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Darboux Transformations and N-soliton Solutions of Two (2+1)-Dimensional Nonlinear Equations*

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Abstract *Two Darboux transformations of the (2+1)-dimensional Caudrey–Dodd–Gibbon–Kotera–Sawaka (CDGKS) equation and (2+1)-dimensional modified Korteweg-de Vries (mKdV) equation are constructed through the Darboux matrix method, respectively. N-soliton solutions of these two equations are presented by applying the Darboux transformations N times. The right-going bright single-soliton solution and interactions of two and three-soliton overtaking collisions of the (2+1)-dimensional CDGKS equation are studied. By choosing different seed solutions, the right-going bright and left-going dark single-soliton solutions, the interactions of two and three-soliton overtaking collisions, and kink soliton solutions of the (2+1)-dimensional mKdV equation are investigated. The results can be used to illustrate the interactions of water waves in shallow water.*

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Key words: Darboux transformation, (2+1)-dimensional Caudrey–Dodd–Gibbon–Kotera–Sawaka equation, (2+1)-dimensional modified Korteweg-de Vries equation, N-soliton solutions

1 Introduction

The study of finding explicit solutions for diverse soliton equations has been extremely important but complicated in the nonlinear science. So far, several systematic methods have been developed to obtain explicit solutions for the 1+1 dimensional soliton equations such as the inverse scattering transformations(IST),[1] Darboux transformation (DT),[2–11] bilinear method,[12–13] symmetry approach,[14–20] nonlinearization approach of Lax pairs,[21] symmetry constraint approach,[22–23] algebraic curve method,[24–25] extended-tanh function method[26–27] and so on. Among them, DT is one of the most effective and powerful approaches, since starting from one trivial seed solution, explicit nontrivial N-soliton solution can be obtained by applying the DT N times.[2–3] But for the (2+1)-dimensional case, the situation is not so good that it becomes more complicated and difficult to find its explicit solutions.[2] Nevertheless, the nonlinearization approach of Lax pairs and symmetry constraint approach provide an effective way to solve the (2+1)-dimensional nonlinear equations, which separate the (2+1)-dimensional nonlinear equations into two (1+1)-dimensional integrable nonlinear equations.[22–23,28–29]

In this paper, we consider two (2+1)-dimensional nonlinear equations, which are both closely related to the modified Korteweg-de Vries equation and have important applications in the study of the shallow water.[10,28–29] The first equation is the (2+1)-dimensional Caudrey–Dodd–Gibbon–Kotera–Sawaka (CDGKS) equation

$$\begin{aligned}36q_t = &-q_{xxxxx} - 15(qq_{xx})_x - 45q^2q_x + 5q_{xxy} + 15qq_y \\ &+ 15q_x\partial_x^{-1}q_y + 5\partial_x^{-1}q_{yy}\,,\end{aligned} \tag{1}$$

and when introducing the constraint[23,28] $q = -2u^2$, Eq. (1) can be decomposed into two (1+1)-dimensional integrable equations, which are the modified Korteweg-de Vries equation and one high-order equation in the mKdV hierarchy

$$u_y = u_{xxx} - 6u^2u_x\,, \tag{2}$$

$$\begin{aligned}4u_t = &\,u_{xxxxx} - 10u^2u_{xxx} - 40uu_xu_{xxx} \\ &- 10u_x^3 + 30u^4u_x\,.\end{aligned} \tag{3}$$

Equation (1) was firstly proposed by Konopelchenko and Dubovsky,[30] and many important results of it have been obtained[23,28,30–35] such as the DT and the Bäcklund transformation based on a third-order linear operator by Geng,[32] quasi-periodic solutions by Cao, Wu, and Geng,[28] nonlocal symmetries by Lou and Hu,[31] multiple soliton solutions by Wazwaz[33] and so on.

The other (2+1)-dimensional nonlinear equation discussed in the present paper is the (2+1)-dimensional modified Korteweg-de Vries (mKdV) equation

$$4q_t = q_{xxx} - 6q^2q_x - 6q_x\partial_x^{-1}q_y + 3\partial_x^{-1}q_{yy} - 18qq_y\,, \tag{4}$$

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which is different from the modified Kadomtsev–Petviashvili (mKP) equation.[10] The equation was proposed by Geng and Cao[29] with the help of first two nontrivial equations in the Kaup-Newell(KN) hierarchy[36]

$$u_y = -u_{xx} - 2(u^2v)_x, \quad v_y = v_{xx} - 2(uv^2)_x, \tag{5}$$

$$u_t = u_{xxx} + 6(uvu_x + u^3v^2)_x,$$
$$v_t = v_{xxx} - 6(uvv_x - u^2v^3)_x, \tag{6}$$

and if u and v are the solutions of Eqs. (5) and (6), the constraint[29] $q = uv$ gives the solution of Eq. (4). Some crucial results have been obtained for the KN system[29,37–44] or the corresponding equations. For instance, the nonlinearization of the KN eigenvalue problem under the Bargmann constraint and the involutive solution of the well-known derivative Schrödinger (NLS) equation were given by Qiao,[40] the quasi-periodic solution of Eq.(4) through introducing the Abel–Jacobi coordinates was obtained by Geng and Cao.[29] Especially, in Ref. [39] some one and two-soliton solutions of Eqs. (5) and (6) were given.

In this paper, based on the DTs in Refs. [32], [37]–[39], two different forms of DTs for Eqs. (1) and (4) are constructed through the Darboux matrix method,[2] respectively. N-soliton solutions of these two (2+1)-dimensional nonlinear equations can be given by applying the DTs N times. By choosing adequate seed solutions and spectral parameters, the right-going bright single-soliton solution and interactions of two and three-soliton overtaking collisions of Eq. (1) are studied, and what is interesting is that the small-amplitude solitons overtake the large-amplitude ones. The right-going bright and left-going dark single-soliton solutions, the interactions of two and three-soliton overtaking collisions of Eq. (4) are researched, the large-amplitude solitons naturally overtake the small-amplitude ones. Moreover, some kink soliton solutions[41–42] of Eq. (4) are given by choosing different kind of seed solution.

The present paper can be organized as follows. In Sec. 2, two DTs of the (2+1)-dimensional CDGKS equation (1) and (2+1)-dimensional mKdV equation (4) are constructed through the Darboux matrix method. In Sec. 3, N-soliton solutions of Eqs. (1) and (4) are given, and some interesting figures are plotted.

2 Darboux Transformation

2.1 *Darboux Transformations of (2+1)-Dimensional CDGKS*

In this section, we establish two DTs for the (2+1)-dimensional CDGKS equation (1) by the Darboux matrix method. Firstly we consider the Lax pairs of it

$$d\phi_x = U(\lambda)\phi, \quad U(\lambda) = \begin{pmatrix} u & \lambda \\ 1 & -u \end{pmatrix}, \tag{7}$$

$$\phi_y = V_1(\lambda)\phi,$$
$$V_1(\lambda) = \begin{pmatrix} 4u\lambda + u_{xx} - 2u^3 & 4\lambda^2 - (2u^2 + 2u_x)\lambda \\ 4\lambda - 2u^2 + 2u_x & -4u\lambda - u_{xx} + 2u^3 \end{pmatrix}, \tag{8}$$

$$\phi_t = V_2(\lambda)\phi, \quad V_2(\lambda) = \begin{pmatrix} V_2^{(11)} & V_2^{(12)} \\ V_2^{(21)} & -V_2^{(11)} \end{pmatrix}, \tag{9}$$

here λ is the spectral parameter, u is the potential, $\phi = (\phi_1, \phi_2)^{\mathrm{T}}$, and

$$V_2^{(11)} = 4u\lambda^2 + (u_{xx} - 2u^3)\lambda + \frac{1}{4}u_{xxxx} - \frac{5}{2}uu_x^2 - \frac{5}{2}u^2u_{xx} + \frac{3}{2}u^5,$$
$$V_2^{(12)} = 4\lambda^3 - (2u^2 + 2u_x)\lambda^2 + \Big(\frac{1}{2}u_x^2 - uu_{xx} + \frac{3}{2}u^4 - \frac{1}{2}u_{xxx} + 3u^2u_x\Big)\lambda,$$
$$V_2^{(21)} = 4\lambda^2 - (2u^2 - 2u_x)\lambda - uu_{xx} + \frac{1}{2}u_x^2 + \frac{3}{2}u^4 + \frac{1}{2}u_{xxx} - 3u^2u_x,$$

by using the compatibility condition of (7), (8), and (9), one can obtain Eqs. (2) and (3) naturally.

Next, in order to construct the Darboux matrix for the above spectral problem, we consider the gauge transformation[2,10–11]

$$\bar{\phi} = T\phi, \tag{10}$$

where $T = T(x, y, t, \lambda)$ is a 2×2 matrix, and under the above transformation the original Lax pairs can be changed into the following new ones

$$\bar{\phi}_x = \bar{U}\bar{\phi}, \quad \bar{U} = (T_x + TU)T^{-1}, \tag{11}$$
$$\bar{\phi}_y = \bar{V}_1\bar{\phi}, \quad \bar{V}_1 = (T_y + TV_1)T^{-1}, \tag{12}$$
$$\bar{\phi}_t = \bar{V}_2\bar{\phi}, \quad \bar{V}_2 = (T_t + TV_2)T^{-1}. \tag{13}$$

If the new Lax pairs have the same type as the original ones, we call the gauge transformation Darboux transformation. To this end, by considering the form of U, two DTs of Eq. (1) can be constructed.

(i) The first Darboux transformation

Denote that

$$T = \begin{pmatrix} -\lambda_1\sigma & \lambda \\ 1 & -\sigma^{-1} \end{pmatrix}, \tag{14}$$

here $\sigma = \phi_2/\phi_1$, and $(\phi_1 = \phi_1(x, y, t, \lambda_1),\ \phi_2 = \phi_2(x, y, t, \lambda_1))^{\mathrm{T}}$ is one basic solution of the spectral problem (7)–(9) with $\lambda = \lambda_1$. Then from the above definitions, we have

$$\det T = -(\lambda - \lambda_1)\,, \tag{15}$$

and the following Riccati equations

$$\sigma_x = 1 - 2\sigma u - \lambda_1 \sigma^2\,, \tag{16}$$

$$\sigma_y = 4\lambda_1 - 2u^2 + 2u_x + (4u^3 - 8\lambda_1 u - 2u_{xx})\sigma - [4\lambda_1^2 - (2u^2 + 2u_x)\lambda_1]\sigma^2\,, \tag{17}$$

$$\begin{aligned}\sigma_t &= 4\lambda_1^2 - 2\lambda_1 u^2 + 2\lambda_1 u_x - u u_{xx} + \frac{1}{2}u_x^2 + \frac{3}{2}u^4 + \frac{1}{2}u_{xxx} - 3u^2 u_x \\ &+ \Big(4\lambda_1 u^3 - 8u\lambda_1^2 - 2\lambda_1 u_{xx} - \frac{1}{2}u_{xxxx} + 5uu_x^2 + 5u^2 u_{xx} - 3u^5\Big)\sigma \\ &- \Big[4\lambda_1^3 - (2u^2 + 2u_x)\lambda_1^2 + \Big(\frac{1}{2}u_x^2 - uu_{xx} + \frac{3}{2}u^4 - \frac{1}{2}u_{xxx} + 3u^2 u_x\Big)\lambda_1\Big]\sigma^2\,.\end{aligned} \tag{18}$$

Based on the above facts, the following theorem can be directly given.

Theorem 1 From a known solution u of Eqs. (2) and (3), the following explicit formula

$$\bar{u} = -u - \lambda_1 \sigma + \sigma^{-1} \tag{19}$$

gives the new special solution of Eqs. (2) and (3), then the new solution of the (2+1)-dimensional CDGKS equation (1) can be obtained by the constraint $\bar{q} = -2\bar{u}^2$.

(ii) The second Darboux transformation

Let

$$T = \begin{pmatrix} \lambda & -\sigma^{-1}\lambda \\ -\lambda_1\sigma & \lambda \end{pmatrix}\,, \tag{20}$$

$$\bar{u} = u + \lambda_1\sigma - \sigma^{-1}\,. \tag{21}$$

Here

$$\det T = \lambda(\lambda - \lambda_1)\,, \tag{22}$$

and σ is the same one defined in (14). Similar to first DT, we can also find that the gauge transformation (10) determined by the above matrix (20) is also the Darboux transformation, the transformation between the old potential and the new one is presented by (21).

2.2 *Darboux Transformations of (2+1)-Dimensional mKdV*

In this section, we study DTs for the (2+1)-dimensional mKdV equation (4). The Lax pairs of it are

$$\phi_x = M(\lambda)\phi, \quad M(\lambda) = \begin{pmatrix} -\frac{1}{2}\lambda & \lambda u \\ v & \frac{1}{2}\lambda \end{pmatrix}\,, \tag{23}$$

$$\phi_y = N_1(\lambda)\phi, \quad N_1(\lambda) = \begin{pmatrix} -\frac{1}{2}\lambda^2 + uv\lambda & u\lambda^2 - (u_x + 2u^2 v)\lambda \\ v\lambda + v_x - 2uv^2 & \frac{1}{2}\lambda^2 - uv\lambda \end{pmatrix}\,, \tag{24}$$

$$\phi_t = N_2(\lambda)\phi, \quad N_2(\lambda) = \begin{pmatrix} N_2^{(11)} & N_2^{(12)} \\ N_2^{(21)} & -N_2^{(11)} \end{pmatrix}\,, \tag{25}$$

with

$$\begin{aligned} N_2^{(11)} &= -\frac{1}{2}\lambda^3 + uv\lambda^2 + (uv_x - u_x v - 3u^2 v^2)\lambda\,, \\ N_2^{(12)} &= u\lambda^3 - (u_x + 2u^2 v)\lambda^2 + (u_{xx} + 6uvu_x + 6u^3 v^2)\lambda\,, \\ N_2^{(21)} &= v\lambda^2 + (v_x - 2uv^2)\lambda + v_{xx} - 6uvv_x + 6u^2 v^3\,. \end{aligned}$$

Here λ is the spectral parameter, u and v are potentials, $\phi = (\phi_1, \phi_2)^{\mathrm{T}}$. By using the compatibility condition of the above Lax pairs, one can directly obtain Eqs. (5) and (6).

Similarly to the above section, two DTs can be obtained according to the form of M.

(i) The first Darboux transformation

Suppose that

$$T = \begin{pmatrix} (1 - \delta u)^{-1}\lambda - \lambda_1 & -u(1 - \delta u)^{-1}\lambda \\ -\delta & 1 \end{pmatrix}\,, \tag{26}$$

here $\delta=\phi_2/\phi_1$, and $(\phi_1=\phi_1(x,y,t,\lambda_1),\ \phi_2=\phi_2(x,y,t,\lambda_1))^{\mathrm{T}}$ is one basic solution of the spectral problem (23)–(25) with $\lambda=\lambda_1$. Then it follows that

$$\det T=(\lambda-\lambda_1)\,, \tag{27}$$

and the Riccati equations

$$\delta_x=v+\lambda_1\delta-\lambda_1 u\delta^2, \tag{28}$$

$$\delta_y=v\lambda_1+v_x-2uv^2+(\lambda_1^2-2uv\lambda_1)\delta-[u\lambda_1^2-(u_x+2u^2v)\lambda_1]\delta^2, \tag{29}$$

$$\begin{aligned}\delta_t&=v\lambda_1^2+(v_x-2uv^2)\lambda_1+v_{xx}-6uvv_x+6u^2v^3+[\lambda_1^3-2uv\lambda_1^2-2(uv_x-u_xv)\lambda_1\\&\quad+6u^2v^2\lambda_1]\delta-[u\lambda_1^3-(u_x+2u^2v)\lambda_1^2+(u_{xx}+6uvu_x+6u^3v^2)\lambda_1]\delta^2.\end{aligned} \tag{30}$$

Based on the above facts, the following theorem can be given similar to the one given in the above section.

Theorem 2 For a known solution (u,v) of Eqs. (5) and (6), the explicit calculation form

$$\bar u=\frac{-u\lambda_1+\lambda_1u^2\delta-u_x-u^2v}{(-1+\delta u)^2},\quad \bar v=\delta(1-\delta u) \tag{31}$$

gives the new special solution of Eqs. (5) and (6), and the new solution of the (2+1)-dimensional mKdV equation (4) can be obtained by the constraint $\bar q=\bar u\bar v$.

(ii) The second Darboux transformation

Assume that

$$T=\begin{pmatrix}-\delta\lambda & \lambda\\ 1 & -\lambda_1^{-1}\delta^{-1}\lambda\end{pmatrix}, \tag{32}$$

$$\bar u=\lambda_1\delta(-1+\delta u),\quad \bar v=\frac{v+\lambda_1\delta}{\lambda_1\delta^2}\,. \tag{33}$$

Here

$$\det T=\lambda\Big(\frac{\lambda}{\lambda_1}-1\Big)\,, \tag{34}$$

and δ is the same one defined in (26). Equally, we can also find that T determined by (32) is the Darboux matrix, the transformation from the old potentials into the new ones is given by (33).

3 Explicit Solutions

In this section, we apply the DTs (14) and (26) given in Subsecs. 2.1 and 2.2 to construct new analytical soliton solutions for Eqs. (1) and (4). The interactions of three-soliton solutions for this two equations are shown. The DTs (20) and (32) can be similarly discussed, here we refrain from presenting it.

3.1 *Explicit Solutions of (2+1)-Dimensional CDGKS*

We start from a constant solution $u=u_0$ $(u_0\neq 0)$ of Eqs. (2) and (3), and substituting it into the linear Lax pairs (7)–(9) with $\lambda=\lambda_j$, two basic solutions can be chosen

$$\phi^{(j)}=\begin{pmatrix}\phi_1^{(j)}\\ \phi_2^{(j)}\end{pmatrix}=\begin{pmatrix}\cosh\xi_j\\ \sqrt{u_0^2+\lambda_j}\sinh\xi_j/\lambda_j-u_0\cosh\xi_j/\lambda_j\end{pmatrix},\quad j\ \text{is odd number}\,,$$

$$\phi^{(j)}=\begin{pmatrix}\phi_1^{(j)}\\ \phi_2^{(j)}\end{pmatrix}=\begin{pmatrix}\sinh\xi_j\\ \sqrt{u_0^2+\lambda_j}\cosh\xi_j/\lambda_j-u_0\sinh\xi_j/\lambda_j\end{pmatrix},\quad j\ \text{is even number}\,,$$

with

$$\xi_j=\sqrt{u_0^2+\lambda_j}\Big[x+2(2\lambda_j-u_0^2)y+\frac{1}{2}(8\lambda_j^2-4u_0^2\lambda_j+3u_0^4)t\Big]\,.$$

Case 1 Using the DT (14) one time, we have

$$\sigma_1=\frac{1}{\lambda_1}\frac{\sqrt{u_0^2+\lambda_1}\sinh\xi_1-u_0\cosh\xi_1}{\cosh\xi_1}=\frac{1}{\lambda_1}\Big(\sqrt{u_0^2+\lambda_1}\tanh\xi_1-u_0\Big)\,.$$

With the aid of (19), we obtain single-soliton solution of Eqs. (2) and (3)

$$u_1=-u_0-\lambda_1\sigma_1+\sigma_1^{-1}=-\sqrt{u_0^2+\lambda_1}\tanh\xi_1+\frac{\lambda_1}{(\sqrt{u_0^2+\lambda_1}\tanh\xi_1-u_0)}\,, \tag{35}$$

and the constraint $q_1=-2u_1^2$ gives the single-soliton solution of Eq. (1).

Case 2 Using DT (14) two times, we have

$$\phi_2[1] = T(\lambda_1, \lambda_2)\phi^{(2)} = \begin{pmatrix} -\lambda_1\sigma_1 & \lambda_2 \\ 1 & -\sigma_1^{-1} \end{pmatrix} \begin{pmatrix} \sinh\xi_2 \\ \sqrt{u_0^2+\lambda_2}\cosh\xi_2/\lambda_2 - u_0\sinh\xi_2/\lambda_2 \end{pmatrix}$$
$$= \begin{pmatrix} -\lambda_1\sigma_1\sinh\xi_2 + \sqrt{u_0^2+\lambda_2}\cosh\xi_2 - u_0\sinh\xi_2 \\ \sinh\xi_2 - (\sqrt{u_0^2+\lambda_2}/\sigma_1\lambda_2)\cosh\xi_2 + (u_0/\sigma_1\lambda_2)\sinh\xi_2 \end{pmatrix},$$
$$\sigma_2 = \frac{\sigma_1\lambda_2\sinh\xi_2 - \sqrt{u_0^2+\lambda_2}\cosh\xi_2 + u_0\sinh\xi_2}{(-\lambda_1\sigma_1\sinh\xi_2 + \sqrt{u_0^2+\lambda_2}\cosh\xi_2 - u_0\sinh\xi_2)\sigma_1\lambda_2} = \frac{\sigma_1\lambda_2\tanh\xi_2 - \sqrt{u_0^2+\lambda_2} + u_0\tanh\xi_2}{(-\lambda_1\sigma_1\tanh\xi_2 + \sqrt{u_0^2+\lambda_2} - u_0\tanh\xi_2)\sigma_1\lambda_2}.$$

By using (19), we obtain two-soliton solution of Eqs. (2) and (3)

$$u_2 = -u_1 - \lambda_2\sigma_2 + \sigma_2^{-1} = \sqrt{u_0^2+\lambda_1}\tanh\xi_1 - \frac{\lambda_1}{(\sqrt{u_0^2+\lambda_1}\tanh\xi_1 - u_0)}$$
$$- \frac{\sigma_1\lambda_2\tanh\xi_2 - \sqrt{u_0^2+\lambda_2} + u_0\tanh\xi_2}{(-\lambda_1\sigma_1\tanh\xi_2 + \sqrt{u_0^2+\lambda_2} - u_0\tanh\xi_2)\sigma_1} + \frac{(-\lambda_1\sigma_1\tanh\xi_2 + \sqrt{u_0^2+\lambda_2} - u_0\tanh\xi_2)\sigma_1\lambda_2}{\sigma_1\lambda_2\tanh\xi_2 - \sqrt{u_0^2+\lambda_2} + u_0\tanh\xi_2}, \tag{36}$$

then the constraint $q_2 = -2u_2^2$ gives the two-soliton solution of Eq. (1).

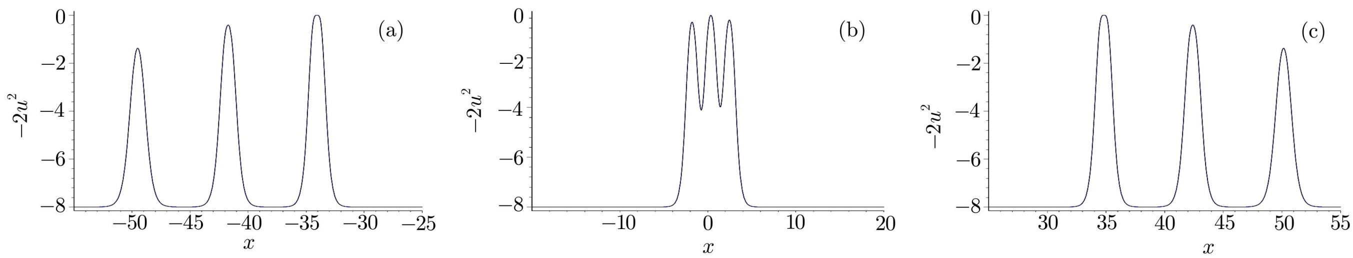

Fig. 1 (a), (b), and (c) Interaction of the right-going bright three-soliton overtaking collision of $u_0 = 2$, $\lambda_1 = -2$, $\lambda_2 = -1.5$, $\lambda_3 = -1$ at $t = 0$, $y = -3$, 0 and 3; The small-amplitude solitons overtake the large-amplitude ones.

Continuing the above process, we get the three-soliton solution of Eq. (1), see Fig. 1. Next we have

$$\phi_j[j-1] = \begin{pmatrix} -\lambda_{j-1}\sigma_{j-1} & \lambda_j \\ 1 & -\sigma_{j-1}^{-1} \end{pmatrix} \begin{pmatrix} -\lambda_{j-2}\sigma_{j-2} & \lambda_j \\ 1 & -\sigma_{j-2}^{-1} \end{pmatrix} \cdots \begin{pmatrix} -\lambda_1\sigma_1 & \lambda_j \\ 1 & -\sigma_1^{-1} \end{pmatrix} \begin{pmatrix} \phi_1^{(j)} \\ \phi_2^j \end{pmatrix},$$
$$\sigma_j = \frac{\phi_{j2}[j-1]}{\phi_{j1}[j-1]}, \quad j = 3, 4, \ldots, N,$$

therefore the following formula can be given.

Case N Using DT (14) N times, N-soliton solution of Eqs. (2) and (3) can be expressed as follows

$$u_N = -u_0 - \sum_{j=1}^{N}\lambda_j\sigma_j + \sum_{j=1}^{N}\sigma_j^{-1}, \tag{37}$$

and the N-soliton solution of Eq. (1) is given by the constraint $q_N = -2u_N^2$.

3.2 *Explicit Solutions of (2+1)-Dimensional mKdV*

In this section, we consider three different kinds of constant seed solutions of Eq. (4), then three different types of soliton solutions of Eq. (4) are obtained.

(i) Choose $(u, v) = (u_0, 1)$ $(u_0 \neq 0)$ as the seed solution, substituting it into the linear equations (23)–(25), two basic solutions can be chosen

$$\phi^{(j)} = \begin{pmatrix} \phi_1^{(j)} \\ \phi_2^{(j)} \end{pmatrix} = \begin{pmatrix} \cosh\xi_j \\ \frac{A_j}{C_j}\sinh\xi_j - \frac{B_j}{C_j}\cosh\xi_j \end{pmatrix}, \quad j \text{ is odd number},$$
$$\phi^{(j)} = \begin{pmatrix} \phi_1^{(j)} \\ \phi_2^{(j)} \end{pmatrix} = \begin{pmatrix} \sinh\xi_j \\ \frac{A_j}{C_j}\cosh\xi_j - \frac{B_j}{C_j}\sinh\xi_j \end{pmatrix}, \quad j \text{ is even number}.$$

Here

$$\xi_j = \frac{\sqrt{(\lambda_j+4u_0)\lambda_j}[x + (\lambda_j - 2u_0)y + (\lambda_j^2 - 2\lambda_j u_0 + 6u_0^2)t]}{2},$$

$$A_j=(6\sqrt{\lambda_j}u_0^2+\lambda_j^{3/2}(\lambda_j-2u_0))\sqrt{\lambda_j+4u_0},\quad B_j=-\lambda_j^3+2u_0\lambda_j^2-6u_0^2\lambda_j,\quad C_j=2u_0\lambda_j^3-4u_0^2\lambda_j^2+12u_0^3\lambda_j\,.$$

Case 1 Using DT (26) one time, we have

$$\delta_1=\frac{A_1\sinh\xi_1-B_1\cosh\xi_1}{C_1\cosh\xi_1}=\frac{A_1}{C_1}\tanh\xi_1-\frac{B_1}{C_1}\,.$$

With the aid of (31), we obtain single-soliton solution of Eqs. (5) and (6)

$$\begin{aligned}u_1&=\frac{-u_0\lambda_1+\lambda_1u_0^2\delta_1-u_0^2}{(-1+\delta_1u_0)^2}=\frac{\lambda_1u_0}{-1+\delta_1u_0}-\frac{u_0^2}{(-1+\delta_1u_0)^2}\,,\\ v_1&=\delta_1(1-\delta_1u_0)=\Big(\frac{A_1}{C_1}\tanh\xi_1-\frac{B_1}{C_1}\Big)\Big[1-u_0\Big(\frac{A_1}{C_1}\tanh\xi_1-\frac{B_1}{C_1}\Big)\Big]\,,\end{aligned}\tag{38}$$

and the constraint $q_1=u_1v_1$ gives the single-soliton solution of Eq. (4).

Case 2 Using DT (26) two times, we get

$$\begin{aligned}\phi_2[1]=T(\lambda_1,\lambda_2)\phi^{(2)}&=\begin{pmatrix}(1/(1-\delta_1u_0))\lambda_2-\lambda_1 & -(u_0/(1-\delta_1u_0))\lambda_2\\ -\delta_1 & 1\end{pmatrix}\begin{pmatrix}\sinh\xi_2\\ (A_2/C_2)\cosh\xi_2-(B_2/C_2)\sinh\xi_2\end{pmatrix}\\ &=\begin{pmatrix}((1/(1-\delta_1u_0))\lambda_2-\lambda_1)\sinh\xi_2-((u_0/(1-\delta_1u_0))\lambda_2)((A_2/C_2)\cosh\xi_2-(B_2/C_2)\sinh\xi_2)\\ -\delta_1\sinh\xi_2+(A_2/C_2)\cosh\xi_2-(B_2/C_2)\sinh\xi_2\end{pmatrix},\end{aligned}$$

then it holds that

$$\begin{aligned}\delta_2&=\frac{-\delta_1\sinh\xi_2+(A_2/C_2)\cosh\xi_2-(B_2/C_2)\sinh\xi_2}{((1(1-\delta_1u_0))\lambda_2-\lambda_1)\sinh\xi_2-((u_0(1-\delta_1u_0))\lambda_2)((A_2/C_2)\cosh\xi_2-(B_2/C_2)\sinh\xi_2)}\\ &\frac{-\delta_1\tanh\xi_2+(A_2/C_2)-(B_2/C_2)\tanh\xi_2}{((1(1-\delta_1u_0))\lambda_2-\lambda_1)\tanh\xi_2-((u_0(1-\delta_1u_0))\lambda_2)((A_2/C_2)-(B_2/C_2)\tanh\xi_2)}\,.\end{aligned}$$

By using (31), we obtain two-soliton solution of Eqs. (5) and (6)

$$u_2=\frac{-u_1\lambda_2+\lambda_2u_1^2\delta_2-u_{1x}-u_1^2v_1}{(-1+\delta_2u_1)^2},\quad v_2=\delta_2(1-\delta_2u_1)\,,\tag{39}$$

then the constraint $q_2=u_2v_2$ gives the two-soliton solution of Eq. (4).

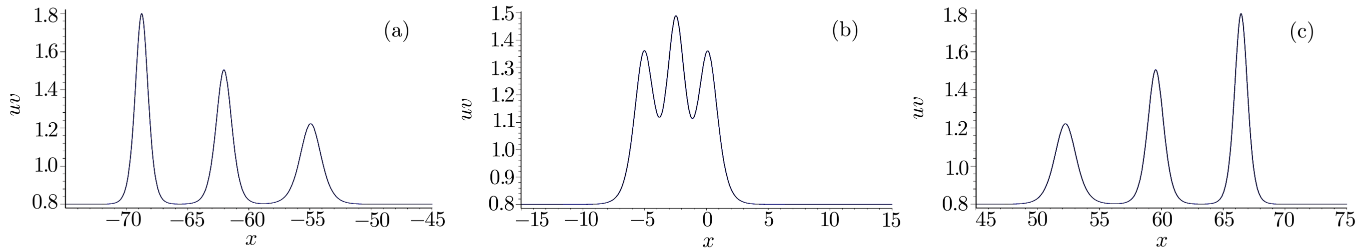

Fig. 2 (a), (b), and (c) Interaction of the right-going bright three-soliton overtaking collision of $u_0=0.8$, $\lambda_1=-4$, $\lambda_2=-4.5$, $\lambda_3=-5$ at $t=0$, $y=-10$, -0.2 and 10; The large-amplitude solitons overtake the small-amplitude ones.

Continuing the above process, three-soliton solution of Eq. (4) can be obtained, see Fig. 2. Next we have

$$\begin{aligned}\phi_j[j-1]=&\begin{pmatrix}(1/(1-\delta_{j-1}u_{j-2}))\lambda_j-\lambda_{j-1} & -(u_{j-2}/(1-\delta_{j-1}u_{j-2}))\lambda_j\\ -\delta_{j-1} & 1\end{pmatrix}\\ &\times\begin{pmatrix}(1/(1-\delta_{j-2}u_{j-3}))\lambda_j-\lambda_{j-2} & -(u_{j-3}/(1-\delta_{j-2}u_{j-3}))\lambda_j\\ -\delta_{j-2} & 1\end{pmatrix}\cdots\\ &\times\begin{pmatrix}(1/(1-\delta_1u_0))\lambda_j-\lambda_1 & -(u_0/(1-\delta_1u_0))\lambda_j\\ -\delta_1 & 1\end{pmatrix}\begin{pmatrix}\phi_1^{(j)}\\ \phi_2^{(j)}\end{pmatrix},\\ \delta_j=&\frac{\phi_{j2}[j-1]}{\phi_{j1}[j-1]},\quad j=3,4,\ldots,N\,,\end{aligned}$$

therefore we have the following formula.

Case N Using DT (26) N times, N-soliton solution of Eqs. (5) and (6) can be obtained

$$u_N=\frac{-u_{N-1}\lambda_N+\lambda_Nu_{N-1}^2\delta_N-u_{N-1,x}-u_{N-1}^2v_{N-1}}{(-1+\delta_Nu_{N-1})^2},$$

$$v_N = \delta_N(1 - \delta_N u_{N-1}), \tag{40}$$

and the N-soliton solution of Eq. (4) is presented by the constraint $q_N = u_N v_N$.

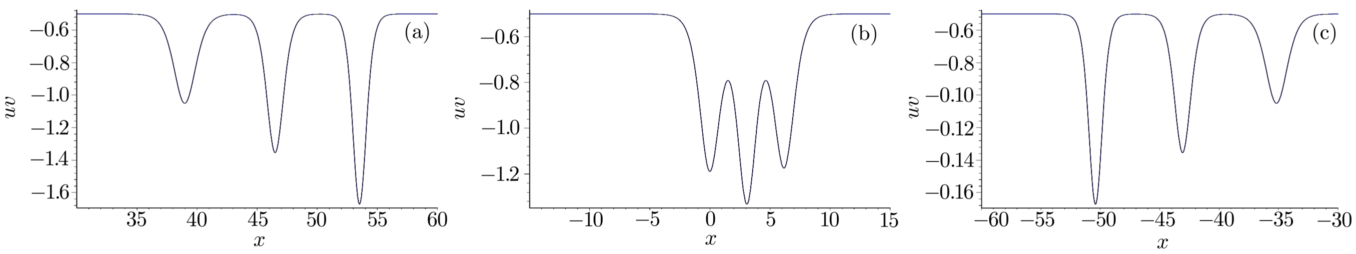

Fig. 3 (a), (b), and (c) Interaction of the left-going dark three-soliton overtaking collision of $u_0 = 0.5$, $\lambda_1 = 3$, $\lambda_2 = 3.5$, $\lambda_3 = 4$ at $t = 0$, $y = -10$, -0.3 and 10; The large-amplitude solitons overtake the small-amplitude ones.

(ii) Choose $(u, v) = (u_0, -1)$ $(u_0 \neq 0)$ as the seed solution of the (2+1)-dimensional mKdV equation (4), the basic solutions of the linear system (23)–(25) can be chosen as

$$\phi^{(j)} = \begin{pmatrix} \phi_1^{(j)} \\ \phi_2^{(j)} \end{pmatrix} = \begin{pmatrix} \cosh\xi_j \\ (\sqrt{(\lambda_j - 4u_0)\lambda_j}/2u_0\lambda_j)\sinh\xi_j + (1/2u_0)\cosh\xi_j \end{pmatrix}, \quad j \text{ is odd number},$$

$$\phi^{(j)} = \begin{pmatrix} \phi_1^{(j)} \\ \phi_2^{(j)} \end{pmatrix} = \begin{pmatrix} \sinh\xi_j \\ (\sqrt{(\lambda_j - 4u_0)\lambda_j}/2u_0\lambda_j)\cosh\xi_j + (1/2u_0)\sinh\xi_j \end{pmatrix}, \quad j \text{ is even number},$$

here

$$\xi_j = \frac{\sqrt{(\lambda_j - 4u_0)\lambda_j}[x + (\lambda_j + 2u_0)y + (\lambda_j^2 + 2\lambda_j u_0 + 6u_0^2)t]}{2}.$$

Based on the above basic solutions, the single, two and three-soliton solutions of Eq. (4) can be directly given similar to the formula (38) and (39). The interaction of the left-going dark three-soliton overtaking collision is shown in Fig. 3.

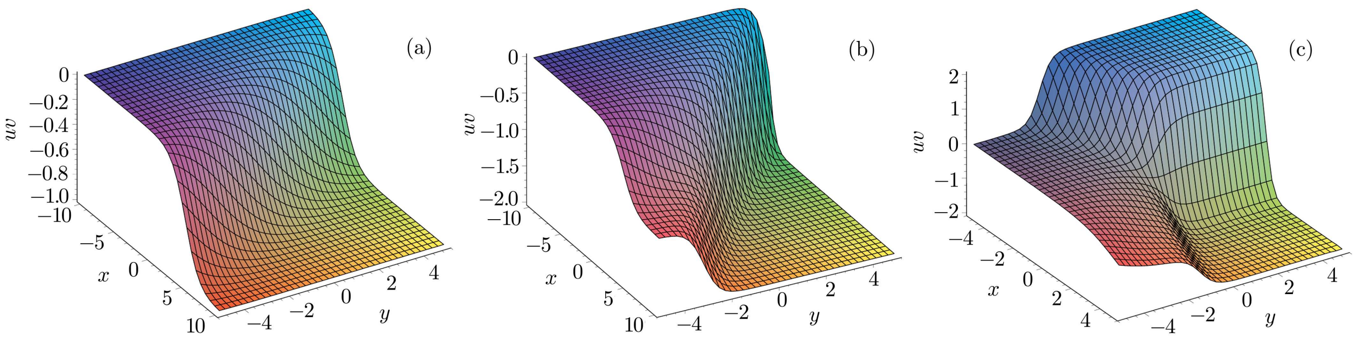

Fig. 4 (a), (b), and (c) The one-kink solution with $u_0 = 1$, $\lambda_1 = 1$, two-kink solution with $u_0 = 1$, $\lambda_1 = 1$, $\lambda_2 = 2$ and three-kink soliton with $u_0 = 1$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = -2$ at $t = 0$.

(iii) Choose $(u, v) = (u_0, 0)$ $(u_0 \neq 0)$ as the seed solution of the (2+1)-dimensional mKdV equation (4), the basic solutions of the linear Lax pairs (25)–(27) can be chosen as

$$\phi^{(j)} = \begin{pmatrix} \phi_1^{(j)} \\ \phi_2^{(j)} \end{pmatrix} = \begin{pmatrix} (u_0\,\mathrm{e}^{(2\lambda_j x + \lambda_j^2 y + \lambda_j^3 t)/2} + \mathrm{e}^{-(\lambda_j^2 y + \lambda_j^3 t)/2})\,\mathrm{e}^{-\lambda_j x/2} \\ \mathrm{e}^{(\lambda_j x + \lambda_j^2 y + \lambda_j^3 t)/2} \end{pmatrix}, \quad j \text{ is odd number},$$

$$\phi^{(j)} = \begin{pmatrix} \phi_1^{(j)} \\ \phi_2^{(j)} \end{pmatrix} = \begin{pmatrix} (u_0\,\mathrm{e}^{(2\lambda_j x + \lambda_j^2 y + \lambda_j^3 t)/2} - \mathrm{e}^{-(\lambda_j^2 y + \lambda_j^3 t)/2})\,\mathrm{e}^{-\lambda_j x/2} \\ \mathrm{e}^{(\lambda_j x + \lambda_j^2 y + \lambda_j^3 t)/2} \end{pmatrix}, \quad j \text{ is even number}.$$

Then some kink soliton solutions of Eq. (4) can be given, see Fig. 4.

4 Discussions

In this paper, we investigate two (2+1)-dimensional nonlinear equations, which are the (2+1)-dimensional CDGKS equation (1) and (2+1)-dimensional mKdV equation (4). Two DTs of these two equations are constructed through the Darboux matrix method, respectively. N-soliton solutions of Eqs. (1) and (4) are obtained by applying the DT N times. For the (2+1)-dimensional CDGKS equation (1), the right-going bright single-soliton solution and interactions of two and three-soliton overtaking collisions are studied, the small-amplitude solitons interestingly overtake the large-amplitude ones. For the (2+1)-dimensional mKdV equation (4), the right-going bright and left-going dark single-solitons, the interactions of two and three-soliton overtaking collisions are investigated, the large-amplitude solitons naturally overtake the small-amplitude ones. Moreover, by choosing different kind of seed solution, some kink soliton solutions of Eq. (4) are plotted. Our results can be used to illustrate the interactions of water waves in shallow water.

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