Chin. Ann. Math.
35B(6), 2014, 841–856
DOI: 10.1007/s11401-014-0868-2

Chinese Annals of Mathematics, Series B

Nonlocal Symmetries and Explicit Solutions of the Boussinesq Equation*

Xiangpeng XIN[1] Junchao CHEN[1] Yong CHEN[1]

Abstract The nonlocal symmetry of the Boussinesq equation is obtained from the known Lax pair. The explicit analytic interaction solutions between solitary waves and cnoidal waves are obtained through the localization procedure of nonlocal symmetry. Some other types of solutions, such as rational solutions and error function solutions, are given by using the fourth Painlevé equation with special values of the parameters. For some interesting solutions, the figures are given out to show their properties.

Keywords Nonlocal symmetry, Lax pair, Prolonged system, Explicit solution
2000 MR Subject Classification 22E05, 34A05, 34A34

1 Introduction

Since Sophus Lie [1] set up the theory of the Lie point symmetry group, the standard method has been widely used to find Lie point symmetry (see [2–10]) for the differential equations (DEs, for short). During the past forty years, the study of symmetries (local and nonlocal) has been connected with the development of soliton theory and in fact, it constitutes an indispensable part of soliton theory.

Because the nonlocal symmetries enlarge the class of symmetries and they are connected with integrable models, therefore, to search for nonlocal symmetries (see [11–16]) of the nonlinear systems is an interesting work. In a number of cases, the nonlocal symmetries may be easily obtained with the help of a recursion operator (see [17]), but sometimes the recursion operators are difficult to obtain. In [18], Akhatov and Gazizov provided a method for constructing nonlocal symmetries of DEs based on the Lie-Bäcklund theory. Bluman introduced the concept of potential symmetry (see [11]) for a differential system by writing the given system in a conserved form. Galas [14] obtained the nonlocal Lie-Bäcklund symmetries by introducing the pseudo-potentials as an auxiliary system. Recently, Lou et al. [19–20] have made some efforts to obtain infinite many nonlocal symmetries by inverse recursion operators, the conformal invariant form (Schwartz form) and Darboux transformation.

As being known, a basic problem for the construction of nonlocal symmetries is the proper choice of nonlocal variables. They are defined by integrable systems of differential equations which relate the nonlocal variables to the original differential variables. The choice of these

Manuscript received June 5, 2013. Revised December 11, 2013.
[1]Shanghai Key Laboratory of Trustworthy Computing, East China Normal University, Shanghai 200062, China. E-mail: xinxiangpeng2012@gmail.com junchaochen85@hotmail.com ychen@sei.ecnu.edu.cn
*This work was supported by the National Natural Science Foundation of China (Nos. 11275072, 11435005), the Research Fund for the Doctoral Program of Higher Education of China (No. 20120076110024), the Innovative Research Team Program of the National Natural Science Foundation of China (No. 61321064), the Shanghai Knowledge Service Platform for Trustworthy Internet of Things (No. ZF1213), the Shanghai Minhang District Talents of High Level Scientific Research Project and the Talent Fund and K. C. Wong Magna Fund in Ningbo University.

differential equations is made on the basis of some additional considerations. In this paper, we intend to give a method (see [21]) to seek the nonlocal symmetries with the auxiliary systems. Different from other methods, we assume that the infinitesimal coefficients of the symmetries have integral terms or high-order derivative terms of nonlocal variables. The fact proved that this method can get nonlocal symmetries quickly and efficiently (see [21]), and the most important thing is that it can realize mechanization on program design.

Moreover, the finite symmetry transformation and similar reduction can not be directly applied to nonlocal symmetries. It is necessary to inquire whether one can transform nonlocal symmetries into local symmetries by extending the original system. The introduction of potential and pseudopotential-type symmetries (see [22–23]) which possess close prolongation extends the applicability of symmetry methods to obtain solutions of differential equations (DEs, for short). In that context, the original given equation(s) can be embedded in some prolonged systems. Hence, these nonlocal symmetries with close prolongation are anticipated (see [14, 24]).

In this paper, we consider the following Boussinesq equation:

$$u_{tt} + \alpha u_{xx} + \beta(u^2)_{xx} + \gamma u_{xxxx} = 0, \tag{1.1}$$

where α, β and γ are constants. This equation was introduced by Boussinesq in 1871 to describe the propagation of long waves in shallow water. The Boussinesq equation also arises in several other physical applications including one-dimensional nonlinear lattice waves, vibrations in a nonlinear string, and ion sound waves in a plasma.

Our renewed interest in the Boussinesq equation is explained mostly by the unusual behavior of the soliton solutions which were discovered in the 1970s. Multiple travelling wave solutions of this equation were obtained (see [25–26]). In [27] Clarkson and Kruskal obtained some new similarity reductions of the Boussinesq equation, including some first, second, and fourth Painlevé equations which can not be obtained using the standard Lie group method. The Wronskian formulation of solutions to the Boussinesq equation was presented by using its bilinear form (see [28]).

Clearly, the solitary waves must interact with other waves, say, the cnoidal waves which are periodic and may be described by Jacobi elliptic functions. However, there are few works in the literature that study the interactions between the periodic cnoidal waves and solitary waves. An application of the Darboux transformation on a cnoidal wave background in the coupled nonlinear Schrödinger equation (see [29]) gives a new solution which describes a soliton moving on a cnoidal wave. In this paper, the explicit expression of cnoidal-solitary wave interaction solutions for the Boussinesq equation is shown by the nonlocal symmetry method.

This paper is arranged as follows: In Section 2, The nonlocal symmetries of the Boussinesq equation are obtained by using both the Darboux transformation and symmetry assumption methods with the Lax pair. In Section 3, the nonlocal symmetries transform into Lie point symmetries by extending the original system. The finite symmetry transformation can be obtained in Section 4. In Section 5, some symmetry reductions and explicit solutions of the Boussinesq can be obtained by using the Lie point symmetry of the extending system. Finally, some conclusions and discussions are given in Section 6.

2 Nonlocal Symmetries of Boussinesq Equation

Without loss of generality, we assume that $\alpha = 0$, $\beta = 1$ and $\gamma = \frac{1}{3}$ in the Boussinesq equation (1.1)

$$u_{tt} + (u^2)_{xx} + \frac{1}{3}u_{xxxx} = 0, \tag{2.1}$$

and (2.1) is equivalent to (1.1) after suitable rescaling and translation of the variables (see [27]).

The corresponding Lax pair is

$$\psi_{xxx} + \frac{3}{2}u\psi_x + \Big(\frac{3}{4}u_x - \frac{3}{4}\partial_x^{-1}u_t\Big)\psi = \lambda\psi, \tag{2.2}$$

$$\psi_t = -\psi_{xx} - u\psi. \tag{2.3}$$

Here we give two methods to search for the nonlocal symmetry of the Boussinesq equation. First, we use the invariant properties of differential equations exhibited by Darboux transformation. Starting from the DT theorem, we have the following proposition.

Proposition 2.1 *Let u be a solution of the Boussinesq equation* (2.1) *with ψ satisfying* (2.2)–(2.3)*. Then $\overline{u} = u + 2\ln(\psi)_{xx}$ is also a solution of* (2.1).

Now using the DT above, one can obtain the following result:

Proposition 2.2 $\sigma = \big(\frac{\widetilde{\varphi}}{\varphi}\big)_{xx}$ *is a symmetry of the Boussinesq equation* (2.1)*, where $\varphi(x,t)$ and $\widetilde{\varphi}(x,t)$ satisfy the following equations:*

$$\varphi_{xxx} + \frac{3}{2}(u - 2\ln(\varphi)_{xx})\varphi_x + \Big(\frac{3}{4}(u_x - 2\ln(\varphi)_{xxx}) - \frac{3}{4}\partial_x^{-1}u_t + 2\ln(\varphi)_{xt}\Big)\varphi = 0, \tag{2.4}$$

$$\varphi_t + \varphi_{xx} + (u - 2\ln(\varphi)_{xx})\varphi = 0, \tag{2.5}$$

$$\widetilde{\varphi}_{xxx} + \frac{3}{2}(u - 2\ln(\varphi)_{xx})\widetilde{\varphi}_x + \Big(\frac{3}{4}(u_x - 2\ln(\varphi)_{xxx}) - \frac{3}{4}\partial_x^{-1}u_t + 2\ln(\varphi)_{xt}\Big)\widetilde{\varphi} = \varphi, \tag{2.6}$$

$$\widetilde{\varphi}_t + \widetilde{\varphi}_{xx} + (u - 2\ln(\varphi)_{xx})\widetilde{\varphi} = 0. \tag{2.7}$$

Proof Set $U = u + 2\ln(\psi(x,t,0))_{xx}$. From Proposition 2.1, we know that U is a solution of the Boussinesq equation (2.1). Now we formally expand $\overline{u}$ in powers of λ,

$$\overline{u} = U + \lambda\Big[\Big(2\frac{\partial}{\partial x^2}\ln\psi\Big)_\lambda\Big|_{\lambda=0}\Big] + \mathrm{O}(\lambda^2).$$

Thus $(2\frac{\partial}{\partial x^2}\ln\psi)_\lambda|_{\lambda=0}$ is a symmetry of (2.1), with respect to U. Finally, we can prove this proposition by substituting $u = U - 2\ln(\psi(x,t,0))_{xx}$ in (2.2)–(2.3) which leads to (2.4)–(2.7), with U replaced by u, $\psi(x,t,0)$ by $\varphi(x,t)$ and $\psi_\lambda(x,t,0)$ by $\widetilde{\varphi}(x,t)$. Thus we have completed the proof of Proposition 2.2.

A direct calculation shows that if φ satisfies (2.4)–(2.5) and φ^* satisfies

$$\varphi^*_{xxx} + \frac{3}{2}(u - 2\ln(\varphi^*)_{xx})\varphi^*_x + \Big(\frac{3}{4}(u_x - 2\ln(\varphi^*)_{xxx}) + \frac{3}{4}\partial_x^{-1}u_t - 2\ln(\varphi^*)_{xt}\Big)\varphi^* = 0, \tag{2.8}$$

$$\varphi^*_t + \varphi^*_{xx} + (u - 2\ln(\varphi^*)_{xx})\varphi^* = 0, \tag{2.9}$$

then

$$\widetilde{\varphi} = \varphi\int_{x_0}^{x}\frac{1}{\varphi\varphi^*}\mathrm{d}x + h(t)\varphi, \tag{2.10}$$

where $h(t)$ is a polynomial of t. Moreover, it can be easily verified that if φ is a solution of (2.4)–(2.5) and φ^* is a solution of (2.8)–(2.9), then $\varphi = \frac{1}{\varphi^*}$ satisfies (2.2)–(2.3) with $\lambda = 0$ and $\varphi^* = \frac{1}{\varphi}$ satisfies

$$\varphi^*_{xxx} + \frac{3}{2}u\varphi^*_x + \Big(\frac{3}{4}u_x u - \frac{3}{4}\partial_x^{-1}u_t\Big)\varphi^* = 0, \tag{2.11}$$

$$\varphi^*_t = \varphi^*_{xx} + u\varphi^*. \tag{2.12}$$

To sum up, we have that $\sigma = (\varphi\varphi^*)_x$ is a nonlocal symmetry of the Boussinesq equation (2.1), where φ satisfies (2.2)–(2.3) with $\lambda = 0$ and φ^* satisfies (2.11)–(2.12).

Next, we use the Boussinesq equation (2.1) and the Lax pair to seek the nonlocal symmetry directly. For the sake of convenience, one can use ϕ instead of φ^*. Then the corresponding Lax pair of (2.1) with $\lambda = 0$ has the form

$$\psi_{xxx} = -\frac{3}{2}u\psi_x - \Big(\frac{3}{4}u_x + \frac{3}{4}\partial_x^{-1}u_t\Big)\psi, \tag{2.13}$$

$$\psi_t = -\psi_{xx} - u\psi, \tag{2.14}$$

and its adjoint version is

$$\phi_{xxx} = -\frac{3}{2}u\phi_x - \Big(\frac{3}{4}u_x - \frac{3}{4}\partial_x^{-1}u_t\Big)\phi, \tag{2.15}$$

$$\phi_t = \phi_{xx} + u\phi. \tag{2.16}$$

That is to say, the integrable conditions of (2.13)–(2.16), $\phi_{xxxt} = \phi_{txxx}$ and $\psi_{xxxt} = \psi_{txxx}$, are just the Boussinesq equation (2.1).

A symmetry σ^u of the Boussinesq equation is defined as a solution of its linearized equation

$$\frac{\partial^2}{\partial t^2}\sigma^u + 4\frac{\partial}{\partial x}u\frac{\partial}{\partial x}\sigma^u + 2u\frac{\partial^2}{\partial x^2}\sigma^u + 2\sigma^u\frac{\partial^2}{\partial x^2}u + \frac{1}{3}\frac{\partial^4}{\partial x^4}\sigma^u = 0, \tag{2.17}$$

which means that (2.1) is form invariant under the transformation

$$u \to u + \varepsilon\sigma^u, \tag{2.18}$$

with the infinitesimal parameter ε.

The symmetry can be written in the form

$$\sigma^u = Xu_x + Tu_t - U, \tag{2.19}$$

and here, we give an assumption that let X, T, U be the functions of the variables $(x, t, u, \phi, \psi, \phi_x, \psi_x, \phi_{xx}, \psi_{xx})$. The assumption shows that this kind of symmetries is neither classical Lie point symmetries nor Lie-Bäcklund symmetries because it is dependent on the auxiliary variables and their high-order partial derivatives. Substituting (2.19) into (2.17) and eliminating u_{tt}, ϕ_{xxx}, ϕ_t, ψ_{xxx} and ψ_t in terms of the closed system, we get the determining equations for the functions X, T, U. Calculated by computer algebra, the general solutions of them take the form

$$X = \frac{1}{2}c_1x + c_3, \quad T = c_1t + c_2, \quad U = c_4\psi_x\phi + c_4\psi\phi_x - c_1u.$$

Then the symmetry of (2.1) satisfies

$$\sigma = \Big(\frac{1}{2}c_1x + c_3\Big)u_x + (c_1t + c_2)u_t + c_1u - c_4\psi_x\phi - c_4\psi\phi_x, \tag{2.20}$$

where c_i $(i = 1, 2, 3, 4)$ are arbitrary constants.

Proposition 2.3 *If we set* $c_1 = c_2 = c_3 = 0$, $c_4 = -1$ *in* (2.20), *i.e.,* (2.1) *has a simple nonlocal symmetry*

$$\sigma = (\phi\psi)_x, \tag{2.21}$$

which is the same result as in [30] *with* ψ *and* ϕ *being the solutions of* (2.13)–(2.16).

Remark 2.1 We found that the highest derivative terms in the Lax pair are ψ_{xxx} and ϕ_{xxx}, so the orders of ψ and ϕ in σ^u must be less than 3. If we assume X, T, U to be the functions of the variables $\{x, t, u, \int \psi \mathrm{d}x, \int \varphi \mathrm{d}x, \phi, \psi, \phi_x, \psi_x, \phi_{xx}, \psi_{xx}\}$, then a more general solution may be obtained.

By comparison, the second method is more simple and effective, and this method can be applied to other kinds of integrable systems.

3 Localization of the Nonlocal Symmetry

We know that nonlocal symmetries can not be directly employed to construct explicit solutions for differential equations. Hence, nonlocal symmetries need to be transformed into local ones. One may extend the original system to a closed prolonged system by introducing some additional dependable variables.

From (2.21), it can be apparently seen that the nonlocal symmetry contains the space derivative of functions ϕ and ψ. Then, to localize the nonlocal symmetry (2.21), we introduce the following transformations:

$$\psi_1 = \psi_x \tag{3.1}$$

and

$$\phi_1 = \phi_x, \tag{3.2}$$

whence the field u has the symmetry transformation $u \to u + \varepsilon\sigma^u$. In other words, we have to solve the linearized equations of (2.14), (2.16) and (3.1)–(3.2),

$$\begin{aligned} &\sigma^\psi_t + \sigma^\psi_{xx} + u\sigma^\psi + \psi\sigma^u = 0, \\ &\sigma^\phi_t - \sigma^\phi_{xx} - u\sigma^\phi + \phi\sigma^u = 0, \\ &\sigma^{\psi_1} = \sigma^\psi_x, \\ &\sigma^{\phi_1} = \sigma^\phi_x, \end{aligned} \tag{3.3}$$

whence σ^u is given by (2.21).

It is not difficult to verify that the solution of (3.3) with (2.21) has the form

$$\begin{aligned} \sigma^\psi &= \frac{1}{2}\psi p, \\ \sigma^\phi &= \frac{1}{2}\phi p, \\ \sigma^{\psi_1} &= \frac{1}{2}(\psi_1 p + \psi^2\phi), \\ \sigma^{\phi_1} &= \frac{1}{2}(\phi_1 p + \phi^2\psi), \end{aligned} \tag{3.4}$$

where the new quantity p is defined as

$$p_x = -\phi\psi. \tag{3.5}$$

The compatibility condition of (3.5) is worth to be mentioned here

$$p_t = \phi\psi_x - \psi\phi_x, \tag{3.6}$$

which means the condition $p_{xt} = p_{tx}$.

Due to the appearance of the quantity p in the symmetry solution (3.4), we have to further solve the linearized equation of (3.5)

$$\sigma_x^p = -(\sigma^\psi \phi + \sigma^\phi \psi) \tag{3.7}$$

with the condition (3.4).

It is easy to solve (3.7) with (3.5), and the σ^p has the simple form

$$\sigma^p = \frac{1}{2}p^2. \tag{3.8}$$

The result (3.8) gives us a hint that p is a solution of the Schwarzian Boussinesq equation

$$\{p; x\}_x + 3\Big(\frac{p_t}{p_x}\Big)_t + 3\Big(\frac{p_t}{p_x}\Big)\Big(\frac{p_t}{p_x}\Big)_x = 0, \tag{3.9}$$

where the Schwarzian derivative $\{p; x\} \equiv \frac{p_{xxx}}{p_x} - \frac{3}{2}\frac{p_{xx}^2}{p_x^2}$, and the quantity $\frac{p_t}{p_x}$ is invariant under the Möbious transformation invariant with the infinitesimal transformation (3.8)

$$p \to \frac{a + bp}{c + dp} \quad (ad \neq cb).$$

The results (3.4) and (3.8) show us that the nonlocal symmetry (2.21) in the original space x, t, u has been successfully localized to a Lie point symmetry in the enlarged space $\{x, t, u, \phi, \psi, \phi_1, \psi_1, p\}$ with the vector form

$$\begin{aligned} V = {} & (\psi\phi_1 + \phi\psi_1)\partial u + \frac{1}{2}\psi p \partial\psi + \frac{1}{2}\phi p \partial\phi \\ & + \frac{1}{2}(\psi_1 p + \psi^2\phi)\partial\psi_1 + \frac{1}{2}(\phi_1 p + \phi^2\psi)\partial\phi_1 + \frac{1}{2}p^2\partial p. \end{aligned} \tag{3.10}$$

4 Finite Symmetry Transformation

After we succeed in making the nonlocal symmetry (2.21) equivalent to Lie point symmetry (3.10) of the related prolonged system, the explicit solutions can be constructed naturally by Lie group theory in two aspects. With the Lie point symmetry (3.10), by solving the following initial value problem:

$$\begin{aligned} & \frac{\mathrm{d}u'(\varepsilon)}{\mathrm{d}\varepsilon} = \psi\phi_1 + \psi_1\phi, \quad u'(0) = u, \\ & \frac{\mathrm{d}\psi'(\varepsilon)}{\mathrm{d}\varepsilon} = \frac{1}{2}\psi'(\varepsilon)p'(\varepsilon), \quad \psi'(0) = \psi, \\ & \frac{\mathrm{d}\phi'(\varepsilon)}{\mathrm{d}\varepsilon} = \frac{1}{2}\phi'(\varepsilon)p'(\varepsilon), \quad \phi'(0) = \phi, \\ & \frac{\mathrm{d}\psi_1'(\varepsilon)}{\mathrm{d}\varepsilon} = \frac{1}{2}(\psi_1'(\varepsilon)p'(\varepsilon) + \psi'(\varepsilon)^2\phi'(\varepsilon)), \quad \psi_1'(0) = \psi_1, \\ & \frac{\mathrm{d}\phi_1'(\varepsilon)}{\mathrm{d}\varepsilon} = \frac{1}{2}(\phi_1'(\varepsilon)p'(\varepsilon) + \phi'(\varepsilon)^2\psi'(\varepsilon)), \quad \phi_1'(0) = \phi_1, \\ & \frac{\mathrm{d}p'(\varepsilon)}{\mathrm{d}\varepsilon} = \frac{1}{2}p'(\varepsilon)^2, \quad p'(0) = p, \end{aligned} \tag{4.1}$$

the finite symmetry transformation can be calculated as

$$
\begin{aligned}
&u'(\varepsilon) = u + \frac{2\varepsilon(\psi\phi_1 + \psi_1\phi)}{2 - \varepsilon p} + \frac{2\varepsilon^2\phi^2\psi^2}{(2-\varepsilon p)^2},\\
&\psi_1'(\varepsilon) = \frac{2\psi_1}{2-\varepsilon p} + \frac{2\varepsilon\phi\psi^2}{(2-\varepsilon p)^2}, \quad \phi_1'(\varepsilon) = \frac{2\phi}{2-\varepsilon p} + \frac{2\varepsilon\phi^2\psi}{(2-\varepsilon p)^2},\\
&\psi'(\varepsilon) = \frac{2\psi}{2-\varepsilon p}, \quad \phi'(\varepsilon) = \frac{2\phi}{2-\varepsilon p}, \quad p'(\varepsilon) = \frac{2p}{2-\varepsilon p}.
\end{aligned} \tag{4.2}
$$

Remark 4.1 For a given solution u of (2.1), the above finite symmetry transformation will arrive at another solution u'.

5 New Symmetry Reductions of the Boussinesq Equation

To search for more similarity reductions (see [31–32]) of (2.1), we study Lie point symmetries of the whole prolonged equation system instead of (2.1). In order to find the Lie point symmetry, we may assume that the symmetries have the vector form

$$
V = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + \Psi\frac{\partial}{\partial \psi} + \Phi\frac{\partial}{\partial \phi} + \Psi_1\frac{\partial}{\partial \psi_1} + \Phi_1\frac{\partial}{\partial \phi_1} + P\frac{\partial}{\partial p}, \tag{5.1}
$$

where $X,\ T,\ U,\ \Psi,\ \Phi,\ \Psi_1,\ \Phi_1$ and P are the functions with respect to $\{x, t, u, \psi, \phi, \psi_1, \phi_1, p\}$, which means that the closed system is invariant under the transformations

$$
\{x, t, u, \psi, \phi, \psi_1, \phi_1, p\} \to (x + \varepsilon X, t + \varepsilon T, u + \varepsilon U, \psi + \varepsilon\Psi, \phi + \varepsilon\Phi, \psi_1 + \varepsilon\Psi_1, \phi_1 + \varepsilon\Phi_1, p + \varepsilon P)
$$

with a small parameter ε. Equivalently, the symmetries in the vector form (5.1) can be written in a function form

$$
\begin{aligned}
&\sigma_u = Xu_x + Tu_t - U, \quad \sigma_\psi = X\psi_x + T\psi_t - \Psi, \quad \sigma_\phi = X\phi_x + T\phi_t - \Phi,\\
&\sigma_{\psi_1} = X\psi_{1,x} + T\psi_{1,t} - \Psi_1, \quad \sigma_{\phi_1} = X\phi_{1,x} + T\phi_{1,t} - \Phi_1, \quad \sigma_p = Xp_x + Tp_t - P.
\end{aligned} \tag{5.2}
$$

In this notation, $\sigma_u,\ \sigma_\phi,\ \sigma_\psi,\ \sigma_{\phi_1},\ \sigma_{\psi_1}$ and σ_p are the solutions of the symmetry equations, i.e., the linearized equations for the closed system

$$
\begin{aligned}
&\sigma_{u,tt} + 4u_x\sigma_{u,x} + 2u\sigma_{u,xx} + 2\sigma_u u_{xx} + \frac{1}{3}\sigma_{u,xxxx} = 0,\\
&\sigma_{\psi,xx} + \sigma_{\psi,t} + u\sigma_\psi + \sigma_u\psi = 0,\\
&\sigma_{\phi,xx} - \sigma_{\phi,t} + u\sigma_\phi + \sigma_u\phi = 0,\\
&\sigma_{\psi,x} - \sigma_{\psi_1} = 0,\\
&\sigma_{\phi,x} - \sigma_{\phi_1} = 0,\\
&\sigma_{p,x} + \psi\sigma_\phi + \sigma_\psi\phi = 0.
\end{aligned} \tag{5.3}
$$

Substituting (5.2) into (5.3) and eliminating $u_{tt},\ \psi_{xx},\ \psi_t,\ \phi_{xx},\ \phi_t$ and p_x in terms of the closed system, we get the determining equations for the functions $X,\ T,\ U,\ \Phi,\ \Psi,\ \Phi_1,\ \Psi_1$ and P. Calculated by computer algebra, the general solutions of them take the form

$$
\begin{aligned}
&X = \frac{c_1}{2}x + c_3, \quad T = c_1 t + c_2, \quad U = -c_1 u + c_4(\psi\phi_1 + \phi\psi_1),\\
&\Psi = \frac{(c_4 p + 2c_6)\psi}{2}, \quad \Phi = \frac{(c_4 p + 2c_5)\phi}{2}, \quad \Psi_1 = \frac{(c_4 p - c_1 + c_6)\psi_1}{2} - \frac{c_4\phi\psi^2}{2},\\
&\Phi_1 = \frac{(c_4 p - c_1 + c_5)\varphi_1}{2} - \frac{c_4\psi\varphi^2}{2}, \quad P = f(t) + \frac{c_4}{2}p^2 + \Big(\frac{c_1}{2} + c_5 + c_6\Big)p,
\end{aligned} \tag{5.4}
$$

where f is an arbitrary function of t and c_i $(i=1,2,\cdots,6)$ are arbitrary constants. Consequently, it is convenient to rewrite symmetries (5.3) as

$$\begin{aligned}
\sigma_u &= \Big(\frac{c_1}{2}x+c_3\Big)u_x+(c_1t+c_2)u_t+c_1u-c_4(\psi\phi_1+\phi\psi_1),\\
\sigma_\psi &= \Big(\frac{c_1}{2}x+c_3\Big)\psi_x+(c_1t+c_2)\psi_t-\frac{(c_4p+2c_6)\psi}{2},\\
\sigma_\phi &= \Big(\frac{c_1}{2}x+c_3\Big)\phi_x+(c_1t+c_2)\phi_t-\frac{(c_4p+2c_5)\phi}{2},\\
\sigma_{\psi_1} &= \Big(\frac{c_1}{2}x+c_3\Big)\psi_{1x}+(c_1t+c_2)\psi_{1t}-\frac{(c_4p-c_1+c_6)\psi_1}{2}+\frac{c_4\phi\psi^2}{2},\\
\sigma_{\varphi_1} &= \Big(\frac{c_1}{2}x+c_3\Big)\phi_{1x}+(c_1t+c_2)\phi_{1t}-\frac{(c_4p-c_1+c_5)\phi_1}{2}+\frac{c_4\psi\phi^2}{2}.
\end{aligned}\tag{5.5}$$

To give more group invariant solutions, we would like to solve the symmetry constraint conditions, by setting $\sigma_u,\ \sigma_\phi,\ \sigma_\psi,\ \sigma_{\phi_1},\ \sigma_{\psi_1}$ and σ_p to be zeros in (5.5), which is equivalent to solving the characteristic equations

$$\begin{aligned}
\frac{\mathrm{d}x}{\frac{c_1}{2}x+c_3} &= \frac{\mathrm{d}t}{c_1t+c_2}=\frac{\mathrm{d}u}{c_4(\psi\phi_1+\phi\psi_1)-c_1u}=\frac{\mathrm{d}\psi}{\frac{(c_4p+2c_6)\psi}{2}}\\
&= \frac{\mathrm{d}\phi_1}{\frac{(c_4p-c_1+c_5)\phi_1}{2}-\frac{c_4\psi\phi^2}{2}}=\frac{\mathrm{d}p}{f(t)+\frac{c_4}{2}p^2+\Big(\frac{c_1}{2}+c_5+c_6\Big)p}\\
&= \frac{\mathrm{d}\phi}{\frac{(c_4p+2c_5)\phi}{2}}=\frac{\mathrm{d}\psi_1}{\frac{(c_4p-c_1+c_6)\psi_1}{2}-\frac{c_4\phi\psi^2}{2}}.
\end{aligned}\tag{5.6}$$

In the following part of the paper, two nontrivial cases under the consideration $c_4\neq 0$ in (5.6) are listed.

Case 1 $c_1\neq 0$ and $c_2=c_3=c_5=c_6=0,\ f(t)=c_7$.

Firstly, we redefine the parameter $c^2=\frac{c_1^2-4c_4c_7}{16c_1^2}$ instead of facilitating the later computation. Two situations with $c\neq 0$ and $c=0$ are given out respectively.

(i) When $c\neq 0$, by solving (5.6), we have

$$\begin{aligned}
p &= -\frac{1}{2}\frac{c_1(1+4c\tanh(\Delta_1))}{c_4},\quad \psi=t^{-\frac14}R(z)\mathrm{e}^{-\frac14P(z)}\mathrm{sech}(\Delta_1),\\
\phi &= t^{-\frac14}Q(z)\mathrm{e}^{-\frac14P(z)}\mathrm{sech}(\Delta_1),\\
\psi_1 &= \frac{1}{2cc_1}t^{-\frac34}\mathrm{e}^{-\frac34P(z)}(2cc_1R_1(z)\mathrm{sech}(\Delta_1)-c_4Q(z)R^2(z)\tanh(\Delta_1)\mathrm{sech}(\Delta_1)),\\
\phi_1 &= \frac{1}{2cc_1}t^{-\frac34}\mathrm{e}^{-\frac34P(z)}(2cc_1Q_1(z)\mathrm{sech}(\Delta_1)-c_4Q^2(z)R(z)\tanh(\Delta_1)\mathrm{sech}(\Delta_1)),\\
u &= \frac{1}{2c^2c_1^2t}(2cc_1c_4\mathrm{e}^{-P(z)}R(z)Q_1(z)\tanh(\Delta_1)+2cc_1c_4\mathrm{e}^{-P(z)}Q(z)R_1(z)\tanh(\Delta_1)\\
&\quad +c_4^2\mathrm{e}^{-P(z)}Q^2(z)R^2(z)\mathrm{sech}^2(\Delta_1))+\frac{U(z)}{t},
\end{aligned}\tag{5.7}$$

where $\Delta_1=c(\ln(t)+P(z)),\ z=\frac{x}{\sqrt{t}}$. $U(z),\ Q(z),\ R(z),\ Q_1(z),\ R_1(z)$ and $P(z)$ in (5.7) represent the group invariants, and substituting (5.7) into the prolonged system yields

$$
\begin{aligned}
Q(z) &= \exp\Big(\frac{1}{4}\int \frac{P_z^2(z)+2P_{zz}(z)-zP_z(z)+2}{P_z(z)}\mathrm{d}z\Big), \quad R(z)=\frac{2c^2c_1\mathrm{e}^{\frac{1}{2}P(z)}P_z(z)}{c_4Q(z)},\\
Q_1(z) &= \frac{1}{4}\frac{(2+2P_{zz}(z)-zP_z(z))Q(z)\mathrm{e}^{\frac{1}{2}P(z)}}{P_z(z)},\\
R_1(z) &= \frac{1}{2}\frac{c^2c_1\mathrm{e}^{P(z)}(-2+zP_z(z)+2P_{zz}(z))}{c_4Q(z)},\\
U(z) &= \frac{1}{16}\frac{-16c^2P_z^4(z)+z^2P_z^2(z)-8P_z(z)P_{zzz}(z)+P_{zz}^2(z)-4}{P_z^2(z)},
\end{aligned} \tag{5.8}
$$

where $P(z)$ satisfies a four-order ordinary differential equation

$$
\begin{aligned}
&12P_z^2-4P_z^2P_{zzzz}-9zP_z^3+16c^2P_z^4P_{zz}+16P_zP_{zz}P_{zzz}\\
&-12zP_zP_{zz}-12P_{zz}^3+12P_{zz}=0.
\end{aligned} \tag{5.9}
$$

First, one can simplify (5.9) by using $w_1(z)$ to replace P_z, and the reduction equation is

$$
\begin{aligned}
&12w_1^2-4w_1^2w_{1zzz}-9zw_1^3-12w_{1z}^3+12w_{1z}+16c^2w_1^4w_{1z}\\
&+16w_1w_{1z}w_{1zz}-12zw_1w_{1z}=0.
\end{aligned} \tag{5.10}
$$

The equation (5.10) can be solved in terms of solutions of the equation

$$
N_{1zz}=\frac{3N_{1z}^2}{2N_1}+2c^2N_1^3-\frac{9N_1z^2}{8}-N_1+3z-\frac{3}{2N_1}. \tag{5.11}
$$

We introduce $M_1(z_1)$ by

$$
N_1=-\frac{2\sqrt{3}}{3M_1(z_1)},\quad z_1=\frac{\sqrt{3}z}{2} \tag{5.12}
$$

which converts (5.11) into the fourth Painlevé equation (PIV, for short):

$$
M_{1z_1z_1}=\frac{1}{2}\frac{M_{1z}^2}{M_1}+\frac{3}{2}M_1^3+4z_1M_1^2+2\Big(z_1^2+\frac{4}{3}C_1\Big)M_1-\frac{32c^2}{9M_1}, \tag{5.13}
$$

where C_1 is an arbitrary constant.

It follows naturally that when $M_1(z_1)$ is solved from (5.13), the explicit solutions of (2.1) would be immediately obtained through (5.7)–(5.8) with (5.12).

Remark 5.1 The fourth Painlevé equations, in common with other integrable equations such as soliton equations, have a lot of good properties (see [33–35]). It can be written as a Hamiltonian system that possesses Bäcklund transformations and many rational solutions, algebraic solutions and solutions expressible in terms of the classical special functions for certain values of the parameters. Since the solutions of the equations are transcendental, a study of the asymptotic behavior of these solutions plays an important role in the application of the equations.

Here we give one kind of rational solutions of the Boussinesq equation, and other forms of solutions can be obtained using the seed solutions in [33].

For $C_1=0,\ c=\frac{3}{4}$, PIV(5.13) has a simple solution

$$
M_{1z_1}=-2z_1, \tag{5.14}
$$

and then

$$P_z = \frac{2}{3}\ln(z), \tag{5.15}$$

which further results in a rational solution of (2.1) by (5.7)–(5.8):

$$u = \frac{1}{x^2}\Big(\frac{4}{9}c^2 - \frac{8}{9}c^2\tanh^2(\Delta_2) - \frac{4}{3}c\tanh(\Delta_2) - \frac{3}{4}\Big) - \frac{t^2}{2x^2}, \tag{5.16}$$

where

$$\Delta_2 = \frac{1}{3}c\Big(3\ln(t) + 2\ln\Big(\frac{x}{\sqrt{t}}\Big)\Big).$$

(ii) When $c = 0$ in (3.9), following the similar steps of the above case $c \neq 0$ and omitting the tedious calculations, the group invariant solutions read

$$u = \frac{1}{2t}\Big(2U(z) - \frac{c_4^2Q^2(z)R^2(z)}{c_1^2\Delta_3^2} - \frac{c_4Q_1(z)R(z)}{c_1(\Delta_3)} - \frac{c_4Q(z)R_1(z)}{c_1(\Delta_3)}\Big), \tag{5.17}$$

where $U(z),\ Q(z),\ R(z),\ Q_1(z),\ R_1(z)$ and $P(z)$ represent the group invariants with $\Delta_3 = \ln(t) + P(z),\ z = \frac{x}{\sqrt{t}}$ and satisfy the following forms:

$$\begin{aligned}
&Q(z) = \exp\Big(-\frac{1}{4}\int\frac{zP_z(z) - 2P_{zz}(z) - 2}{P_z(z)}\mathrm{d}z\Big), \quad R(z) = -\frac{2c_1P_z(z)}{c_4Q(z)},\\
&Q_1(z) = -\frac{1}{4}\frac{(zP_z(z) - 2 - 2P_{zz}(z))Q(z)}{P_z(z)},\\
&R_1(z) = -\frac{1}{2}\frac{c_1(-2 + zP_z(z) + 2P_{zz}(z))}{c_4Q(z)},\\
&U(z) = \frac{1}{16}\frac{4P_{zz}^2(z) + z^2P_z^2(z) - 8P_z(z)P_{zzz}(z) - 4}{P_z^2(z)},
\end{aligned} \tag{5.18}$$

where $P(z)$ satisfies a three-order ordinary differential equation

$$12P_{zz}^3 - 16P_zP_{zz}P_{zzz} + 4P_z^2P_{zzzz} + 12zP_zP_{zz} + 9zP_z^3 - 12P_{zz} - 12P_z^2 = 0.$$

Using $w_2(z)$ to replace P_z, the reduction equation is

$$12w_{2z}^3 - 16w_2w_{2z}w_{2zz} + 4w_2^2w_{2zzz} + 12zw_2w_{2z} + 9zw_2^3 - 12w_{2z} - 12w_2^2 = 0. \tag{5.19}$$

The equation (5.19) can be solved in terms of solutions of the equation

$$N_{2zz} = \frac{3N_{2z}^2}{2N_2} - \frac{9z^2N_2}{8} - N_2 + 3z - \frac{3}{2N_2}. \tag{5.20}$$

To deal with the above equation, we introduce $M_2(z_1)$ by

$$N = -\frac{2\sqrt{3}}{3M_2(z_1)}, \quad z_1 = \frac{\sqrt{3}z}{2}, \tag{5.21}$$

which converts (5.20) into the fourth Painlevé equation

$$M_{2z_1z_1} = \frac{1}{2}\frac{M_{2z_1}^2}{M_2} + \frac{3}{2}M_2^3 + 4z_1M_2^2 + 2\Big(z_1^2 + \frac{4}{3}C_1\Big)M_2, \tag{5.22}$$

where C_1 is an arbitrary constant.

When $C_1 = -\frac{3}{4}$, (5.22) has a solution (see [33]) which is written in terms of an erfc function,

$$M_2 = -\frac{2C_2 \exp(-z_1^2)}{\sqrt{\pi}(C_3 + C_2 \mathrm{erfc}(z_1))}, \tag{5.23}$$

where C_2 and C_3 are arbitrary constants.

When $C_1 = \frac{3}{4}$, the solution of (5.22) has the form

$$M_2 = \frac{2\mathrm{i}C_4 \exp(-z_1^2)}{\sqrt{\pi}(C_5 + C_4 \mathrm{erfc}(\mathrm{i}z_1))}, \tag{5.24}$$

where C_4 and C_5 are arbitrary constants and the erfc function has the following form:

$$\mathrm{erfc}(z_1) = \frac{2}{\sqrt{\pi}} \int_{z_1}^{\infty} \exp(-\xi^2)\mathrm{d}\xi.$$

Using the seed solutions, (5.17)–(5.18), (5.21) and (5.23)–(5.24), one can easily obtain the solutions of the Boussinesq equation (2.1). Here we omitted.

Case 2 $c_1 = 0$

Without loss of generality, we let $f(t) = c_7$, $c_2 \equiv 1$ and $c_3 \equiv k$. For simplicity, we redefine the parameter $d^2 = \frac{c_6^2 + c_5^2 + 2c_5c_6 - 2c_7c_4}{4}$. Next, two cases $d \neq 0$ and $d = 0$ are both taken into account.

(iii) When $d \neq 0$, by solving (5.6), we have

$$\begin{aligned}
&p = -\frac{c_5 + c_6 + 2d\tanh(\Delta_4)}{c_4}, \quad \psi = \frac{R(z)\mathrm{e}^{\frac{-(c_5-c_6)t}{2}}}{\cosh(\Delta_4)}, \quad \phi = \frac{Q(z)\mathrm{e}^{\frac{(c_5-c_6)t}{2}}}{\cosh(\Delta_4)}, \\
&\psi_1 = \frac{-\mathrm{e}^{\frac{-(c_5-c_6)t}{2}}(c_4Q(z)R^2(z)\sinh(\Delta_4) - 2dR_1(z)\cosh(\Delta_4)}{2d\cosh^2(\Delta_4)}, \\
&\phi_1 = \frac{-\mathrm{e}^{\frac{(c_5-c_6)t}{2}}(c_4Q^2(z)R(z)\sinh(\Delta_4) - 2dQ_1(z)\cosh(\Delta_4))}{2d\cosh^2(\Delta_4)}, \\
&u = U(z) + \frac{c_4}{d}Q(z)R_1(z)\tanh(\Delta_4) + \frac{c_4}{d}R(z)Q_1(z)\tanh(\Delta_4) \\
&\quad + \frac{c_4^2}{d^2}R(z)^2Q^2(z)\mathrm{sech}^2(\Delta_4),
\end{aligned} \tag{5.25}$$

where $\Delta_4 = d(t + P(z))$, $z = x - kt$.

Substituting (5.25) into the prolonged system yields

$$\begin{aligned}
&P_z = M(z), \quad Q(z) = \exp\Big(-\frac{1}{2}\int \frac{kP_z(z) - P_{zz}(z) - 1}{P_z(z)}\mathrm{d}z\Big), \quad R(z) = \frac{2d^2P_z(z)}{c_4Q(z)}, \\
&Q_1(z) = -\frac{1}{2}\frac{(kP_z(z) - P_{zz}(z) - 1)Q(z)}{P_z(z)}, \quad R_1(z) = \frac{d^2(kP_z(z) + P_{zz}(z) - 1)}{c_4Q(z)}, \\
&U(z) = \frac{4d^2P_z^4(z) + 2P_z(z)P_{zzz}(z) - k^2P_z^2(z) + 2c_6P_z^2(z) - 2c_5P_z^2(z) - P_{zz}^2(z) + 1}{-4P_z^2(z)},
\end{aligned}$$

where $M(z)$ satisfies

$$M_z^2 = 1 - 6kM + b_2M^2 + b_3M^3 + 4d^2M^4 \tag{5.26}$$

with arbitrary constants b_2 and b_3. The solution of the above ODE can be written in terms of the Jacobi elliptic function. Hence, the solution expressed by (5.25) is just the explicit exact interaction between the soliton and cnoidal periodic waves. Here we give two kinds of solutions. The first kind is (5.26) that has an elliptic function in the form of

$$M(z) = a_0 + a_1 \mathrm{sn}(lz, m). \tag{5.27}$$

Then by virtue of (5.27), we can obtain the following solution of (2.1) through (5.26), saying

$$\begin{aligned} u = & \frac{-\frac{1}{4}(l_4 \mathrm{sn}^4(lz,m) + l_3 \mathrm{sn}^3(lz,m) + l_2 \mathrm{sn}^2(lz,m) + l_1 \mathrm{sn}(lz,m) + l_0)}{(a_0 + a_1 \mathrm{sn}(lz,m))^2} \\ & + 2lda_1 \mathrm{cn}(lz,m)\mathrm{dn}(lz,m)\tanh(\Theta) + 2d^2(a_0 + a_1 \mathrm{sn}^2(lz,m)\mathrm{sech}^2(\Theta)) \end{aligned} \tag{5.28}$$

with

$$\Theta = td + d\int_{z_0}^{z} (a_0 + a_1 \mathrm{sn}(lz', m))\mathrm{d}z', \quad z = x - \frac{a_0(2m^2a_0^2 - m^2a_1^2 - a_1^2)}{3(a_1^4 - a_0^2a_1^2 - m^2a_0^2a_1^2 + m^2a_0^4)}t,$$

and

$$\begin{aligned} l_0 &= 1 - k^2a_0^2 + 2c_6a_0^2 - 2c_5a_0^2 + 4d^2a_0^4 - a_1^2l^2, \\ l_1 &= 4c_6a_0a_1 - 4c_5a_0a_1 + 16d^2a_0^3a_1 - 2a_0a_1l^2m^2 - 2k^2a_0a_1 - 2a_0a_1l^2, \\ l_2 &= 2c_6a_1^2 + 24d^2a_0^2a_1^2 - k^2a_1^2 - 2c_5a_1^2 - a_1^2l^2 - a_1^2l^2m^2, \\ l_3 &= 16d^2a_0a_1^3 + 4a_0a_1l^2m^2, \\ l_4 &= 4d^2a_1^4 + 3a_1^2l^2m^2, \\ d &= \frac{m}{2\sqrt{a_1^4 - a_0^2a_1^2 - m^2a_0^2a_1^2 + m^2a_0^4}}, \quad k = \frac{a_0(2m^2a_0^2 - m^2a_1^2 - a_1^2)}{3(a_1^4 - a_0^2a_1^2 - m^2a_0^2a_1^2 + m^2a_0^4)}, \\ l &= \frac{a_1}{\sqrt{a_1^4 - a_0^2a_1^2 - m^2a_0^2a_1^2 + m^2a_0^4}}, \end{aligned}$$

where $\mathrm{sn}, \mathrm{cn}, \mathrm{dn}$ are usual Jacobian elliptic functions with modulus m while $a_0,\ a_1$ and z_0 are independent constants.

The solution given in (5.28) denotes the analytic interaction solution between the single-soliton and the periodic solution, which have not been found in the Boussinesq equation before. It can be easily applied to the analysis of physically interesting processes, which seems rather rare in the literature of physics.

In the analytic solution of (5.28), the first term

$$\frac{-\frac{1}{4}(l_4 \mathrm{sn}^4(lz,m) + l_3 \mathrm{sn}^3(lz,m) + l_2 \mathrm{sn}^2(lz,m) + l_1 \mathrm{sn}(lz,m) + l_0)}{(a_0 + a_1 \mathrm{sn}(lz,m))^2}$$

exhibits a pure periodic property while the second and third terms

$$2lda_1 \mathrm{cn}(lz,m)\mathrm{dn}(lz,m)\tanh(\Theta) + 2d^2(a_0 + a_1 \mathrm{sn}^2(lz,m)\mathrm{sech}^2(\Theta))$$

are presenting the complicated interactions between the single-soliton and periodic waves. The composition of the usual periodic wave and the solitary wave may be a possible explanation for some strange phenomena in the ocean such as tsunami. Through the graphics, one can understand the process clearly.

In the following figures, we plot the analytic solitary-period wave solution expressed by (5.28) with $a_0 = 1,\ a_1 = 0.08,\ m = 0.1,\ c_5 = 0.5,\ c_6 = 0.5$.

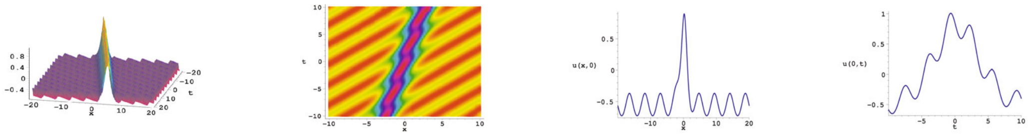

Figure 1 Interaction solution to the Boussinesq equation

An interaction wave to the Boussinesq equation with parameters: $a_0 = 1,\ a_1 = 0.08,\ m = 0.1,\ c_5 = 0.5,\ c_6 = 0.5$. This figure shows that the solitary-periodic wave is a spatially solitary wave and a periodic wave in two directions. The first figure is the corresponding two-dimensional image. The second figure is an overhead view of the wave with the contour plot shown. The bright lines are crests and the dark lines are troughs. The third figure and the forth figure are related to $t = 0$ and $x = 0$ of the solitary-period wave solution (5.28) with (5.27), respectively.

The second kind of solution $M(z)$ of (5.26) has the form

$$M(z) = \widetilde{a}_0 + \widetilde{a}_1 \tanh(\widetilde{l}z), \tag{5.29}$$

and one can think of this situation as degradation of the first case, i.e., to take $m = 1$ in (5.27). The exact solution can be obtained by substituting $m = 1$ into the (5.28),

$$\begin{aligned} u = -\Big\{ & (8d\widetilde{l}\widetilde{a}_1^3 \tanh(\Theta) - 8d^2\mathrm{sech}^2(\Theta)\widetilde{a}_1^4 + \widetilde{l}_4) \tanh^4(\widetilde{l}z) \\ & + (16d\widetilde{l}\widetilde{a}_0\widetilde{a}_1^2 \tanh(\Theta) - 32d^2\widetilde{a}_0\widetilde{a}_1^3\mathrm{sech}^2(\Theta) + \widetilde{l}_3) \tanh^3(\widetilde{l}z) \\ & + (8d\widetilde{l}\widetilde{a}_0^2\widetilde{a}_1 \tanh(\Theta) - 48d^2\widetilde{a}_0^2\widetilde{a}_1^2\mathrm{sech}^2(\Theta) - 8\widetilde{l}d\widetilde{a}_1^3 \tanh(\Theta) + \widetilde{l}_2) \tanh^2(\widetilde{l}z) \\ & + (\widetilde{l}_1 - 16d\widetilde{l}\widetilde{a}_0\widetilde{a}_1^2 \tanh(\Theta) - 32d^2\widetilde{a}_0^2\widetilde{a}_1\mathrm{sech}^2(\Theta)) \tanh(\widetilde{l}z) - 8d^2\widetilde{a}_0^4\mathrm{sech}^2(\Theta) \\ & - \frac{8d\widetilde{l}\widetilde{a}_1\widetilde{a}_0^2 \tanh(\Theta) + \widetilde{l}_0}{4(\widetilde{a}_0 + \widetilde{a}_1 \tanh(\widetilde{l}z))^2} \Big\} \end{aligned} \tag{5.30}$$

with

$$\Theta = td + d\int_{z_0}^{z} (a_0 + a_1 \tanh(lz'))\mathrm{d}z', \quad z = x - \frac{a_0(2^2a_0^2 - a_1^2 - a_1^2)}{3(a_1^4 - a_0^2a_1^2 - a_0^2a_1^2 + a_0^4)}t,$$

and

$$\begin{aligned} &\widetilde{l}_0 = 1 - \widetilde{a}_1^2\widetilde{l}^2 - k^2\widetilde{a}_0^2 + 2c_6\widetilde{a}_0^2 + 4d^2\widetilde{a}_0^4 - 2c_5\widetilde{a}_0^2, \\ &\widetilde{l}_1 = 16d^2\widetilde{a}_0^3a_1 - 4\widetilde{a}_0\widetilde{a}_1\widetilde{l}^2 + 4c_6\widetilde{a}_0\widetilde{a}_1 - 2k^2\widetilde{a}_0\widetilde{a}_1 - 4c_5\widetilde{a}_0\widetilde{a}_1, \\ &\widetilde{l}_2 = -2\widetilde{a}_1^2\widetilde{l}^2 + 2c_6\widetilde{a}_1^2 + 24d^2\widetilde{a}_0^2a_1^2 - 2c_5\widetilde{a}_1^2 - k^2\widetilde{a}_1^2 \\ &\widetilde{l}_3 = 4\widetilde{a}_0\widetilde{a}_1\widetilde{l}^2 + 16d^2\widetilde{a}_0\widetilde{a}_1^3, \quad \widetilde{l}_4 = 4d^2\widetilde{a}_1^4 + 3\widetilde{a}_1^2\widetilde{l}^2, \\ &d = \frac{1}{2(\widetilde{a}_0 - \widetilde{a}_1)(\widetilde{a}_0 + \widetilde{a}_1)}, \quad k = \frac{2\widetilde{a}_0}{3(\widetilde{a}_0 - \widetilde{a}_1)(\widetilde{a}_0 + \widetilde{a}_1)}, \quad l = \frac{\widetilde{a}_1}{(\widetilde{a}_0 + \widetilde{a}_1)(\widetilde{a}_0 - \widetilde{a}_1)}, \end{aligned}$$

where $\widetilde{a}_0, \widetilde{a}_1$ are independent constants. In order to study the structure of this solution, we give some pictures as following:

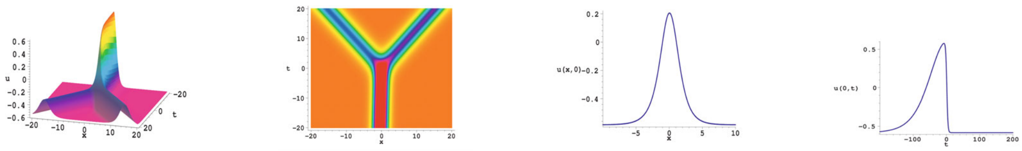

Figure 2 Resonance soliton solution to the Boussinesq equation

The above figures present a kind of minus resonance interactions of the two-soliton solution possessing three arms, each of which is a one-soliton profile. The interaction soliton with the highest amplitude is related to two other solitons and the amplitude of the newly produced soliton at the resonance becomes four times the amplitude of the initial soliton. This phenomenon can be observed on the sea surface, and has important applications in maritime security and coastal engineering.

(iv) When $d = 0$ in (5.6), following the above line, we obtain that

$$u = U(\xi) + \frac{(-c_4^2R^2(\xi)Q^2(\xi) - 2c_4tQ_1(\xi)R(\xi) - 2c_4Q_1(\xi)R(\xi)P(\xi) - 2c_4tR_1(\xi)Q(\xi) - 2c_4R_1(\xi)Q(\xi)P(\xi))}{2(t+P(\xi))^2} \tag{5.31}$$

and the related reduced equations are

$$P_z(z) = N(z), \quad Q(z) = \exp\Big(-\frac{1}{2}\int\frac{kP_z(z) - P_{zz}(z) - 1}{P_z(z)}\mathrm{d}z\Big), \quad R(z) = -\frac{2P_z(z)}{c_4Q(z)},$$
$$Q_1(z) = -\frac{1}{2}\frac{(kP_z(z) - P_{zz}(z) - 1)Q(z)}{P_z(z)}, \quad R_1(z) = -\frac{kP_z(z) + P_{zz}(z) - 1}{c_4Q(z)},$$
$$U(z) = \frac{1}{4}\frac{-2c_6P_z^2(z) - 2P_z(z)P_{zzz}(z) + P_z^2(z) + k^2P_z^2(z) + 2c^5P_z^2(z) - 1}{P_z^2(z)},$$

where N is the solution of the elliptic equation

$$N_z^2 = 1 - 6kN + b_2N^2 + b_3N^3. \tag{5.32}$$

Remark 5.2 From (5.32), we know that since $N(z)$ can be expressed as an elliptic integration, (5.31) denotes the interactions among cnoidal periodic waves and rational waves for the Boussinesq equation.

6 Summaries and Discussions

In this paper, the nonlocal symmetry of the Boussinesq equation is obtained by using both the invariant properties of differential equations exhibited by DT and a symmetry assumption method with the Lax pair. The nonlocal symmetry can be localized when the five potentials, the spectral function ψ, the adjoint spectral function ϕ, the x-derivatives of the spectral functions $\psi_1 = \psi_x$ and $\phi_1 = \phi_x$, and the singularity manifold function $p = -\int\psi\phi\mathrm{d}x$ are introduced. In this case, the primary nonlocal symmetry is equivalent to a Lie point symmetry of a prolonged system, on the basis of which one can find nonlocal groups as well as the explicit similarity solutions.

Our next objective focused on using the closed prolonged system to obtain a diversity of exact explicit solutions of the Boussinesq equation, which can not be obtained by the classical Lie group method. For example, the soliton-cnoidal wave solution which describes solitons moving on a cnoidal wave background instead of the plane continuous wave background can be easily applied to the analysis of physically interesting processes. If the module degenerates to 1, the soliton-cnoidal wave solution degenerates to a resonance soliton solution which the amplification of the amplitude has been experimentally observed and has practical applications in maritime security and coastal engineering. Some other types of solutions, such as rational solutions, and error function solutions, are given by using the fourth Painlevé equation with special values of the parameters.

To search for nonlocal symmetries of integrable DEs and to apply the nonlocal symmetries to construct explicit solutions are both of considerable interest and value. The method should and can be applied to other kinds of integrable systems, especially for supersymmetric models and discrete ones, to find interaction solutions among different kinds of nonlinear waves. However, in general, the prolongation is not close, neither for the local nor for the nonlocal variables. There is not a universal way to estimate what kind of nonlocal symmetries can be spread to the Lie point symmetries of a related prolonged system. To calculate the moving direction of a soliton on a cnoidal background and the shift of the crest of a cnoidal wave is also an interesting topic. The above topics will be discussed in the future series of research works.

Acknowledgement The authors extend their gratitude to the referees for the comments and suggestions and to Professors S. Y. Lou, E. G. Fan, and Z. Y. Yan for their helpful discussions.

References

[1] Lie, S., Über die Integration durch bestimmte integrale von einer Klasse linearer partieller differentialgleichungen, *Arch. Math.*, **6**, 1881, 328–368.

[2] Ovsiannikov, L. V., Group Analysis of Differential Equations, Academic, New York, 1982.

[3] Olver, P. J., Applications of Lie Groups to Differential Equations, Springer-Verlag, Berlin, 1986.

[4] Ibragimov, N. H., Transformation Groups Applied to Mathematical Physics Boston, Reidel, MA, 1985.

[5] Bluman, G. W. and Kumei, S., Symmetries and Differential Equations, Springer-Verlag, New York, 1989.

[6] Dong, Z. Z., Huang, F. and Chen, Y., Symmetry reductions and exact solutions of the two-layer model in atmosphere, *Z. Naturforsch*, **66**, 2011, 75–86.

[7] Hu, X. R., Huang, F. and Chen, Y., Symmetry reductions and exact solutions of the (2+1)-dimensional Navier-Stokes equations, *Z. Naturforsch*, **65**, 2010, 1–7.

[8] Chen, Y. and Dong, Z. Z., Symmetry reduction and exact solutions of the generalized Nizhnik-Novikov-Veselov equation, *Nonlinear Anal.*, **71**, 2009, 810–817.

[9] Temuer, C. L., Eerdun, B. H. and Xia, T. C., Nonclassical symmetry of the wave equation with source, *Chin. Ann. Math.*, **33A**(2), 2012, 193–204. (in Chinese)

[10] Xia, T. C., Zhang, D. P. and Temuer, C. L., Nonclassical symmetry classification of the wave equation with a nonlinear source term, *Acta Math. Sci.*, **32**, 2012, 941–949.

[11] Bluman, G. W., Cheviakov, A. F. and Anco, S. C., Applications of Symmetry Methods to Partial Differential Equations, Springer-Verlag, New York, 2010.

[12] Bluman, G. W. and Cheviakov, A. F., Framework for potential systems and nonlocal symmetries: algorithmic approach, *J. Math. Phys.*, **46**, 2005, 1–19.

[13] Bluman, G. W., Reid, G. I. and Kumei, S., New classes of Symmetries for partial differential equations, *J. Math. Phys.*, **29**(4), 1988, 806–811.

[14] Galas, F., New nonlocal symmetries with pseudopotentials, *J. Phys. A: Math. Gen.*, **25**, 1992, L981–L986.

[15] Lou, S. Y., Hu, X. R. and Chen, Y., Nonlocal symmetries related to Bäcklund transformation and their applications, *J. Phys. A: Math. Theor.*, **45**, 2012, 155209–155209.

[16] Hu, X. R., Lou, S. Y. and Chen, Y., Explicit solutions from eigenfunction symmetry of the Korteweg-de Vries equation, *Phys. Rev. E*, **85**, 2012, 056607-1–8.

[17] Guthrie, G. A., More non-local symmetries of the KdV equation, *J. Phys. A: Math.* Gen., **26**, 1993, L905–L908.

[18] Akhatov, I. S. and Gazizov, R. K., Nonlocal symmetries: a heuristic approach, *J. Math. Sci.*, **55**, 1991, 1401–1450.

[19] Lou, S. Y., Conformal invariance and integrable models, *J. Phys. A: Math. Phys.*, **30**, 1997, 4803–4813.

[20] Lou, S. Y. and Hu, X. B., Non-local symmetries via Darboux transformations, *J. Phys. A: Math. Gen.*, **30**, 1997, L95–L100.

[21] Xin, X. P. and Chen, Y., A method to construct the high order nonlocal symmetries, *Chin. Phys. Lett.*, **30**, 2013, 100202 1–4.

[22] Edelen, D. G., Isovector Methods for Equations of Balance with Programs for Assistance in Operator Calculations and Exposition of Practical Topics of Eaterior Calculus, Alphen aam den Rijn: Sijthoff and Noordhoff, 1980.

[23] Krasilshchik, I. S. and Vinogradov, A., Nonlocal symmetries and the theory of coverings: an addendum to A. M. Vinogradov's 'local symmetries and conservation laws', *Acta Appl. Math.*, **2**, 1984, 79–96.

[24] Lou, S. Y. and Hu, X. B., Nonlocal Lie-Bäcklund symmetries and Olver symmetries of the KdV equation, *Chin. Phys. Lett.*, **10**, 1993, 577–580.

[25] Fan, E. G., Extended tanh-function method and its applications to nonlinear equations, *Phys. Lett. A*, **277**, 2000, 212–218.

[26] Fan, E. G., Two new applications of the homogeneous balance method, *Phys. Lett. A*, **265**, 2000, 353–357.

[27] Clarkson, P. A. and Kruskal, M. D., New similarity reductions of the Boussinesq equation, *J. Math. Phys.*, **30**, 1989, 2201–2213.

[28] Li, C. X., Ma, W. X., Liu, X. J. and Zeng, Y. B., Wronskian solutions of the Boussinesq equation-solitons, negatons, positons and complexitons, *Inverse Prob.*, **23**, 2007, 279–296.

[29] Shin, H. J., Soliton on a cnoidal wave background in the coupled nonlinear Schrödinger equation, *J. Phys. A: Math. Gen.*, **37**, 2004, 8017–8030.

[30] Hu, X. B. and Lou, S. Y., Nonlocal symmetries of nonlinear integrable models, *Proceedings of Institute of Mathematics of NAS of Ukraine*, **30**, 2000, 120–126.

[31] Shen, S. F., Qu, C. Z., Jin, Y. Y. and Ji, L. N., Maximal dimension of invariant subspaces to systems of nonlinear evolution equations, *Chin. Ann. Math.*, **33**B(2), 2012, 161–178.

[32] Dong, Z. Z., Chen, Y., Kong, D. X. and Wang, Z. G., Symmetry reduction and exact solutions of a hyperbolic Monge-Ampere equation, *Chin. Ann. Math.*, **33**B(2), 2012, 309–316.

[33] Clarkson, P. A., The fourth Painlevé transcendent, IMSAS Technical Report, UKC/IMS, 2008.

[34] Clarkson, P. A., The fourth Painlevé equation and associated special polynomials, *J. Math. Phys.*, **44**, 2003, 5350–5374.

[35] Airault, H., Rational solutions of Painlevé equations, *Stud. Appl. Math.*, **61**, 1979, 31–53.