{aligned} \tag{24}$$

Noting that

$$\begin{aligned}\bar{G}(p)&=\sum_{n=-\infty}^{\infty}G[2\pi\mathrm{i}(2n-p)\kappa,2\pi\mathrm{i}(2n-p)\iota,2\pi\mathrm{i}(2n-p)\omega]\,\mathrm{e}^{\pi\mathrm{i}(n^2+(p-n)^2)\tau}\\&=\sum_{h=-\infty}^{\infty}G[2\pi\mathrm{i}(2h-(p-2))\kappa,2\pi\mathrm{i}(2h-(p-2))\iota,2\pi\mathrm{i}(2h-(p-2))\omega]\,\mathrm{e}^{\pi\mathrm{i}((h+1)^2+(p-h-1)^2)\tau}\\&=\sum_{h=-\infty}^{\infty}G[2\pi\mathrm{i}(2h-(p-2))\kappa,2\pi\mathrm{i}(2h-(p-2))\iota,2\pi\mathrm{i}(2h-(p-2))\omega]\,\mathrm{e}^{\pi\mathrm{i}(h^2+(p-h-2)^2)\tau}\cdot\mathrm{e}^{2\pi\mathrm{i}(p-1)\tau}\\&=\bar{G}(p-2)\,\mathrm{e}^{2\pi\mathrm{i}(p-1)\tau}\,,\end{aligned} \tag{25}$$

where $p=m+n$.

In view of Eq. (25) and by induction method, we can get that

$$\bar{G}(p)=\begin{cases}\bar{G}(0)\,\mathrm{e}^{\pi\mathrm{i}np\tau}\,, & p=2n\,,\\ \bar{G}(1)\,\mathrm{e}^{\pi\mathrm{i}(2n+2n^2)(p+1)\tau}\,, & p=2n+1\,.\end{cases} \tag{26}$$

In this way, we may let

$$\begin{aligned}&\bar{G}(0)=\sum_{n=-\infty}^{\infty}(-16n^2\pi^2\kappa\omega+256n^4\pi^4\kappa^3\iota-\lambda)\,\mathrm{e}^{2\pi\mathrm{i}n^2\tau}=0\,,\\&\bar{G}(1)=\sum_{n=-\infty}^{\infty}(-4(2n-1)^2\pi^2\kappa\omega+16(2n-1)^4\pi^4\kappa^3\iota-\lambda)\\&\qquad\times\,\mathrm{e}^{\pi\mathrm{i}(2n^2-2n+1)\tau}=0\,.\end{aligned} \tag{27}$$

For the sake of convenience, if we denote that

$$\begin{aligned}&q_1(n)=\mathrm{e}^{2\pi in^2\tau}\,,\quad q_2(n)=\mathrm{e}^{\pi\mathrm{i}(2n^2-2n+1)\tau}\,,\\&a_{11}=\sum_{n=-\infty}^{\infty}-16n^2\pi^2\kappa q_1(n)\,,\quad a_{12}=\sum_{n=-\infty}^{\infty}q_1(n)\,,\\&a_{21}=\sum_{n=-\infty}^{\infty}-4(2n-1)^2\pi^2\kappa q_2(n)\,,\quad a_{22}=\sum_{n=-\infty}^{\infty}q_2(n)\,,\\&b_1=\sum_{n=-\infty}^{\infty}(256n^4\pi^4\kappa^3\iota)q_1(n)\,,\\&b_2=\sum_{n=-\infty}^{\infty}(16(2n-1)^4\pi^4\kappa^3\iota)q_2(n)\,.\end{aligned} \tag{28}$$

Then Eq. (27) can be written as

$$a_{11}\omega + b_1 - \lambda a_{12} = 0\,, \quad a_{21}\omega + b_2 - \lambda a_{22} = 0\,. \tag{29}$$

Solving this system, we obtain

$$\omega = \frac{a_{12}b_2 - b_1 a_{22}}{a_{11}a_{22} - a_{21}a_{12}}\,, \quad \lambda = \frac{a_{11}b_2 - b_1 a_{21}}{a_{11}a_{22} - a_{21}a_{12}}\,. \tag{30}$$

Thus, we obtain the periodic wave solution

$$u = 2\ln(F)_{2x}\,, \tag{31}$$

where F and ω are given by Eqs. (23) and (30), respectively.

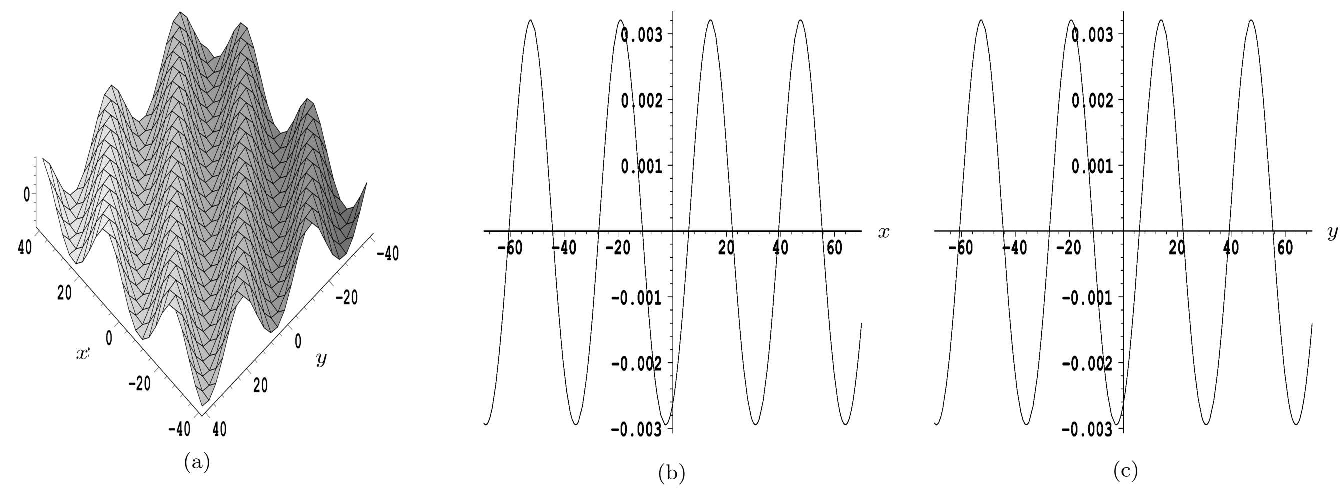

Fig. 1 (a) The figure represents periodic solution (31) with $\kappa = 0.03$, $\tau = i$, $\iota = 0.03$; (b) Along x-axis; (c) Along y-axis.

We are interested in the asymptotic properties of the periodic wave solutions of Eq. (6). From Eq. (23), we write F as

$$F = 1 + \eta(\mathrm{e}^{2\pi\mathrm{i}\xi} + \mathrm{e}^{-2\pi\mathrm{i}\xi}) + \eta^4(\mathrm{e}^{4\pi\mathrm{i}\xi} + \mathrm{e}^{-4\pi i\xi}) + \cdots \tag{32}$$

where $\eta = \mathrm{e}^{\pi i\tau}$.

Setting

$$\kappa' = 2\pi\mathrm{i}\kappa\,, \quad \iota' = 2\pi\mathrm{i}\iota\,, \quad \omega' = 2\pi\mathrm{i}\omega\,,$$
$$\xi' = \kappa' x + \iota' y + \omega' t + \pi\mathrm{i}\tau\,,$$

we get

$$\begin{aligned} F &= 1 + \eta(\mathrm{e}^{2\pi\mathrm{i}\xi} + \mathrm{e}^{-2\pi\mathrm{i}\xi}) + \eta^4(\mathrm{e}^{4\pi\mathrm{i}\xi} + \mathrm{e}^{-4\pi\mathrm{i}\xi}) + \cdots \\ &= 1 + \mathrm{e}^{\xi'} + \eta^2(\mathrm{e}^{-\xi'} + \mathrm{e}^{2\xi'}) + \eta^6(\mathrm{e}^{-2\xi'} + \mathrm{e}^{3\xi'}) + \cdots \\ &\to 1 + \mathrm{e}^{\xi'}\,, \quad \text{as} \quad \eta \to 0\,. \end{aligned} \tag{33}$$

It is interesting that if we can prove that

$$\omega' \to -\kappa'^2\iota'\,, \tag{34}$$

then the periodic wave solutions (31) turns to the soliton solution

$$\begin{aligned} &u = 2\ln(F)_x\,, \quad F = 1 + \mathrm{e}^{\xi'}\,, \\ &\xi' = \kappa' x + \iota' y + \omega' t + \pi\mathrm{i}\tau\,, \quad \omega' = -\kappa'^2\iota'\,. \end{aligned} \tag{35}$$

In fact, it is easy to see that

$$\begin{aligned} &a_{11} = -32\pi^2\kappa(\eta^2 + 4\eta^4 + \cdots)\,, \\ &a_{12} = 1 + 2\eta^2 + 2\eta^8 + \cdots\,, \\ &a_{21} = -8\pi^2\kappa(\eta + 9\eta^5 + \cdots)\,, \\ &a_{22} = 2\eta + 2\eta^5 + \cdots\,, \quad b_1 = 2\cdot 256\pi^4\kappa^3\iota\eta^2 + \cdots\,, \\ &b_2 = 2(16\pi^4\kappa^3\iota)\eta + 2(16\cdot 3^4\pi^4\kappa^3\iota)\eta^5 + \cdots\,, \end{aligned} \tag{36}$$

which lead to

$$\begin{aligned} a_{12}b_2 - b_1 a_{22} &= 32\pi^4\kappa^3\iota\eta + o(\eta), a_{11}a_{22} - a_{12}a_{21} \\ &= 8\pi^2\kappa\eta + o(\eta)\,, \end{aligned} \tag{37}$$

so we have $\omega \to 4\pi^2\kappa^2\iota$, as $\eta \to 0$, which is equivalent to $\omega' \to -\kappa'^2\iota'$, as $\eta \to 0$.

4.2 *Extended (2+1)-Dimensional Shallow Water Wave Equation*

With the silimiar calculation process as subsec. 4.1, we have

$$\begin{aligned} \bar{G}(0) &= \sum_{n=-\infty}^{\infty}(-16n^2\pi^2\iota\omega + 256n^4\pi^4\kappa^3\iota - 16\alpha n^2\pi^2\kappa\iota - \gamma) \\ &\quad\times \mathrm{e}^{2\pi i n^2\tau} = 0\,, \\ \bar{G}(1) &= \sum_{n=-\infty}^{\infty}(-4(2n-1)^2\pi^2\iota\omega + 16(2n-1)^4\pi^4\kappa^3\iota \\ &\quad - 4\alpha(2n-1)^2\pi^2\kappa\iota - \gamma)\,\mathrm{e}^{\pi\mathrm{i}(2n^2-2n+1)\tau} = 0\,. \end{aligned} \tag{38}$$

For the sake of convenience, we denote that

$$q_1(n) = \mathrm{e}^{2\pi i n^2\tau}\,, \quad q_2(n) = \mathrm{e}^{\pi\mathrm{i}(2n^2-2n+1)\tau}\,,$$

$$a_{11} = \sum_{n=-\infty}^{\infty} -16n^2\pi^2\iota q_1(n)\,, \quad a_{12} = \sum_{n=-\infty}^{\infty} q_1(n)\,,$$

$$a_{21} = -\sum_{n=-\infty}^{\infty} 4(2n-1)^2\pi^2\iota q_2(n)\,, \quad a_{22} = \sum_{n=-\infty}^{\infty} q_2(n)\,,$$

$$b_1=\sum_{n=-\infty}^{\infty}(256n^4\pi^4\kappa^3\iota-16\alpha n^2\pi^2\kappa\iota)q_1(n)\,,$$

$$b_2=\sum_{n=-\infty}^{\infty}(16(2n-1)^4\pi^4\kappa^3\iota-4\alpha(2n-1)^2\pi^2\kappa\iota)q_2(n)\,,\quad(39)$$

then Eq. (38) can be written as

$$a_{11}\omega+b_1-\varsigma a_{12}=0\,,\quad a_{21}\omega+b_2-\varsigma a_{22}=0\,.\quad(40)$$

Solving this system, we obtain

$$\omega=\frac{a_{12}b_2-b_1a_{22}}{a_{11}a_{22}-a_{21}a_{12}}\,,\quad \varsigma=\frac{a_{11}b_2-b_1a_{21}}{a_{11}a_{22}-a_{21}a_{12}}\,.\quad(41)$$

Thus, we obtain the periodic wave solution

$$u=-2\ln(F)_x\,,\quad(42)$$

where F and ω are given by Eqs. (23) and (41), respectively.

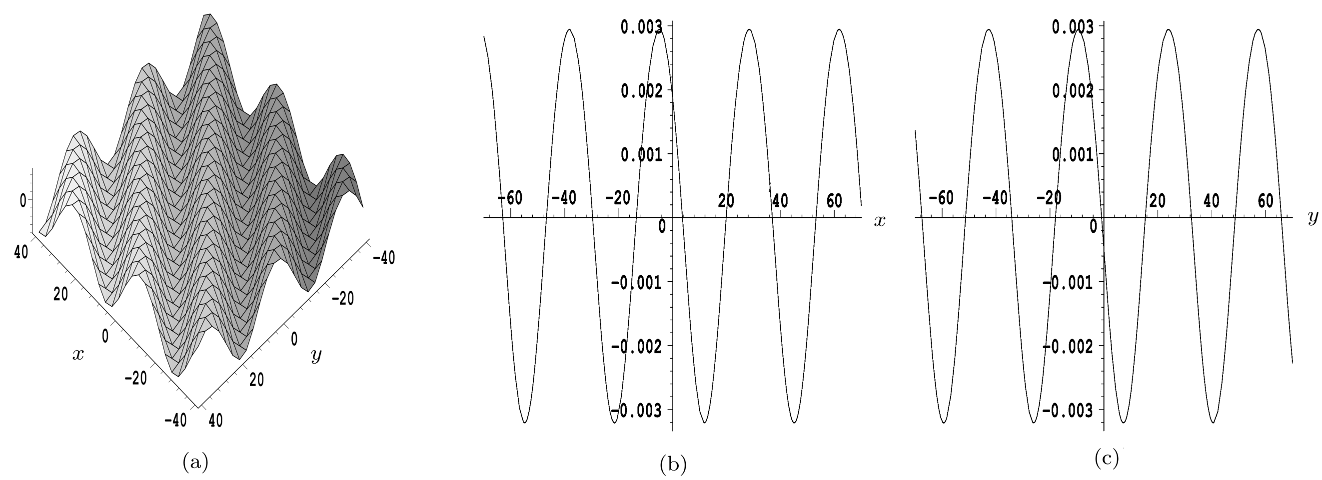

Fig. 2 (a) The figure represents periodic solution (42) with $\alpha=4$, $\kappa=0.03$, $\tau=i$, $\iota=0.03$; (b) Along x-axis; (c) Along y-axis.

From Eq. (23), we write F as

$$F=1+\delta(\mathrm{e}^{2\pi\mathrm{i}\xi}+\mathrm{e}^{-2\pi\mathrm{i}\xi})+\delta^4(\mathrm{e}^{4\pi\mathrm{i}\xi}+\mathrm{e}^{-4\pi\mathrm{i}\xi})+\cdots\,,\quad(43)$$

where $\delta=\mathrm{e}^{\pi\mathrm{i}\tau}$.

Setting $\kappa'=2\pi\mathrm{i}\kappa$, $\iota'=2\pi\mathrm{i}\iota$, $\omega'=2\pi\mathrm{i}\omega$, $\xi'=\kappa'x+\iota'y+\omega't+\pi\mathrm{i}\tau$, we get

$$\begin{aligned}F&=1+\delta(\mathrm{e}^{2\pi\mathrm{i}\xi}+\mathrm{e}^{-2\pi\mathrm{i}\xi})+\delta^4(\mathrm{e}^{4\pi\mathrm{i}\xi}+\mathrm{e}^{-4\pi\mathrm{i}\xi})+\cdots\\&=1+\mathrm{e}^{\xi'}+\delta^2(\mathrm{e}^{-\xi'}+\mathrm{e}^{2\xi'})+\delta^6(\mathrm{e}^{-2\xi'}+\mathrm{e}^{3\xi'})+\cdots\\&\to 1+\mathrm{e}^{\xi'}\,,\quad \text{as}\quad \delta\to 0\,.\end{aligned}\quad(44)$$

It is interesting that if we can prove that

$$\omega'\to-\kappa'(\kappa'^2+\alpha)\,,\quad(45)$$

then the periodic wave solutions (42) turns to the soliton solution

$$\begin{aligned}&u=-2\ln(F)_x\,,\quad F=1+\mathrm{e}^{\xi'}\,,\\&\xi'=\kappa'x+\iota'y+\omega't+\pi\mathrm{i}\tau\,,\quad \omega'=-\kappa'(\kappa'^2+\alpha)\,.\end{aligned}\quad(46)$$

In fact, it is easy to see that

$$\begin{aligned}&a_{11}=-32\pi^2\iota(\delta^2+4\delta^4+\cdots)\,,\quad a_{12}=1+2\delta^2+2\delta^8+\cdots\,,\\&a_{21}=-8\pi^2\iota(\delta+9\delta^5+\cdots)\,,\quad a_{22}=2\delta+2\delta^5+\cdots\,,\\&b_1=(256\pi^4\kappa^3\iota-16\alpha\pi^2\kappa\iota)\delta^2+\cdots\,,\\&b_2=2(16\pi^4\kappa^3\iota-4\alpha\pi^2\kappa\iota)\delta+2(16\cdot3^4\pi^4\kappa^3\iota\\&\quad-4\alpha\cdot3^2\pi^2\kappa\iota)\delta^5+\cdots\,,\end{aligned}\quad(47)$$

which lead to

$$\begin{aligned}&a_{12}b_2-b_1a_{22}=(32\pi^4\kappa^3\iota-8\alpha\pi^2\kappa\iota)\delta+o(\delta)\,,\\&a_{11}a_{22}-a_{12}a_{21}=8\pi^2\iota\delta+o(\delta)\,,\end{aligned}\quad(48)$$

so we have $\omega\to-4\pi^2\kappa^3+\alpha\kappa$, as $\delta\to0$, which is equivalent to $\omega'\to-\kappa'(\kappa'^2+\alpha)$, as $\delta\to0$.

4.3 (2+1)-*Dimensional Sawada–Kotera Equation*

With the silimiar calculation process as Sec. 4.1, we have

$$\begin{aligned}\bar G(0)=&\sum_{n=-\infty}^{\infty}(-16n^2\pi^2\kappa\omega-1280n^4\pi^4\kappa^3\iota-80n^2\pi^2\iota^2\\&+4096n^6\pi^6\kappa^6-\gamma)\,\mathrm{e}^{2\pi\mathrm{i}n^2\tau}=0\,,\\ \bar G(1)=&\sum_{n=-\infty}^{\infty}((-4(2n-1)^2\pi^2\kappa\omega\\&-80(2n-1)^4\pi^4\kappa^3\iota-4(2n-1)^2\pi^2\iota^2\\&+64(2n-1)^6\pi^6\kappa^6)-\gamma)\,\mathrm{e}^{\pi\mathrm{i}(2n^2-2n+1)\tau}=0\,.\end{aligned}\quad(49)$$

For the sake of convenience, we denote that

$$q_1(n)=\mathrm{e}^{2\pi\mathrm{i}n^2\tau}\,,\quad q_2(n)=\mathrm{e}^{\pi\mathrm{i}(2n^2-2n+1)\tau}\,,$$

$$a_{11}=\sum_{n=-\infty}^{\infty}-16n^2\pi^2\kappa q_1(n)\,,\quad a_{12}=\sum_{n=-\infty}^{\infty}q_1(n)\,,$$

$$a_{21} = \sum_{n=-\infty}^{\infty} -4(2n-1)^2\pi^2\kappa q_2(n)\,, \quad a_{22} = \sum_{n=-\infty}^{\infty} q_2(n)\,,$$

$$b_1 = \sum_{n=-\infty}^{\infty} (-1280n^4\pi^4\kappa^3\iota - 80n^2\pi^2\iota^2 + 4096n^6\pi^6\kappa^6)q_1(n)\,,$$

$$b_2 = \sum_{n=-\infty}^{\infty} (-80(2n-1)^4\pi^4\kappa^3\iota - 20(2n-1)^2\pi^2\iota^2 + 64(2n-1)^6\pi^6\kappa^6)q_2(n)\,, \tag{50}$$

then Eq. (49) can be written as

$$a_{11}\omega + b_1 - \varsigma a_{12} = 0\,, \quad a_{21}\omega + b_2 - \varsigma a_{22} = 0\,. \tag{51}$$

Solving this system, we obtain

$$\omega = \frac{a_{12}b_2 - b_1a_{22}}{a_{11}a_{22} - a_{21}a_{12}}\,, \quad \varsigma = \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}}\,. \tag{52}$$

Thus, we obtain the periodic wave solution

$$u = 6\ln(F)_x\,, \tag{53}$$

where F and ω are given by Eqs. (23) and (52), respectively.

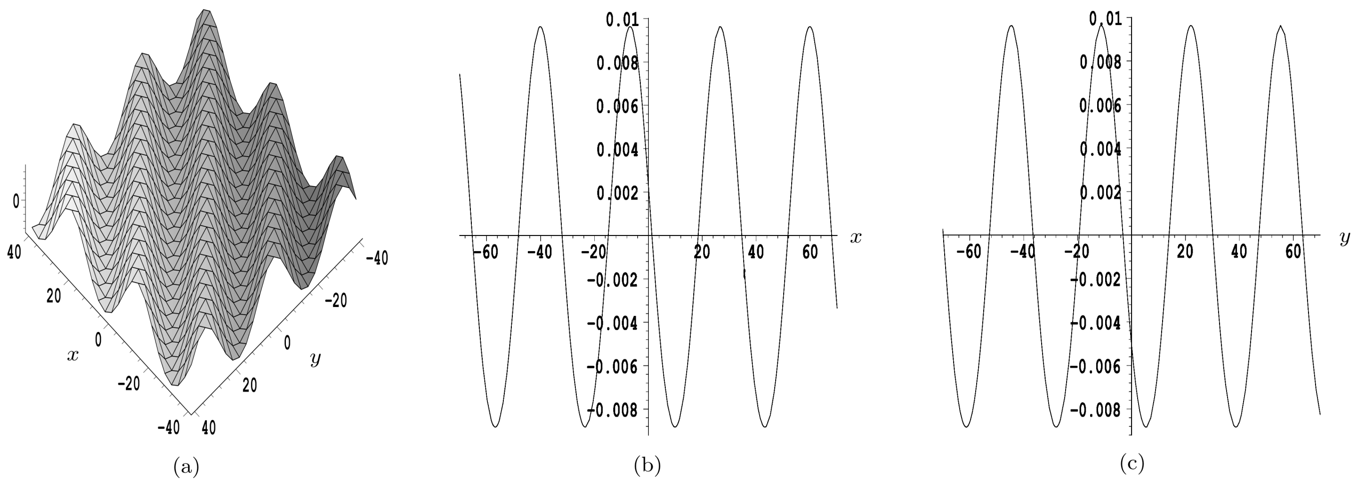

Fig. 3 (a) The figure represents periodic solution (53) with $\kappa = 0.03$, $\tau = i$, $\iota = 0.03$; (b) Along x-axis; (c) Along $y-$axis.

From Eq. (23), we write F as

$$F = 1 + \delta(\mathrm{e}^{2\pi\mathrm{i}\xi} + \mathrm{e}^{-2\pi\mathrm{i}\xi}) + \delta^4(\mathrm{e}^{4\pi\mathrm{i}\xi} + \mathrm{e}^{-4\pi\mathrm{i}\xi}) + \cdots\,, \tag{54}$$

where $\delta = \mathrm{e}^{\pi\mathrm{i}\tau}$.

Setting

$$\kappa' = 2\pi\mathrm{i}\kappa\,, \quad \iota' = 2\pi\mathrm{i}\iota\,, \quad \omega' = 2\pi\mathrm{i}\omega\,,$$
$$\xi' = \kappa' x + \iota' y + \omega' t + \pi\mathrm{i}\tau\,,$$

we get

$$\begin{aligned} F &= 1 + \delta(\mathrm{e}^{2\pi\mathrm{i}\xi} + \mathrm{e}^{-2\pi\mathrm{i}\xi}) + \delta^4(\mathrm{e}^{4\pi i\xi} + \mathrm{e}^{-4\pi\mathrm{i}\xi}) + \cdots \\ &= 1 + \mathrm{e}^{\xi'} + \delta^2(\mathrm{e}^{-\xi'} + \mathrm{e}^{2\xi'}) + \delta^6(\mathrm{e}^{-2\xi'} + \mathrm{e}^{3\xi'}) + \cdots \\ &\to 1 + \mathrm{e}^{\xi'}\,, \quad \text{as} \quad \delta \to 0\,. \end{aligned} \tag{55}$$

It is interesting that if we can prove that

$$\omega' \to 5\kappa'^2 - \frac{5\iota'^2}{\kappa'} + \kappa'^5\,, \tag{56}$$

then the periodic wave solutions (53) turns to the soliton solution

$$\begin{aligned} &u = -2\ln(F)_x\,, \quad F = 1 + \mathrm{e}^{\xi'}\,, \\ &\xi' = \kappa' x + \iota' y + \omega' t + \pi\mathrm{i}\tau\,, \\ &\omega' = 5\kappa'^2 - \frac{5\iota'^2}{\kappa'} + \kappa'^5\,. \end{aligned} \tag{57}$$

In fact, it is easy to see that

$$\begin{aligned} &a_{11} = -32\pi^2\kappa(\delta^2 + 4\delta^4 + \cdots)\,, \\ &a_{12} = 1 + 2\delta^2 + 2\delta^8 + \cdots\,, \\ &a_{21} = -8\pi^2\kappa(\delta + 9\delta^5 + \cdots)\,, \\ &a_{22} = 2\delta + 2\delta^5 + \cdots\,, \\ &b_1 = (-1280\pi^4\kappa^3\iota - 80\pi^2\iota^2 + 4096\pi^6\kappa^6)\delta^2 + \cdots\,, \\ &b_2 = 2(-80\pi^4\kappa^3\iota - 20\pi^2\iota^2 + 64\pi^6\kappa^6)\delta + 2(-80\cdot 3^4\pi^4\kappa^3\iota \\ &\qquad - 4\cdot 3^2\pi^2\iota^2 + 64\cdot 3^6\pi^6\kappa^6)\delta^5 + \cdots\,, \end{aligned} \tag{58}$$

which lead to

$$\begin{aligned} &a_{12}b_2 - b_1a_{22} = (-160\pi^4\kappa^3\iota - 40\pi^2\iota^2 + 128\pi^6\kappa^6)\delta + o(\delta)\,, \\ &a_{11}a_{22} - a_{12}a_{21} = 8\pi^2\kappa\delta + o(\delta)\,, \end{aligned} \tag{59}$$

so we have

$$\omega \to -20\pi^2\kappa^2\iota - \frac{5\iota^2}{\kappa} + 16\pi^4\kappa^5\,, \quad \text{as} \quad \delta \to 0\,,$$

which is equivalent to $\omega' \to 5\kappa'^2 - 5\iota'^2/\kappa' + \kappa'^5$, as $\delta \to 0$.

5 Conclusions

In this paper, we investigate $(2+1)$-dimensional KdV equation, extended (2+1)-dimensional shallow water wave equation, and $(2+1)$-dimensional Sawada–Kotera equation. Their bilinear forms are given by applying binary

Bell polynomials which has proved to be a quick and simple method. Then, we get their periodic solutions with the help of Riemann theta function and Hirota method. Futhermore, we obtain the corresponding soliton solutions via asymptotic analysis for their periodic wave solutions.

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