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Publisher: Taylor & Francis

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Journal of Nonlinear Mathematical Physics

Publication details, including instructions for authors and subscription information:
<http://www.tandfonline.com/loi/tnmp20>

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Published online: 14 Oct 2014.

To cite this article: Hongmin Li, Yuqi Li & Yong Chen (2014) Bi-Hamiltonian structure of multi-component Novikov equation, Journal of Nonlinear Mathematical Physics, 21:4, 509-520, DOI: [10.1080/14029251.2014.975522](https://doi.org/10.1080/14029251.2014.975522)

To link to this article: <http://dx.doi.org/10.1080/14029251.2014.975522>

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Bi-Hamiltonian structure of multi-component Novikov equation

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Received 28 March 2014

Accepted 12 June 2014

In this paper, we present the multi-component Novikov equation and derive its bi-Hamiltonian structure.

Keywords: bi-Hamiltonian structure; multi-component Novikov equation.

2000 Mathematics Subject Classification: 37K10, 35Q51, 35Q58

1. Introduction

The Camassa-Holm (CH) equation

$$m_t + um_x + 2u_x m = 0, \quad m = u - u_{xx}, \quad (1.1)$$

was derived by Camassa and Holm from an approximation to the incompressible Euler equations [3], and found to be completely integrable with a Lax pair and associated bi-Hamiltonian structure [4]. The CH equation has been studied in a large number of papers [1, 2, 5, 6, 9, 10, 14–18]. Interestingly, the CH equation is linked with the first negative flow of the KdV hierarchy by reciprocal transformation [10]. But unlike KdV equation, the CH equation admits peaked soliton solutions [1–4]. Besides the CH equation, many other systems with peaked soliton solutions have been constructed.

In 1999, Degasperis-Procesi presented a new equation with peaked solutions

$$m_t + um_x + 3u_x m = 0, \quad m = u - u_{xx}, \quad (1.2)$$

which is known as DP equation [7]. The DP equation is integrable with a bi-Hamiltonian structure and a Lax pair associated with a third-order spectral problem [8]. Both CH equation and DP equation have nonlinear terms that are quadratic.

Recently, Vladimir Novikov found a new equation with cubic nonlinearity

$$m_t + u^2 m_x + 3uu_x m = 0, \quad m = u - u_{xx}, \quad (1.3)$$

from his symmetry classification study of nonlocal partial differential equations [20]. In [12], Hone and Wang gave a matrix Lax pair, infinitely many conserved quantities as well as a bi-Hamiltonian

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structure of the Eq. (1.3) which is also named Novikov equation. Very recently, Geng and Xue constructed a two-component generalization for the Novikov equation (1.3)

$$\begin{aligned} m_t + 3u_x vm + uvm_x &= 0, \\ n_t + 3v_x un + uvn_x &= 0, \\ m &= u - u_{xx}, \quad n = v - v_{xx}, \end{aligned} \tag{1.4}$$

which was associated with a 3×3 matrix spectral problem, they also gave the N peakons, infinite sequence of conserved quantities and a Hamiltonian structure [11]. In 2013, Li and Liu showed the system (1.4) was indeed a bi-Hamiltonian structure and got the Hamiltonian operators found by Hone and Wang for the Novikov equation (1.3) using the proper Dirac reduction [19].

The purpose of this paper is to construct the bi-Hamiltonian system for the multi-component Novikov equation

$$\begin{aligned} q_{it} &= \sum_{j=1}^n [-2q_i u_{jx} v_j - q_i u_j v_{jx} - q_{ix} u_j v_j - u_{ix} q_j v_j + u_i q_j v_{jx}], \\ r_{it} &= \sum_{j=1}^n [-2r_i u_j v_{jx} - r_i u_{jx} v_j - r_{ix} u_j v_j - v_{ix} r_j u_j + v_i r_j u_{jx}], \\ q_i &= u_i - u_{ixx}, \quad r_i = v_i - v_{ixx}, \quad i = 1, 2, \dots, n, \end{aligned} \tag{1.5}$$

where $q_{it} = \frac{\partial q_i(x,t)}{\partial t}$, $r_{it} = \frac{\partial r_i(x,t)}{\partial t}$, $u_{jx} = \frac{\partial u_j(x,t)}{\partial x}$, $v_{jx} = \frac{\partial v_j(x,t)}{\partial x}$, $i, j = 1, 2, \dots, n$, and so on. When $n = 1, q = m, r = n$ the Eq. (1.5) reduces to the two-component system (1.4). Moreover the system (1.5) can reduce to the Eqs. (1.2) and (1.3) as $n = 1, q = m, v = 1$ and $n = 1, q = m, v = u$ respectively. It's worthwhile to note that there is also research on other multi-component CH-type equations [13, 22, 23].

2. Bi-Hamiltonian structure of multi-component Novikov equation

Possession of the bi-Hamiltonian structure is an important property of soliton equations and all soliton equations are turn out to be bi-Hamiltonian systems. In this section, we derive the bi-Hamiltonian structure of multi-component Novikov equation (1.5).

The multi-component Novikov equation (1.5) has the equivalent form

$$\begin{aligned} Q_t &= -2\langle U_x, V \rangle Q - \langle U, V_x \rangle Q - \langle U, V \rangle Q_x - \langle Q, V \rangle U_x + \langle Q, V_x \rangle U, \\ R_t &= -2\langle U, V_x \rangle R - \langle U_x, V \rangle R - \langle U, V \rangle R_x - \langle R, U \rangle V_x + \langle R, U_x \rangle V, \end{aligned} \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product and Q, R, U, V are the n -component vector potentials defined as

$$\begin{aligned} Q &= (q_1, q_2, \dots, q_n)^T, \quad R = (r_1, r_2, \dots, r_n)^T, \quad U = (u_1, u_2, \dots, u_n)^T, \quad V = (v_1, v_2, \dots, v_n)^T, \\ Q &= U - U_{xx}, \quad R = V - V_{xx}, \end{aligned}$$

and T is the transpose of a vector, and $U_x^T = \frac{\partial U^T}{\partial x}$ as well.

The system (2.1) arises as a zero-curve equation

$$M_t - N_x + [M, N] = 0, \tag{2.2}$$

this being the compatibility condition of the $(n + 2) \times (n + 2)$ matrix spectral problem

$$\varphi_x = M\varphi, \quad \varphi_t = N\varphi, \tag{2.3}$$

with

$$M = \begin{pmatrix} 0 & \lambda Q^T & 1 \\ \mathbf{0}^T & \mathbf{0}_{n \times n} & \lambda R \\ 1 & \mathbf{0} & 0 \end{pmatrix}, \quad N = \begin{pmatrix} -U_x^T V & \frac{U_x^T}{\lambda} - \lambda U^T V Q^T & U_x^T V_x \\ \frac{V}{\lambda} & -\frac{I_n}{\lambda^2} + V U_x^T - V_x U^T & -\frac{V_x}{\lambda} - \lambda U^T V R \\ -U^T V & \frac{U_x^T}{\lambda} & U^T V_x \end{pmatrix}, \tag{2.4}$$

where $\mathbf{0}$ and $\mathbf{0}_{n \times n}$ are respectively n dimension row vector and $n \times n$ zero matrix, λ is a constant spectral parameter and I_n denotes the $n \times n$ identity matrix. It is worth noting that the matrix spectral problem (2.3) is the vector prolongation of the spectral problem in [11], so the system (2.1) is also a negative flow in the hierarchy.

To compute the bi-Hamiltonian structure of the system (2.1), we consider an integrable hierarchy which consists of (2.1), i.e., the N in time part of (2.3) is

$$N = \begin{pmatrix} N_{1,1} & A & N_{1,n+2} \\ B & S & C \\ N_{n+2,1} & D & N_{n+2,n+2} \end{pmatrix}, \tag{2.5}$$

where B , C and A , D are respectively n dimension column and row vectors depending on vector potentials Q , R and spectral parameter λ . S and the remaining entries are respectively $n \times n$ matrix and functions depending on vector potentials Q , R and spectral parameter λ .

Substituting M and N respectively in (2.4) and (2.5) into (2.2), we get

$$\begin{aligned} C &= -B_x + \lambda R N_{n+2,1}, \quad A = D_x + \lambda N_{n+2,1} Q^T, \quad N_{11} = (N_{n+2,1})_x + N_{n+2,n+2}, \\ S &= \lambda \partial^{-1} (R D - B Q^T), \quad N_{n+2,1} = \lambda (\partial^3 - 4\partial)^{-1} (3Q^T B_x + Q_x^T B + 3D_x R + D R_x), \\ (N_{1,n+2})_x + (N_{n+2,1})_x + \lambda (Q^T B_x + D_x R) &= 0, \quad N_{n+2,n+2} = -\frac{1}{2} (N_{n+2,1})_x + \frac{1}{2} \lambda \partial^{-1} (Q^T B - D R), \end{aligned}$$

and

$$\begin{pmatrix} Q \\ R \end{pmatrix}_t = (\lambda^{-1} \mathcal{H} + \lambda \mathcal{J}) \begin{pmatrix} B \\ D^T \end{pmatrix}, \tag{2.6}$$

where

$$\mathcal{H} = \begin{pmatrix} 0 & (\partial^2 - 1) I_n \\ (1 - \partial^2) I_n & 0 \end{pmatrix}, \tag{2.7}$$

$$\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2, \tag{2.8}$$

with

$$\begin{aligned} \mathcal{J}_1 &= \begin{pmatrix} \frac{3}{2} Q \partial + Q_x \\ \frac{3}{2} R \partial + R_x \end{pmatrix} (\partial^3 - 4\partial)^{-1} (3Q^T \partial + Q_x^T 3R^T \partial + R_x^T), \\ \mathcal{J}_2 &= \begin{pmatrix} \frac{1}{2} Q \partial^{-1} Q^T + (Q \partial^{-1} Q^T)^T & -\frac{1}{2} Q \partial^{-1} R^T - Q^T \partial^{-1} R I_n \\ -\frac{1}{2} R \partial^{-1} Q^T - R^T \partial^{-1} Q I_n & \frac{1}{2} R \partial^{-1} R^T + (R \partial^{-1} R^T)^T \end{pmatrix}. \end{aligned}$$

Obviously, the operators \mathcal{K} and \mathcal{J} are skew-symmetric, furthermore the operator \mathcal{K} is a Hamiltonian operator. In the following, we show how the Jacobi identity for the operator \mathcal{J} and compatibility for the operators \mathcal{K} and \mathcal{J} may be checked by the multivector approach to Hamiltonian systems in infinite dimensions, as described in the work of Olver [21].

Our main results are summarized as

Theorem 2.1. *The multi-component Novikov system (2.1) may be reformulated as a bi-Hamiltonian system*

$$\begin{pmatrix} Q \\ R \end{pmatrix}_t = \mathcal{K} \begin{pmatrix} \frac{\delta H_0}{\delta m} \\ \frac{\delta H_0}{\delta n} \end{pmatrix} = \mathcal{J} \begin{pmatrix} \frac{\delta H_1}{\delta m} \\ \frac{\delta H_1}{\delta n} \end{pmatrix} \tag{2.9}$$

where the operators \mathcal{K} and \mathcal{J} are given by (2.7) and (2.8) respectively, and

$$\begin{aligned} H_0 &= \frac{1}{2} \int \langle Q, V \rangle \langle U_x, V \rangle - \langle R, U \rangle \langle V_x, U \rangle + (\langle R, U_x \rangle - \langle Q, V_x \rangle) \langle U, V \rangle dx, \\ H_1 &= \frac{1}{2} \int \langle Q, V \rangle + \langle R, U \rangle dx. \end{aligned}$$

Before the proof of the Theorem 2.1, please allow us give a brief explanation of the Olver’s technique [21]. Let θ denote the basic uni-vector corresponding to potential, \mathcal{D} is any skew-symmetry operator depending on a spatial variable x and the potential. In the proof procedure, we have mainly used the following three properties:

- the basic property of wedge product

$$\int \xi \wedge \eta dx = (-1)^{mn} \int \eta \wedge \xi dx, \tag{2.10}$$

for any m -form ξ and n -form η .

- the skew-symmetry of the operator \mathcal{D}

$$\int \xi \wedge \mathcal{D}\eta dx = - \int (\mathcal{D}\xi) \wedge \eta dx. \tag{2.11}$$

- the prolongation

$$-\text{PrV}_{\mathcal{D}\theta}(\theta \wedge \mathcal{D}\theta) = \theta \wedge \text{PrV}_{\mathcal{D}\theta}(\mathcal{D}) \wedge \theta, \tag{2.12}$$

the minus sign coming from the fact that we have interchanged a wedge product of θ ’s using the formula (2.10).

Proof. Assume that $\theta_1 = (\theta_{11}, \theta_{12}, \dots, \theta_{1n})^T, \theta_2 = (\theta_{21}, \theta_{22}, \dots, \theta_{2n})^T$ are the basic uni-vectors corresponding to Q and R respectively. We know that the operator \mathcal{J} is the Hamiltonian if and only if

$$\text{PrV}_{\mathcal{J}\theta}(\Theta_{\mathcal{J}}) = \text{PrV}_{\mathcal{J}\theta}(\Theta_{\mathcal{J}_1}) + \text{PrV}_{\mathcal{J}\theta}(\Theta_{\mathcal{J}_2}) = 0, \tag{2.13}$$

where $\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$ and

$$\Theta_{\mathcal{J}} = \frac{1}{2} \int (\theta \wedge \mathcal{J}\theta) dx = \Theta_{\mathcal{J}_1} + \Theta_{\mathcal{J}_2} = \frac{1}{2} \int (\theta \wedge \mathcal{J}_1\theta) dx + \frac{1}{2} \int (\theta \wedge \mathcal{J}_2\theta) dx,$$

is the associated bi-vector of \mathcal{J} .

To check whether \mathcal{H} and \mathcal{J} form a bi-Hamiltonian pair, we only need to prove

$$\text{PrV}_{\mathcal{H}\theta}(\Theta_{\mathcal{J}}) = \text{PrV}_{\mathcal{H}\theta}(\Theta_{\mathcal{J}_1}) + \text{PrV}_{\mathcal{H}\theta}(\Theta_{\mathcal{J}_2}) = 0. \quad (2.14)$$

The proof of the Theorem 2.1 is rather technical and lengthy, so are given in Appendix A. \square

According to the bi-Hamiltonian theory, the Hamiltonian pair \mathcal{H}, \mathcal{J} gives rise to the hereditary recursion operator $\mathcal{R} = \mathcal{J}\mathcal{H}^{-1}$. The recursion operator acting on a seed symmetry of the soliton equation can generate an infinite sequence of symmetries. Assume the seed symmetry of (2.1) is $(\mathbf{0}, \mathbf{0})^T$, then we get the sequence of symmetries

$$\sigma_n = \mathcal{R}^n(\mathbf{0}, \mathbf{0})^T, \quad n = 0, 1, 2, \dots \quad (2.15)$$

As $n = 1$, the above expression (2.15) is the local symmetry

$$\sigma_1 = \begin{pmatrix} -2\langle U_x, V \rangle Q - \langle U, V_x \rangle Q - \langle U, V \rangle Q_x - \langle Q, V \rangle U_x + \langle Q, V_x \rangle U \\ -2\langle U, V_x \rangle R - \langle U_x, V \rangle R - \langle U, V \rangle R_x - \langle R, U \rangle V_x + \langle R, U_x \rangle V \end{pmatrix}$$

which is just the right side of the equality (2.1). This is natural.

But when $n = 2$, the recursion formula (2.15) leads to the nonlocal symmetry

$$\sigma_2 = \begin{pmatrix} \sigma_{21} \\ \sigma_{22} \end{pmatrix}, \quad (2.16)$$

where

$$\begin{aligned} \sigma_{21} &= \left(\frac{3}{2}Q\partial + Q_x\right)(\partial^3 - 4\partial)^{-1}(3\langle Q, \Phi_{1x} \rangle + \langle Q_x, \Phi_1 \rangle - 3\langle R, \Phi_{2x} \rangle - \langle R_x, \Phi_2 \rangle) \\ &\quad + \frac{1}{2}Q\partial^{-1}(\langle Q, \Phi_1 \rangle + \langle R, \Phi_2 \rangle) + (Q\partial^{-1}Q^T)^T\Phi_1 + Q^T\partial^{-1}R I_n\Phi_2, \\ \sigma_{22} &= \left(\frac{3}{2}R\partial + R_x\right)(\partial^3 - 4\partial)^{-1}(3\langle Q, \Phi_{1x} \rangle + \langle Q_x, \Phi_1 \rangle - 3\langle R, \Phi_{2x} \rangle - \langle R_x, \Phi_2 \rangle) \\ &\quad - \frac{1}{2}R\partial^{-1}(\langle Q, \Phi_1 \rangle + \langle R, \Phi_2 \rangle) - (R\partial^{-1}R^T)^T\Phi_2 - R^T\partial^{-1}Q I_n\Phi_1, \end{aligned}$$

and Φ_1, Φ_2 are nonlocal variables defined by

$$\Phi_1 = (1 - \partial^2)^{-1}R_t, \quad \Phi_2 = (1 - \partial^2)^{-1}Q_t.$$

For example, let us consider the reduction: the Q and R are both one-dimensional scalar functions, and $R = Q$ as well. A local symmetry under the reduction must be local. We will demonstrate the symmetry σ_2 under the reduction is nonlocal. Set the symmetry σ_2 in (2.16) under the constraint is $\widehat{\sigma}_2$, then

$$\widehat{\sigma}_2 = \begin{pmatrix} 3Q\partial^{-1}(Q\Psi) \\ -3Q\partial^{-1}(Q\Psi) \end{pmatrix},$$

where $\Psi = (1 - \partial^2)^{-1}Q_t$ with $Q_t = -4u^2u_x + u^2u_{xxx} + 3uu_xu_{xx}$. If $\widehat{\sigma}_2$ is local, Ψ must be a local variable, i.e., there is a function $f(x, u, u_x)$ that satisfies

$$(1 - \partial^2)^{-1}(-4u^2u_x + u^2u_{xxx} + 3uu_xu_{xx}) = f(x, u, u_x). \quad (2.17)$$

But after calculation, we find there is no function $f(x, u, u_x)$ that satisfies the equality (2.17), so $\widehat{\sigma}_2$ is nonlocal and then σ_2 is a nonlocal symmetry.

Therefore, from the recursion formula (2.15), we can obtain an infinite sequence of higher order nonlocal symmetries of the system (2.1).

Acknowledgments

The project is supported by the National Natural Science Foundation of China (Grants No. 11275072 and No. 11375090), Research Fund for the Doctoral Program of Higher Education of China (No. 20120076110024), Global Change Research Program of China (No. 2015CB953900), National Natural Science Foundation of China (No. 11435005), the Innovative Research Team Program of the National Natural Science Foundation of China (Grant No. 61321064), Shanghai Knowledge Service Platform for Trustworthy Internet of Things under Grant No. ZF1213, Talent Fund and K.C. Wong Magna Fund in Ningbo University.

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Appendix A.

First, we prove that the operator \mathcal{J} is Hamiltonian, namely to verify (2.13). To simplify the presentation and calculations, we introduce \tilde{Q} and \tilde{R} as

$$\tilde{Q} = (\partial^3 - 4\partial)^{-1}(3Q^T \theta_{1x} + Q_x^T \theta_1), \quad \tilde{R} = (\partial^3 - 4\partial)^{-1}(3R^T \theta_{2x} + R_x^T \theta_2). \tag{A.1}$$

From (2.8), we have

$$\mathcal{J}_1 \theta = \begin{pmatrix} \frac{3}{2}Q(\tilde{Q} + \tilde{R})_x + Q_x(\tilde{Q} + \tilde{R}) \\ \frac{3}{2}R(\tilde{Q} + \tilde{R})_x + R_x(\tilde{Q} + \tilde{R}) \end{pmatrix} = \begin{pmatrix} \frac{3}{2}q_1(\tilde{Q} + \tilde{R})_x + q_{1x}(\tilde{Q} + \tilde{R}) \\ \vdots \\ \frac{3}{2}q_n(\tilde{Q} + \tilde{R})_x + q_{nx}(\tilde{Q} + \tilde{R}) \\ \frac{3}{2}r_1(\tilde{Q} + \tilde{R})_x + r_{1x}(\tilde{Q} + \tilde{R}) \\ \vdots \\ \frac{3}{2}r_n(\tilde{Q} + \tilde{R})_x + r_{nx}(\tilde{Q} + \tilde{R}) \end{pmatrix}, \tag{A.2}$$

and

$$\begin{aligned} \mathcal{J}_2 \theta &= \begin{pmatrix} \frac{1}{2}Q\partial^{-1}(Q^T \theta_1 - R^T \theta_2) + (Q\partial^{-1}Q^T)^T \theta_1 - Q^T \partial^{-1}R I_n \theta_2 \\ -\frac{1}{2}R\partial^{-1}(Q^T \theta_1 - R^T \theta_2) - R^T \partial^{-1}Q I_n \theta_1 + (R\partial^{-1}R^T)^T \theta_2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}q_1\partial^{-1}(Q^T \theta_1 - R^T \theta_2) + \sum_{i=1}^n q_i\partial^{-1}(q_1\theta_{1i} - r_i\theta_{2i}) \\ \vdots \\ \frac{1}{2}q_n\partial^{-1}(Q^T \theta_1 - R^T \theta_2) + \sum_{i=1}^n q_i\partial^{-1}(q_n\theta_{1i} - r_i\theta_{2n}) \\ -\frac{1}{2}r_1\partial^{-1}(Q^T \theta_1 - R^T \theta_2) - \sum_{i=1}^n r_i\partial^{-1}(q_i\theta_{11} - r_1\theta_{2i}) \\ \vdots \\ -\frac{1}{2}r_n\partial^{-1}(Q^T \theta_1 - R^T \theta_2) - \sum_{i=1}^n r_i\partial^{-1}(q_i\theta_{1n} - r_n\theta_{2i}) \end{pmatrix}. \end{aligned} \tag{A.3}$$

Then the associated bi-vectors for \mathcal{J}_1 and \mathcal{J}_2 are respectively

$$\begin{aligned} \Theta_{\mathcal{J}_1} &= \frac{1}{2} \int (\theta \wedge \mathcal{J}_1 \theta) dx \\ &= \frac{1}{2} \int \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \wedge \begin{pmatrix} \frac{3}{2}Q(\tilde{Q} + \tilde{R})_x + Q_x(\tilde{Q} + \tilde{R}) \\ \frac{3}{2}R(\tilde{Q} + \tilde{R})_x + R_x(\tilde{Q} + \tilde{R}) \end{pmatrix} dx \\ &= \frac{1}{2} \sum_{j=1}^n \int \left[\frac{3}{2}(\theta_{1j} \wedge q_j + \theta_{2j} \wedge r_j)(\tilde{Q} + \tilde{R})_x + (\theta_{1j} \wedge q_{jx} + \theta_{2j} \wedge r_{jx})(\tilde{Q} + \tilde{R}) \right] dx \\ &= -\frac{1}{4} \sum_{j=1}^n \int (q_{jx}\theta_{1j} + 3q_j\theta_{1jx} + r_{jx}\theta_{2j} + 3r_j\theta_{2jx}) \wedge (\tilde{Q} + \tilde{R}) dx, \end{aligned} \tag{A.4}$$

and

$$\begin{aligned}
 \Theta_{\mathcal{J}_2} &= \frac{1}{2} \int (\theta \wedge \mathcal{J}_2 \theta) dx \\
 &= \frac{1}{2} \int \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \wedge \begin{pmatrix} \frac{1}{2} Q \partial^{-1} (Q^T \theta_1 - R^T \theta_2) + (Q \partial^{-1} Q^T)^T \theta_1 - Q^T \partial^{-1} R I_n \theta_2 \\ -\frac{1}{2} R \partial^{-1} (Q^T \theta_1 - R^T \theta_2) - R^T \partial^{-1} Q I_n \theta_1 + (R \partial^{-1} R^T)^T \theta_2 \end{pmatrix} dx \\
 &= \frac{1}{2} \int \left[\frac{1}{2} (\theta_1 \wedge Q - \theta_2 \wedge R) \partial^{-1} (Q^T \theta_1 - R^T \theta_2) + \theta_1 \wedge (Q \partial^{-1} Q^T)^T \theta_1 - \theta_1 \wedge Q^T \partial^{-1} R I_n \theta_2 \right. \\
 &\quad \left. + \theta_2 \wedge (R \partial^{-1} R^T)^T \theta_2 - \theta_2 \wedge R^T \partial^{-1} Q I_n \theta_1 \right] dx \\
 &= \frac{1}{2} \sum_{i,j=1}^n \int \left[\frac{1}{2} (\theta_{1j} \wedge q_j - \theta_{2j} \wedge r_j) \partial^{-1} (q_i \theta_{1i} - r_i \theta_{2i}) + \theta_{1j} \wedge q_i \partial^{-1} q_j \theta_{1i} + \theta_{2j} \wedge r_i \partial^{-1} r_j \theta_{2i} \right. \\
 &\quad \left. - \theta_{1j} \wedge q_i \partial^{-1} r_i \theta_{2j} - \theta_{2j} \wedge r_i \partial^{-1} q_i \theta_{1j} \right] dx \\
 &= \frac{1}{2} \sum_{i,j=1}^n \int \left[\frac{1}{2} (\theta_{1j} \wedge q_j - \theta_{2j} \wedge r_j) \partial^{-1} (q_i \theta_{1i} - r_i \theta_{2i}) + \theta_{1j} \wedge q_i \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) \right. \\
 &\quad \left. + \theta_{2j} \wedge r_i \partial^{-1} (r_j \theta_{2i} - q_i \theta_{1j}) \right] dx. \tag{A.5}
 \end{aligned}$$

In the equality (A.4), we have applied integration by parts which is a special case of (2.11) to the terms which contain explicitly $(\tilde{Q} + \tilde{R})_x$.

We calculate

$$\begin{aligned}
 \text{PrV}_{\mathcal{J}\theta}(\Theta_{\mathcal{J}_1}) &= \sum_{j=1}^n \int \left[-\frac{3}{2} \theta_{1jx} \wedge \left(\frac{3}{2} q_j (\tilde{Q} + \tilde{R})_x + q_{jx} (\tilde{Q} + \tilde{R}) + \frac{1}{2} q_j \partial^{-1} (Q^T \theta_1 - R^T \theta_2) \right) \right. \\
 &\quad \left. + \sum_{i=1}^n q_i \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) - \frac{1}{2} \theta_{1j} \wedge \left(\frac{3}{2} q_j (\tilde{Q} + \tilde{R})_x + q_{jx} (\tilde{Q} + \tilde{R}) \right) \right. \\
 &\quad \left. + \frac{1}{2} q_j \partial^{-1} (Q^T \theta_1 - R^T \theta_2) + \sum_{i=1}^n q_i \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) \right]_x \\
 &\quad - \frac{3}{2} \theta_{2jx} \wedge \left(\frac{3}{2} r_j (\tilde{Q} + \tilde{R})_x + r_{jx} (\tilde{Q} + \tilde{R}) - \frac{1}{2} r_j \partial^{-1} (Q^T \theta_1 - R^T \theta_2) \right) \\
 &\quad - \sum_{i=1}^n r_i \partial^{-1} (q_i \theta_{1j} - r_j \theta_{2i}) - \frac{1}{2} \theta_{2j} \wedge \left(\frac{3}{2} r_j (\tilde{Q} + \tilde{R})_x + r_{jx} (\tilde{Q} + \tilde{R}) \right) \\
 &\quad \left. - \frac{1}{2} r_j \partial^{-1} (Q^T \theta_1 - R^T \theta_2) - \sum_{i=1}^n r_i \partial^{-1} (q_i \theta_{1j} - r_j \theta_{2i}) \right] \wedge (\tilde{Q} + \tilde{R}) dx \\
 &= \sum_{j=1}^n \int \left[\left(-\frac{9}{4} q_j \theta_{1jx} - \frac{5}{4} q_{jx} \theta_{1j} - \frac{9}{4} r_j \theta_{2jx} - \frac{5}{4} r_{jx} \theta_{2j} \right) \wedge (\tilde{Q} + \tilde{R})_x \right. \\
 &\quad \left. + \left(-\frac{3}{4} q_j \theta_{1j} - \frac{3}{4} r_j \theta_{2j} \right) \wedge (\tilde{Q} + \tilde{R})_{xx} - \frac{1}{4} (q_j \theta_{1j} - r_j \theta_{2j}) \wedge (Q^T \theta_1 - R^T \theta_2) \right. \\
 &\quad \left. + \left(-\frac{3}{4} q_j \theta_{1jx} - \frac{1}{4} q_{jx} \theta_{1j} + \frac{3}{4} r_j \theta_{2jx} + \frac{1}{4} r_{jx} \theta_{2j} \right) \wedge \partial^{-1} (Q^T \theta_1 - R^T \theta_2) \right. \\
 &\quad \left. + \sum_{i=1}^n \left(\left(-\frac{3}{2} q_i \theta_{1jx} - \frac{1}{2} q_{ix} \theta_{1j} \right) \wedge \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) - \frac{1}{2} q_i \theta_{1j} \wedge q_j \theta_{1i} \right. \right. \\
 &\quad \left. \left. + \left(\frac{3}{2} r_i \theta_{2jx} + \frac{1}{2} r_{ix} \theta_{2j} \right) \wedge \partial^{-1} (q_i \theta_{1j} - r_j \theta_{2i}) - \frac{1}{2} r_i \theta_{2j} \wedge r_j \theta_{2i} \right) \right] \wedge (\tilde{Q} + \tilde{R}) dx
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j=1}^n \int [(-\frac{9}{4}q_j\theta_{1jx} - \frac{5}{4}q_{jx}\theta_{1j} - \frac{9}{4}r_j\theta_{2jx} - \frac{5}{4}r_{jx}\theta_{2j}) \wedge (\tilde{Q} + \tilde{R})_x \\
 &\quad + (\frac{3}{4}q_{jx}\theta_{1j} + \frac{3}{4}q_j\theta_{1jx} + \frac{3}{4}r_{jx}\theta_{2j} + \frac{3}{4}r_j\theta_{2jx}) \wedge (\tilde{Q} + \tilde{R})_x \\
 &\quad + (-\frac{3}{4}q_j\theta_{1jx} - \frac{1}{4}q_{jx}\theta_{1j} + \frac{3}{4}r_j\theta_{2jx} + \frac{1}{4}r_{jx}\theta_{2j}) \wedge \partial^{-1}(Q^T\theta_1 - R^T\theta_2) \\
 &\quad + \sum_{i=1}^n ((-\frac{3}{2}q_i\theta_{1jx} - \frac{1}{2}q_{ix}\theta_{1j}) \wedge \partial^{-1}(q_j\theta_{1i} - r_i\theta_{2j}) \\
 &\quad + (\frac{3}{2}r_i\theta_{2jx} + \frac{1}{2}r_{ix}\theta_{2j}) \wedge \partial^{-1}(q_i\theta_{1j} - r_j\theta_{2i}))] \wedge (\tilde{Q} + \tilde{R})dx.
 \end{aligned}$$

In the above, we have used the formula (2.12) and dropped some terms through (2.10), as well as the expression $\sum_{j=1}^n (q_j\theta_{1j} - r_j\theta_{2j}) = Q^T\theta_1 - R^T\theta_2$. Moreover, we have integrated the terms which contain explicitly $(\tilde{Q} + \tilde{R})_{xx}$ by parts.

Owing to

$$\begin{aligned}
 &\sum_{j=1}^n \int [(-\frac{9}{4}q_j\theta_{1jx} - \frac{5}{4}q_{jx}\theta_{1j} - \frac{9}{4}r_j\theta_{2jx} - \frac{5}{4}r_{jx}\theta_{2j}) \wedge (\tilde{Q} + \tilde{R})_x \\
 &\quad + (\frac{3}{4}q_{jx}\theta_{1j} + \frac{3}{4}q_j\theta_{1jx} + \frac{3}{4}r_{jx}\theta_{2j} + \frac{3}{4}r_j\theta_{2jx}) \wedge (\tilde{Q} + \tilde{R})_x] \wedge (\tilde{Q} + \tilde{R})dx \\
 &= \sum_{j=1}^n \int [-\frac{1}{2}(3q_j\theta_{1jx} + q_{jx}\theta_{1j} + 3r_j\theta_{2jx} + r_{jx}\theta_{2j}) \wedge (\tilde{Q} + \tilde{R})_x] \wedge (\tilde{Q} + \tilde{R})dx \\
 &= \int -\frac{1}{2}(\partial^3 - 4\partial)(\tilde{Q} + \tilde{R}) \wedge (\tilde{Q} + \tilde{R})_x \wedge (\tilde{Q} + \tilde{R})dx \\
 &= 0,
 \end{aligned} \tag{A.6}$$

we obtain

$$\begin{aligned}
 \text{PrV } \mathcal{J}_\theta(\Theta_{\mathcal{J}_1}) &= \sum_{j=1}^n \int [(-\frac{3}{4}q_j\theta_{1jx} - \frac{1}{4}q_{jx}\theta_{1j} + \frac{3}{4}r_j\theta_{2jx} + \frac{1}{4}r_{jx}\theta_{2j}) \wedge \partial^{-1}(Q^T\theta_1 - R^T\theta_2) \\
 &\quad + \sum_{i=1}^n ((-\frac{3}{2}q_i\theta_{1jx} - \frac{1}{2}q_{ix}\theta_{1j}) \wedge \partial^{-1}(q_j\theta_{1i} - r_i\theta_{2j}) \\
 &\quad + (\frac{3}{2}r_i\theta_{2jx} + \frac{1}{2}r_{ix}\theta_{2j}) \wedge \partial^{-1}(q_i\theta_{1j} - r_j\theta_{2i}))] \wedge (\tilde{Q} + \tilde{R})dx.
 \end{aligned} \tag{A.7}$$

On the other hand, we have

$$\begin{aligned}
 \text{PrV } \mathcal{J}_\theta(\Theta_{\mathcal{J}_2}) &= \sum_{j=1}^n \int \{ \frac{1}{2}[\theta_{1j} \wedge (\frac{3}{2}q_j(\tilde{Q} + \tilde{R})_x + q_{jx}(\tilde{Q} + \tilde{R})) + \frac{1}{2}q_j\partial^{-1}(Q^T\theta_1 - R^T\theta_2) \\
 &\quad + \sum_{k=1}^n q_k\partial^{-1}(q_j\theta_{1k} - r_k\theta_{2j}) - \theta_{2j} \wedge (\frac{3}{2}r_j(\tilde{Q} + \tilde{R})_x + r_{jx}(\tilde{Q} + \tilde{R})) \\
 &\quad - \frac{1}{2}r_j\partial^{-1}(Q^T\theta_1 - R^T\theta_2) + \sum_{k=1}^n r_k\partial^{-1}(r_j\theta_{2k} - q_k\theta_{1j})] \wedge \partial^{-1}(Q^T\theta_1 - R^T\theta_2) \\
 &\quad + \theta_{1j} \wedge \sum_{i=1}^n [\frac{3}{2}q_i(\tilde{Q} + \tilde{R})_x + q_{ix}(\tilde{Q} + \tilde{R})) + \frac{1}{2}q_i\partial^{-1}(Q^T\theta_1 - R^T\theta_2)
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{k=1}^n q_k \partial^{-1}(q_i \theta_{1k} - r_k \theta_{2i}) \wedge \partial^{-1}(q_j \theta_{1i} - r_i \theta_{2j}) \\
 & + \theta_{2j} \wedge \sum_{i=1}^n \left[\frac{3}{2} r_i (\tilde{Q} + \tilde{R})_x + r_{ix} (\tilde{Q} + \tilde{R}) - \frac{1}{2} r_i \partial^{-1}(Q^T \theta_1 - R^T \theta_2) \right. \\
 & \left. - \sum_{k=1}^n r_k \partial^{-1}(q_k \theta_{1i} - r_i \theta_{2k}) \wedge \partial^{-1}(r_j \theta_{2i} - q_i \theta_{1j}) \right] dx. \tag{A.8}
 \end{aligned}$$

In order to understand, we divide the equality (A.8) into two parts **I** and **II**. The part **I** is the terms which contain explicitly only $\tilde{Q} + \tilde{R}$ or $(\tilde{Q} + \tilde{R})_x$, i.e.,

$$\begin{aligned}
 \mathbf{I} & = \sum_{j=1}^n \int \left[\left(\frac{3}{4} q_j \theta_{1j} - \frac{3}{4} r_j \theta_{2j} \right) \wedge (\tilde{Q} + \tilde{R})_x + \left(\frac{1}{2} q_{jx} \theta_{1j} - \frac{1}{2} r_{jx} \theta_{2j} \right) \wedge (\tilde{Q} + \tilde{R}) \wedge \partial^{-1}(Q^T \theta_1 - R^T \theta_2) \right. \\
 & \quad + \sum_{i=1}^n \left(\frac{3}{2} q_i \theta_{1j} \wedge (\tilde{Q} + \tilde{R})_x + q_{ix} \theta_{1j} \wedge (\tilde{Q} + \tilde{R}) \wedge \partial^{-1}(q_j \theta_{1i} - r_i \theta_{2j}) \right. \\
 & \quad \left. \left. + \left(\frac{3}{2} r_i \theta_{2j} \wedge (\tilde{Q} + \tilde{R})_x + r_{ix} \theta_{2j} \wedge (\tilde{Q} + \tilde{R}) \wedge \partial^{-1}(r_j \theta_{2i} - q_i \theta_{1j}) \right) \right] dx \\
 & = \sum_{j=1}^n \int \left[\left(-\frac{3}{4} q_j \theta_{1jx} - \frac{1}{4} q_{jx} \theta_{1j} + \frac{3}{4} r_j \theta_{2jx} + \frac{1}{4} r_{jx} \theta_{2j} \right) \wedge (\tilde{Q} + \tilde{R}) \wedge \partial^{-1}(Q^T \theta_1 - R^T \theta_2) \right. \\
 & \quad + \sum_{i=1}^n \left(\left(-\frac{3}{2} q_i \theta_{1jx} - \frac{1}{2} q_{ix} \theta_{1j} \right) \wedge (\tilde{Q} + \tilde{R}) \wedge \partial^{-1}(q_j \theta_{1i} - r_i \theta_{2j}) \right. \\
 & \quad \left. \left. + \left(\frac{3}{2} r_i \theta_{2jx} + \frac{1}{2} r_{ix} \theta_{2j} \right) \wedge (\tilde{Q} + \tilde{R}) \wedge \partial^{-1}(q_i \theta_{1j} - r_j \theta_{2i}) \right) \right] dx. \tag{A.9}
 \end{aligned}$$

The rest of (A.8) is as follows

$$\begin{aligned}
 \mathbf{II} & = \sum_{j=1}^n \int \left\{ \frac{1}{2} \sum_{k=1}^n [\theta_{1j} \wedge q_k \partial^{-1}(q_j \theta_{1k} - r_k \theta_{2j}) + \theta_{2j} \wedge r_k \partial^{-1}(q_k \theta_{1j} - r_j \theta_{2k})] \wedge \partial^{-1}(Q^T \theta_1 - R^T \theta_2) \right. \\
 & \quad + \theta_{1j} \wedge \sum_{i=1}^n \left[\frac{1}{2} q_i \partial^{-1}(Q^T \theta_1 - R^T \theta_2) + \sum_{k=1}^n q_k \partial^{-1}(q_i \theta_{1k} - r_k \theta_{2i}) \wedge \partial^{-1}(q_j \theta_{1i} - r_i \theta_{2j}) \right. \\
 & \quad \left. \left. - \theta_{2j} \wedge \sum_{i=1}^n \left[-\frac{1}{2} r_i \partial^{-1}(Q^T \theta_1 - R^T \theta_2) - \sum_{k=1}^n r_k \partial^{-1}(q_k \theta_{1i} - r_i \theta_{2k}) \wedge \partial^{-1}(q_i \theta_{1j} - r_j \theta_{2i}) \right] \right\} dx \\
 & = \sum_{j=1}^n \int \left[\theta_{1j} \wedge \sum_{i,k=1}^n q_k \partial^{-1}(q_i \theta_{1k} - r_k \theta_{2i}) \wedge \partial^{-1}(q_j \theta_{1i} - r_i \theta_{2j}) \right. \\
 & \quad \left. + \theta_{2j} \wedge \sum_{i,k=1}^n r_k \partial^{-1}(q_k \theta_{1i} - r_i \theta_{2k}) \wedge \partial^{-1}(q_i \theta_{1j} - r_j \theta_{2i}) \right] dx \\
 & = \sum_{i,j,k=1}^n \int \left[q_k \theta_{1j} \wedge \partial^{-1}(q_i \theta_{1k} - r_k \theta_{2i}) \wedge \partial^{-1}(q_j \theta_{1i} - r_i \theta_{2j}) \right. \\
 & \quad - r_j \theta_{2k} \wedge \partial^{-1}(q_i \theta_{1k} - r_k \theta_{2i}) \wedge \partial^{-1}(q_j \theta_{1i} - r_i \theta_{2j}) \\
 & \quad + r_k \theta_{2j} \wedge \partial^{-1}(q_k \theta_{1i} - r_i \theta_{2k}) \wedge \partial^{-1}(q_i \theta_{1j} - r_j \theta_{2i}) \\
 & \quad \left. + r_j \theta_{2k} \wedge \partial^{-1}(q_i \theta_{1k} - r_k \theta_{2i}) \wedge \partial^{-1}(q_j \theta_{1i} - r_i \theta_{2j}) \right] dx
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i,j,k=1}^n \int (q_k \theta_{1j} - r_j \theta_{2k}) \wedge \partial^{-1} (q_i \theta_{1k} - r_k \theta_{2i}) \wedge \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) dx \\
 &= 0.
 \end{aligned} \tag{A.10}$$

From (A.9) and (A.10), we have

$$\text{PrV}_{\mathcal{J}\theta}(\Theta_{\mathcal{J}_2}) = \mathbf{I}. \tag{A.11}$$

Combining (A.7) and (A.11) gives

$$\text{PrV}_{\mathcal{J}\theta}(\Theta_{\mathcal{J}}) = 0, \tag{A.12}$$

so the operator \mathcal{J} is Hamiltonian.

Secondly, we will show the compatibility of the operators \mathcal{K} and \mathcal{J} , i.e., the equality (2.14).

Notice that

$$\mathcal{K}\theta = \begin{pmatrix} \theta_{2xx} - \theta_2 \\ \theta_1 - \theta_{1xx} \end{pmatrix}, \tag{A.13}$$

so from the equalities (A.4) and (A.5), we obtain

$$\begin{aligned}
 \text{PrV}_{\mathcal{K}\theta}(\Theta_{\mathcal{J}_1}) &= \sum_{j=1}^n \int \left[-\frac{3}{2} \theta_{1jx} \wedge (\theta_{2jxx} - \theta_{2j}) - \frac{1}{2} \theta_{1j} \wedge (\theta_{2jxx} - \theta_{2j})_x \right. \\
 &\quad \left. - \frac{3}{2} \theta_{2jx} \wedge (\theta_{1j} - \theta_{1jxx}) - \frac{1}{2} \theta_{2j} \wedge (\theta_{1j} - \theta_{1jxx})_x \right] \wedge (\tilde{Q} + \tilde{R}) dx \\
 &= \sum_{j=1}^n \int \left[-\frac{1}{2} (\partial^3 - 4\partial) (\theta_{1j} \wedge \theta_{2j}) \wedge (\tilde{Q} + \tilde{R}) \right] dx \\
 &= \frac{1}{2} \sum_{j=1}^n \int (\theta_{1j} \wedge \theta_{2j}) \wedge (3Q^T \theta_{1x} + Q_x^T \theta_1 + 3R^T \theta_{2x} + R_x^T \theta_2) dx \\
 &= \frac{1}{2} \sum_{i,j=1}^n \int (\theta_{1j} \wedge \theta_{2j}) \wedge (3q_i \theta_{1ix} + q_{ix} \theta_{1i} + 3r_i \theta_{2ix} + r_{ix} \theta_{2i}) dx,
 \end{aligned} \tag{A.14}$$

and

$$\begin{aligned}
 \text{PrV}_{\mathcal{K}\theta}(\Theta_{\mathcal{J}_2}) &= \sum_{i,j=1}^n \int \left[\frac{1}{2} (\theta_{1j} \wedge (\theta_{2jxx} - \theta_{2j}) - \theta_{2j} \wedge (\theta_{1j} - \theta_{1jxx})) \wedge \partial^{-1} (q_i \theta_{1i} - r_i \theta_{2i}) \right. \\
 &\quad \left. + \theta_{1j} \wedge (\theta_{2ixx} - \theta_{2i}) \wedge \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) \right. \\
 &\quad \left. + \theta_{2j} \wedge (\theta_{1i} - \theta_{1ixx}) \wedge \partial^{-1} (r_j \theta_{2i} - q_i \theta_{1j}) \right] dx \\
 &= \sum_{i,j=1}^n \int \left[\frac{1}{2} (\theta_{1j} \wedge \theta_{2jxx} - \theta_{1jxx} \wedge \theta_{2j}) \wedge \partial^{-1} (q_i \theta_{1i} - r_i \theta_{2i}) \right. \\
 &\quad \left. + (\theta_{1j} \wedge \theta_{2ixx} - \theta_{1jxx} \wedge \theta_{2i}) \wedge \partial^{-1} (q_j \theta_{1i} - r_i \theta_{2j}) \right] dx \\
 &= \sum_{i,j=1}^n \int \left[-\frac{1}{2} (\theta_{1j} \wedge \theta_{2jx} - \theta_{1jx} \wedge \theta_{2j}) \wedge (q_i \theta_{1i} - r_i \theta_{2i}) \right. \\
 &\quad \left. - (\theta_{1j} \wedge \theta_{2ix} - \theta_{1jx} \wedge \theta_{2i}) \wedge (q_j \theta_{1i} - r_i \theta_{2j}) \right] dx
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i,j=1}^n \int \left[\frac{1}{2} (\boldsymbol{\theta}_{1j} \wedge \boldsymbol{\theta}_{2jx} + \boldsymbol{\theta}_{1jx} \wedge \boldsymbol{\theta}_{2j}) \wedge (q_i \boldsymbol{\theta}_{1i} + r_i \boldsymbol{\theta}_{2i}) \right. \\
 &\quad \left. - \boldsymbol{\theta}_{1j} \wedge \boldsymbol{\theta}_{2j} \wedge (q_i \boldsymbol{\theta}_{1ix} + r_i \boldsymbol{\theta}_{2ix}) \right] dx \\
 &= -\frac{1}{2} \sum_{i,j=1}^n \int [(\boldsymbol{\theta}_{1j} \wedge \boldsymbol{\theta}_{2j}) \wedge (3q_i \boldsymbol{\theta}_{1ix} + q_{ix} \boldsymbol{\theta}_{1i} + 3r_i \boldsymbol{\theta}_{2ix} + r_{ix} \boldsymbol{\theta}_{2i})] dx. \tag{A.15}
 \end{aligned}$$

The Eqs. (A.14) and (A.15) lead to

$$\text{PrV}_{\mathcal{H}\theta}(\Theta_{\mathcal{J}}) = \text{PrV}_{\mathcal{H}\theta}(\Theta_{\mathcal{J}_1}) + \text{PrV}_{\mathcal{H}\theta}(\Theta_{\mathcal{J}_2}) = 0, \tag{A.16}$$

so the operators \mathcal{H} and \mathcal{J} are compatible Hamiltonian operators.

Thus, we complete the proof of the Theorem 2.1.