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Darboux Transformations via Lie Point Symmetries: KdV Equation *

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By localizing the nonlocal symmetries of a nonlinear model to local symmetries of an enlarged system, we find Darboux-Bäcklund transformations for both the original and prolonged systems. The idea is explicitly realized for the well-known KdV equation.

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Soliton equations connect rich histories of exactly solvable systems constructed in mathematics, cosmology, fluid physics and field theory, and powerfully demonstrate the unity of nonlinear concepts across disciplines and scales from micro-physics and biology to cosmology. Among these equations, the Korteweg de-Vries (KdV) equation

$$u_t + u_{xxx} - 6uu_x = 0 \tag{1}$$

is one of the most ubiquitous.[1−5]

The Darboux transformation (DT) and the symmetry method are two of the most effective approaches to solve the problem of integrable nonlinear systems. Recently, it has been found that these two approaches are deeply related. In particular, the infinitesimal forms of DTs are nonlocal symmetries of the original nonlinear systems.[6] In this Letter, we concentrate on the following interesting and important problem: *Can we obtain DTs from Lie point symmetry theory?* To answer this question, we take the celebrated KdV equation as an example. The result is valid to other integrable systems.

For Eq. (1), its Lax pair has the form

$$\psi_{xx} - u\psi + \lambda\psi = 0, \tag{2a}$$
$$\psi_t + 4\psi_{xxx} - 6u\psi_x - 3u_x\psi = 0, \tag{2b}$$

with the spectral parameter λ.

A symmetry of Eq. (1) is defined as its solution of the linearized equation

$$\sigma_t + \sigma_{xxx} - 6(u\sigma)_x = 0, \tag{3}$$

which means that Eq. (1) is form invariant under the transformation $u \to u + \epsilon\sigma$ with infinitesimal parameter ϵ.

In Ref. [6], it has been proved that the known square spectral function symmetry

$$\sigma = (\psi^2)_x \tag{4}$$

is just the infinitesimal form of the first type of DT for Eq. (1) by taking the spectral parameter as the group parameter. It is clear that the symmetry of Eq. (4) is nonlocal for Eq. (1) because ψ is related to u by the differential equation system of Eq. (2).

Obviously, by solving the initial value problem

$$\frac{du_1(\lambda_1)}{d\lambda_1} = [\varPsi(\lambda_1)^2]_x, \quad u_1(\lambda) = u, \tag{5}$$

where $\varPsi(\lambda_1)$ is the spectral function of the Lax pair in Eq. (2) with the spectral parameter λ_1 and the potential $u_1(\lambda_1)$, we can re-obtain the well-known first type of DT of the KdV equation because the nonlocal symmetry of Eq. (4) just comes from the limiting procedure

$$\lambda \to \lambda_1 = \lambda + d\lambda_1, \tag{6}$$

for the first type of DT.[6]

Since the nonlocal symmetry of Eq. (4) can also be derived by many other approaches (such as, by applying the inverse recursion operator on the identity symmetry $\sigma = 0$ without using the limiting procedure in Eq. (6))[7] we can introduce a new group parameter ϵ that is independent of the spectral parameter. In other words, in addition to the initial value problem in Eq. (5), we can also introduce the second type of the initial value problem

$$\frac{dU(\epsilon)}{d\epsilon} = [\varPsi(\epsilon)^2]_x, \quad U(0) = u, \tag{7}$$

where $\varPsi(\epsilon)$ is the spectral function of the Lax pair in Eq. (2) with the spectral parameter λ and the potential $U(\epsilon)$ while the group parameter ϵ is independent of λ.

To solve the initial value problem of Eq. (7), we have to localize the nonlocal symmetry of Eq. (4). By introducing the auxiliary variables g and f with the relations

$$g = \psi_x, \quad f_x = \psi^2, \tag{8}$$

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the nonlocal symmetry of Eq. (4) of the original KdV equation becomes a local Lie point symmetry

$$\sigma=\begin{pmatrix}\sigma^u\\ \sigma^\psi\\ \sigma^g\\ \sigma^f\end{pmatrix}=\begin{pmatrix}2\psi g\\ \frac{1}{2}f\psi\\ \frac{1}{2}(\psi^3+fg),\\ \frac{1}{2}f^2\end{pmatrix}, \tag{9}$$

for the enlarged KdV system (1), (2), and (8); that is, Eq. (9) is a solution of the following linearized system

$$\sigma^u_t+\sigma^u_{xxx}-6(u\sigma^u)_x=0, \tag{10a}$$
$$\sigma^\psi_{xx}-\sigma^u\psi-u\sigma^\psi+\lambda\sigma^\psi=0 \tag{10b}$$
$$\sigma^g=\sigma^\psi_x, \tag{10c}$$
$$\sigma^f_x=2\psi\sigma^\psi. \tag{10d}$$

Correspondingly, the initial value problem of Eq. (7) is changed to

$$\frac{dU(\epsilon)}{d\epsilon}=2\Psi(\epsilon)G(\epsilon),\quad U(0)=u, \tag{11a}$$
$$\frac{d\Psi(\epsilon)}{d\epsilon}=\frac{1}{2}\Psi(\epsilon)F(\epsilon),\quad \Psi(0)=\psi, \tag{11b}$$
$$\frac{dG(\epsilon)}{d\epsilon}=\frac{1}{2}[F(\epsilon)G(\epsilon)+\Psi^3(\epsilon)],\quad G(0)=g, \tag{11c}$$
$$\frac{dF(\epsilon)}{d\epsilon}=\frac{1}{2}F^2(\epsilon),\quad F(0)=f. \tag{11d}$$

After solving the initial value problem of Eq. (11) we have the following theorem[8,9] for the second type of the DT.

Theorem 1. If $\{u,\psi,g,f\}$ is a solution of the enlarged KdV system (1), (2) and (8), so is $\{U(\epsilon),\ \Psi(\epsilon),\ G(\epsilon),\ F(\epsilon)\}$ with

$$U(\epsilon)=u+2\epsilon\psi\frac{\epsilon\psi^3-2g(\epsilon f-2)}{(\epsilon f-2)^2}, \tag{12a}$$
$$\Psi(\epsilon)=-\frac{2\psi}{\epsilon f-2}, \tag{12b}$$
$$G(\epsilon)=2\frac{\epsilon\psi^3-g(\epsilon f-2)}{(\epsilon f-2)^2}, \tag{12c}$$
$$F(\epsilon)=-\frac{2f}{\epsilon f-2}. \tag{12d}$$

It is clear that, because of the intrusion of the spectral parameter λ in the nonlocal symmetry of Eq. (4), we have infinitely many nonlocal symmetries for the single field u,

$$\sigma^u_n=\sum_{i=1}^n c_i(\psi_i^2)_x,\ n=1,2,\ldots, \tag{13}$$

where $\psi_i, i=1,2,\ldots,n$ are spectral functions of the Lax pair in Eq. (2) with different spectral parameters $\lambda_i\neq\lambda_j,\forall j\neq i$.

The nonlocal symmetries of Eq. (13) are used to find algebraic-geometric solutions by the so-called nonlinearization procedure of Lax pairs.[10] Now, our question is how to solve the initial value problem related to the nonlocal symmetry of Eq. (13) for any fixed n:

$$\frac{du_n(\epsilon)}{d\epsilon}=\sum_{i=1}^n c_i[\Psi_i(\epsilon)^2]_x,\quad u_n(0)=u, \tag{14}$$

where $\Psi_i(\epsilon),\ i=1,2,\ldots,n$ are spectral functions of the Lax pair in Eq. (2) with spectral parameters λ_i and potential $u_n(\epsilon)$.

As in the $n=1$ case of Eq. (7), to solve the initial value problem in Eq. (14), we have to further prolong the KdV equation to a larger system such that Eq. (13) becomes a Lie point symmetry. It is remarkable that if we introduce the following enlarged system, $i=1,2,\cdots,n$,

$$u_t+u_{xxx}-6uu_x=0, \tag{15a}$$
$$\psi_{i,xx}-u\psi_i+\lambda_i\psi_i=0,$$
$$\psi_{i,t}+4\psi_{i,xxx}-6u\psi_{i,x}-3u_x\psi_i=0, \tag{15b}$$
$$g_i=\psi_{i,x},\quad f_{i,x}=\psi_i^2, \tag{15c}$$

the nonlocal symmetry Eq. (14) of Eq. (1) becomes a local Lie point symmetry of the enlarged system Eq. (15),

$$\sigma^u=2\sum_{i=1}^n c_i\psi_i g_i,$$
$$\sigma^{\psi_j}=\frac{c_j}{2}\psi_j f_j+\sum_{i\neq j}^n\frac{c_i}{2}\frac{\psi_i(g_j\psi_i-g_i\psi_j)}{\lambda_i-\lambda_j},$$
$$\sigma^{g_j}=\frac{c_j}{2}(\psi_j^3+f_jg_j)+\sum_{i\neq j}^n\frac{c_i}{2}[\psi_i^2\psi_j+g_i\frac{\psi_ig_j-\psi_jg_i}{\lambda_i-\lambda_j}],$$
$$\sigma^{f_j}=\frac{c_j}{2}f_j^2+\sum_{i\neq j}^n\frac{c_i}{2}\frac{(\psi_ig_j-\psi_jg_i)^2}{(\lambda_i-\lambda_j)^2},\ j=1,2,\ldots,n, \tag{16}$$

which is the solution of the linearized system ($i=1,2,\cdots,\ n$)

$$\sigma^u_t+\sigma^u_{xxx}-6(u\sigma^u)_x=0, \tag{17a}$$
$$\sigma^{\psi_i}_{xx}-\sigma^u\psi_i-u\sigma^{\psi_i}+\lambda_i\sigma^{\psi_i}=0, \tag{17b}$$
$$\sigma^{g_i}=\sigma^{\psi_i}_x,\quad \sigma^{f_i}_x=2\psi_i\sigma^{\psi_i}. \tag{17c}$$

To verify the correctness of Eq. (16), it is enough to fix $c_j\neq0,\ c_k=0,\ k\neq j$ with $\sigma^u=c_j(\psi_j^2)_x=2c_j\psi_jg_j$, which is a known solution of Eq. (17b). Substituting $\sigma^u=c_j(\psi_j^2)_x$ into Eq. (17c) for $i=j$, we have

$$\sigma^{\psi_j}_{xx}-c_j(\psi_j^2)_x\psi_j-u\sigma^{\psi_j}+\lambda_j\sigma^{\psi_j}=0. \tag{18}$$

Canceling u in Eq. (18) via the x part of Lax pair; that is, $u=\lambda_j+\psi_{jxx}/\psi_j$, we have

$$\sigma^{\psi_j}_{xx}-c_j(\psi_j^2)_x\psi_j-\frac{\psi_{jxx}}{\psi_j}\sigma^{\psi_j}=0. \tag{19}$$

Now it is a trivial work to verify

$$\sigma^{\psi_j} = \frac{c_j}{2}\psi_j f_j, \quad f_{jx} = \psi_j^2. \tag{20}$$

To find σ^{ψ_i} with $i \neq j$ from Eq. (17c) and the Lax pair, we have

$$\sigma^{\psi_i}_{xx} - c_j(\psi_j^2)_x\psi_i - \Big(\frac{\psi_{jxx}}{\psi_j} + \lambda_j - \lambda_i\Big)\sigma^{\psi_i} = 0. \tag{21}$$

Now, to verify the result

$$\sigma^{\psi_i} = \frac{c_j}{2}\frac{\psi_j(g_i\psi_j - g_j\psi_i)}{\lambda_j - \lambda_i}, \quad i \neq j, \tag{22}$$

from Eq. (21), one has to use the relation

$$\frac{\psi_{jxx}}{\psi_j} + \lambda_j = \frac{\psi_{ixx}}{\psi_i} + \lambda_i, \tag{23}$$

which comes from the Lax pairs by canceling the field u. After obtaining the symmetry components σ^{ψ_j} and σ^{ψ_i}, $i \neq j$, to obtain the other components $\{\sigma^{g_j},\ \sigma^{f_j}\}$ and $\{\sigma^{g_i},\ \sigma^{f_i},\ i \neq j\}$ from Eq. (17d) is straightforward after considering Eq. (23).

Thus, the initial value problem of Eq. (14) is changed to

$$\frac{dU(\epsilon)}{d\epsilon} = 2\sum_{i=1}^{n} c_i\,\Psi_i(\epsilon)G_i(\epsilon), \tag{24a}$$

$$\frac{d\Psi_j(\epsilon)}{d\epsilon} = \frac{c_j}{2}\Psi_j(\epsilon)F_j(\epsilon) + \sum_{i\neq j}^{n}\frac{c_i}{2}\frac{\Psi_i(\epsilon)(G_j(\epsilon)\Psi_i(\epsilon) - G_i(\epsilon)\Psi_j(\epsilon))}{\lambda_i - \lambda_j}, \tag{24b}$$

$$\frac{dG_j(\epsilon)}{d\epsilon} = \frac{c_j}{2}(\Psi_j^3(\epsilon) + F_j(\epsilon)G_j(\epsilon)) + \sum_{i\neq j}^{n}\frac{c_i}{2}\Big[\Psi_i^2(\epsilon)\Psi_j(\epsilon) + G_i(\epsilon) \cdot \frac{\Psi_i(\epsilon)G_j(\epsilon) - \Psi_j(\epsilon)G_i(\epsilon)}{\lambda_i - \lambda_j}\Big], \tag{24c}$$

$$\frac{dF_j(\epsilon)}{d\epsilon} = \frac{c_j}{2}F_j(\epsilon)^2 + \sum_{i\neq j}^{n}\frac{c_i}{2} \cdot \frac{(\Psi_i(\epsilon)G_j(\epsilon) - \Psi_j(\epsilon)G_i(\epsilon))^2}{(\lambda_i - \lambda_j)^2}, \tag{24d}$$

$$\{U(0), \Psi_1(0), \ldots, \Psi_j(0), \ldots, \Psi_n(0), G_1(0), \ldots, G_j(0), \ldots, G_n(0), F_1(0), \ldots, F_j(0), \ldots, F_n(0)\} = \{u, \psi_1, \ldots, \psi_j, \ldots, \psi_n, g_1, \ldots, g_j, \ldots, g_n, f_1, \ldots, f_j, \ldots, f_n\}. \tag{24e}$$

Although it is quite difficult solve the initial value problem of Eq. (24), it is not difficult to verify that the final result can be summarized to the following theorem.

Theorem 2. If $\{u, \psi_i, g_i, f_i,\ i = 1, 2, \cdots, n\}$ is a solution of the enlarged KdV system Eq. (15), so is $\{U(\epsilon),\ \Psi_i(\epsilon),\ G_i(\epsilon),\ F_i(\epsilon)\}$ with

$$U(\epsilon) = u - 2(\ln\Delta)_{xx}, \tag{25a}$$

$$\Psi_i(\epsilon) = -2\frac{\Gamma_i}{\Delta}, \tag{25b}$$

$$G_i(\epsilon) = \Psi_{ix}(\epsilon), \tag{25c}$$

$$F_i(\epsilon) = -2\frac{\Delta_i}{\Delta}, \tag{25d}$$

where Δ, Δ_i and Γ_i are determinants of the matrices M, M_i and N_i, respectively, with

$$M = \begin{pmatrix} c_1\epsilon f_1 - 2 & c_1\epsilon w_{12} & \cdots & c_1\epsilon w_{1j} & \cdots & c_1\epsilon w_{1n} \\ c_2\epsilon w_{12} & c_2\epsilon f_2 - 2 & \cdots & c_2\epsilon w_{2j} & \cdots & c_2\epsilon w_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_j\epsilon w_{1j} & c_j\epsilon w_{2j} & \cdots & c_j\epsilon f_j - 2 & \cdots & c_j\epsilon w_{jn} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_n\epsilon w_{1n} & c_n\epsilon w_{2n} & \cdots & c_n\epsilon w_{jn} & \cdots & c_n\epsilon f_n - 2 \end{pmatrix}, \tag{26}$$

$$M_i = \begin{pmatrix} c_1\epsilon f_1 - 2 & c_1\epsilon w_{12} & \cdots & c_1\epsilon w_{1,i-1} & c_1\epsilon w_{1i} & c_1\epsilon w_{1,i+1} & \cdots & c_1\epsilon w_{1n} \\ c_2\epsilon w_{12} & c_2\epsilon f_2 - 2 & \cdots & c_2\epsilon w_{2,i-1} & c_2\epsilon w_{2i} & c_2\epsilon w_{2,i+1} & \cdots & c_2\epsilon w_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{i-1}\epsilon w_{1,i-1} & c_{i-1}\epsilon w_{2,i-1} & \cdots & c_{i-1}\epsilon f_{i-1} - 2 & c_{i-1}\epsilon w_{i-1,i} & c_{i-1}\epsilon w_{i-1,i+1} & \cdots & c_{i-1}\epsilon w_{i-1,n} \\ w_{1i} & w_{2i} & \cdots & w_{i,i-1} & f_i & w_{i,i+1} & \cdots & w_{in} \\ c_{i+1}\epsilon w_{1,i+1} & c_{i+1}\epsilon w_{2,i+1} & \cdots & c_{i+1}\epsilon w_{i-1,i+1} & c_{i+1}\epsilon w_{i,i+1} & c_{i+1}\epsilon f_{i+1} - 2 & \cdots & c_{i+1}\epsilon w_{i+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_n\epsilon w_{1n} & c_n\epsilon w_{2n} & \cdots & c_n\epsilon w_{i-1,n} & c_n\epsilon w_{in} & c_n\epsilon w_{i+1,n} & \cdots & c_n\epsilon f_n - 2 \end{pmatrix}, \tag{27}$$

and

$$N_i = \begin{pmatrix} c_1\epsilon f_1 - 2 & c_1\epsilon w_{12} & \cdots & c_1\epsilon w_{1,i-1} & c_1\epsilon w_{1i} & c_1\epsilon w_{1,i+1} & \cdots & c_1\epsilon w_{1n} \\ c_2\epsilon w_{12} & c_2\epsilon f_2 - 2 & \cdots & c_2\epsilon w_{2,i-1} & c_2\epsilon w_{2i} & c_2\epsilon w_{2,i+1} & \cdots & c_2\epsilon w_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{i-1}\epsilon w_{1,i-1} & c_{i-1}\epsilon w_{2,i-1} & \cdots & c_{i-1}\epsilon f_{i-1} - 2 & c_{i-1}\epsilon w_{i-1,i} & c_{i-1}\epsilon w_{i-1,i+1} & \cdots & c_{i-1}\epsilon w_{i-1,n} \\ \psi_1 & \psi_2 & \cdots & \psi_{i-1} & \psi_i & \psi_{i+1} & \cdots & \psi_n \\ c_{i+1}\epsilon w_{1,i+1} & c_{i+1}\epsilon w_{2,i+1} & \cdots & c_{i+1}\epsilon w_{i-1,i+1} & c_{i+1}\epsilon w_{i,i+1} & c_{i+1}\epsilon f_{i+1} - 2 & \cdots & c_{i+1}\epsilon w_{i+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_n\epsilon w_{1n} & c_n\epsilon w_{2n} & \cdots & c_n\epsilon w_{i-1,n} & c_n\epsilon w_{in} & c_n\epsilon w_{i+1,n} & \cdots & c_n\epsilon f_n - 2 \end{pmatrix}, \tag{28}$$

and

$$w_{ij} = \frac{\psi_i g_j - g_i \psi_j}{\lambda_i - \lambda_j}. \tag{29}$$

Since ψ_i is the spectral function with spectral parameter λ_i, one can find that

$$w_{ijx} = \psi_i \psi_j,\ f_{ix} = \psi_i^2. \tag{30}$$

On the other hand, the enlarged system is trivially invariant under the transformation $f_i \to a_i f_i + b_i$, $\psi \to \sqrt{a_i}\psi$ for arbitrary constants a_i and b_i. Thus, the DT shown in Eq. (25e) of the theorem 2 is equivalent to the second type of DT (i.e., the binary DT or Levi transformation) known in literature for the KdV equation.[11] Thus, the multiple solitons obtained from theorem 2 are the same as the known ones.

In summary, starting from the nonlocal symmetries obtained from the first type of DT, the square spectral function symmetry, the binary DT can be obtained simply by solving an initial value problem via introducing a suitable prolonged system. The idea and the method used here is universal for other integrable systems and will be further investigated.

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References

[1] Kivshar Yu S and Malomed B A 1989 *Rev. Mod. Phys.* **61** 763
Kivshar Yu S and Luther-Davies B 1998 *Phys. Rep.* **298** 81
[2] Korteweg D J and de Vries F 1895 *Philos. Mag.* **39** 422
[3] Russell J S 1844 *Report Waves: Report 14th Meeting of the British Association for the Advancement of Science* (London: John Murray) pp 311–390
[4] Crighton D G 1995 *Acta Appl. Math.* **39** 39
[5] Zabusky N J and Kruskal M D 1965 *Phys. Rev. Lett.* **15** 240
[6] Lou S Y and Hu X B 1997 *J. Phys.* A **30** L95
Lou S Y and Hu X B 1997 *J. Math. Phys.* **38** 6401
[7] Lou S Y 1994 *J. Math. Phys.* **35** 2390
[8] Lou S Y, Hu X R and Chen Y 2012 *J. Phys. A: Math. Theor.* **45** 155209
Hu X R, Lou S Y and Chen Y 2012 *Phys. Rev.* E **85** 056607
[9] Lou S Y and Hu X B 1993 *Chin. Phys. Lett.* **10** 577
[10] Cao C W 1987 *Henan Sci.* **5** 1
Cao C W 1988 *Chin. Q. J. Math.* **3** 90
Cao C W 1990 *Sci. Chin.* A **33** 528
Cao C W and Geng X G 1990 *J. Phys.* A **23** 4117
Cao C W and Geng X G 1991 *J. Math. Phys.* **32** 2323
[11] Li Y S 1992 *Sci. Chin.* A **35** 600 (in Chinese)
Boiti M, Pempinelli F and Pogrebkov A K 1991 *Inverse Probl.* **7** 43