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New explicit solitary wave solutions for $(2+1)$-dimensional Boussinesq equation and $(3+1)$-dimensional KP equation

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Abstract

In this Letter, we study $(2+1)$-dimensional Boussinesq equation and $(3+1)$-dimensional KP equation by using the new generalized transformation in Homogeneous Balance Method (HBM). As a result, many explicit exact solutions, which contain new solitary wave solutions, periodic wave solutions and the combined formal solitary wave solutions and periodic wave solutions, are obtained.

Keywords: Boussinesq equation; KP equation; HBM; Riccati equation; Solitary wave solution; Periodic wave solution

1. Introduction

In recent years, searching for explicit exact solutions, in particular, solitary wave solutions, of nonlinear evolution equations (NEEs) in mathematical physics plays an important role in soliton theory [1–11,14, 21]. Particularly, various powerful methods have been presented, such as, Backlund transformation, Darboux transformation, Cole–Hopf transformation, tanh method, sine–cosine method, Painlevé method, homogeneous balance method (HBM), Hirota method [12], Lie group analysis, similarity reduced method and so on. Based upon the well-known Riccati equation, homogeneous balance method (HBM) proposed by Wang et al. [6,7] is to find exact solutions of certain nonlinear PDEs. Fan and Zhang [13,15] improved considerably the key steps of the HBM. Particularly, more general ansatz have been proposed in order to obtain new form of solutions. Recently, Senthilvelan [16] studied the travelling wave solutions for $(2+1)$-dimensional Boussinesq equation and $(3+1)$-dimensional KP equation by homogeneous balance method (HBM) and explored certain new solution of the equations. In this Letter, we would like to discuss further $(2+1)$-dimensional Boussinesq equation and $(3+1)$-dimensional KP equation by our improved method, in which we presented a new generalized transformation [17]. As a result, more new exact solutions, which include the solutions obtained by Senthilvel [16], are obtained.

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2. Method

Our method is summed up as follows.

For the given nonlinear evolution equations, say, in three variables, x, y, t

$$F(u, u_t, u_x, u_y, u_{xt}, u_{yt}, u_{tt}, u_{xy}, u_{yy}, \ldots) = 0, \tag{1}$$

we seek the following formal travelling wave solutions

$$u(x,t) = u(\xi), \quad \xi = \alpha x + \beta y - \lambda t, \tag{2}$$

where α, β, λ are all constants to be determined later. Then (2) reduces to a nonlinear ordinary differential equation

$$F_0(u, u', u'', \ldots) = 0, \tag{3}$$

where "$'$" denotes $\frac{d}{d\xi}$. In order to seek the travelling wave solutions of (3), we take the following transformations

$$u(\xi) = \sum_{i=1}^{m} \omega^{i-1}(\xi)\Big[a_i\omega(\xi) + a_i\sqrt{\mu_1\big(1+\mu_2\omega^2(\xi)\big)}\Big] + a_0, \tag{4}$$

and the new variable $\omega = \omega(\xi)$ satisfying

$$\omega' - R\big(1+\mu_2\omega^2\big) = \frac{d\omega}{d\xi} - R\big(1+\mu_2\omega^2\big) = 0, \tag{5}$$

where $\mu_j = \pm 1$ $(j = 1, 2)$; m is an integer to be determined and a_i, a_i $(i = 0, 1, 2, \ldots, m;\ j = 0, 1, 2, \ldots, n)$, R are constants to be determined later.

There exist the following steps to be considered further.

Step 1. Determine the values of m of (4) by respectively balancing the highest order partial derivative term and the nonlinear term in (2), it is easy to get the value of m.

Step 2. With the aid of MATHEMATICA, substituting system (4) along with the condition (5) into (3), yields a system of algebraic equations w.r.t. $\omega^i(\mu_1 + \mu_1\mu_2\omega^2)^{j/2}$ $(j = 0, 1; i = 0, 1, 2, \ldots)$.

Step 3. Collect all terms with the same power in $\omega^i(\mu_1 + \mu_1\mu_2\omega^2)^{j/2}$ $(j = 0, 1; i = 0, 1, 2, \ldots)$. Setting the coefficients of the terms $\omega^i(\mu_1 + \mu_1\mu_2\omega^2)^{j/2}$ $(j = 0, 1; i = 0, 1, 2, \ldots)$ to zero their coefficients to get a over-determined system of nonlinear algebraic equations w.r.t. the unknown variables $\lambda, \alpha, \beta, R, a_0, a_j, b_j$ $(i = 1, 2, \ldots, m)$.

Step 4. With the aid of MATHEMATICA, we apply Wu-elimination method [18,19] to solve the above over-determined system of nonlinear algebraic equations obtained in Step 4, yields the values of $\lambda, \alpha, \beta, R, a_0, a_j, b_j$ $(i = 1, 2, \ldots, m)$.

Step 5. It is well known that the general solutions of (5) are

(1) When taking $\mu_2 = -1$,

$$\omega = \omega(\xi) = \frac{A - B\exp(-2R\xi)}{A + B\exp(-2R\xi)} = \begin{cases} 1 & \text{for } B = 0, \\ -1 & \text{for } A = 0, \\ \tanh\big[R\xi - \frac{1}{2}\ln\big(\frac{A}{B}\big)\big] & \text{for } AB > 0, \\ \coth\big[R\xi - \frac{1}{2}\ln\big(-\frac{A}{B}\big)\big] & \text{for } AB < 0. \end{cases} \tag{6}$$

When A, B are arbitrary constants satisfying $A^2 + B^2 \neq 0$. This solution may be obtained by three tricks: a Möbius transformation, a Cole–Hopf transformation or a relation

$$\frac{(\omega_1 - \omega_2)(\omega_3 - \omega_4)}{(\omega_1 - \omega_3)(\omega_2 - \omega_4)} = C = \text{const}$$

of the solutions ω_i, $1 \leqslant i \leqslant 4$, beginning with three known solutions $1, -1, \tanh(R\xi)$.

(2) When $\mu_2 = 1$,

$$\omega = \omega(\xi) = \begin{cases} \tan(R\xi + \xi_0), \\ -\cot(R\xi + \xi_0). \end{cases} \tag{7}$$

Thus according to (2), (3), (6), (7) and the conclusions in Step 4, we can obtain more travelling wave solutions of (3).

3. Solitary wave solution and periodic wave solution

3.1. $(2+1)$-dimensional Boussinesq equation

Let us consider a $(2+1)$-dimensional generalization of Boussinesq equation [20]

$$u_{tt} - u_{xx} - u_{yy} - \left(u^2\right)_{xx} - u_{xxxx} = 0. \tag{8}$$

According to the above steps, we firstly make the following formal travelling wave transformation:

$$u(x, y, t) = u(\xi), \quad \xi = \alpha x + \beta y - \lambda t, \tag{9}$$

where α, β, λ are constants to be determined.

Substituting (9) into (8) and integrating it twice reads

$$\alpha^4 u'' + \alpha^2 u^2 + \left(\alpha^2 + \beta^2 - \lambda^2\right)u = 0. \tag{10}$$

According to Step 1 in Section 2, we support that (10) has the following formal solutions

$$u = a_0 + a_1\omega + b_1\sqrt{\mu_1\left(1 + \mu_2\omega^2\right)} + a_2\omega^2 + b_2\omega\sqrt{\mu_1\left(1 + \mu_2\omega^2\right)} \tag{11}$$

and $\omega = \omega(\xi)$ satisfying Eq. (5), where a_0, a_1, a_2, b_1, b_2 are constants to be determined later.

With the aid of MATHEMATICA, substituting (11) into (10) along with (5) and collecting all terms with the same power in $\omega^i(\mu_1 + \mu_1\mu_2\omega^2)^{j/2}$ $(j = 0, 1; i = 0, 1, 2, 3, 4)$, yield a system of equations w.r.t. $\omega^i(\mu_1 + \mu_1\mu_2\omega^2)^{j/2}$. Setting the coefficients of $\omega^i(\mu_1 + \mu_1\mu_2\omega^2)^{j/2}$ $(j = 0, 1; i = 0, 1, 2, 3, 4)$ in the obtained system of equations to zero, we can deduce the following set of over-determined algebraic polynomials with the respect the unknowns

$$2a_2R^2\alpha^4 + \left(a_0^2 + b_1^2\mu_1\right)\alpha^2 + a_0\left(\alpha^2 - \lambda^2 + \beta^2\right) = 0, \tag{12.1}$$

$$2a_1\mu_2R^2\alpha^4 + (2a_0a_1 + 2b_1b_2\mu_1)\alpha^2 + a_1\left(\alpha^2 + \beta^2 - \lambda^2\right) = 0, \tag{12.2}$$

$$8a_2\mu_2R^2\alpha^4 + \left(a_1^2 + 2a_0a_2 + b_2^2\mu_1 + b_1^2\mu_1\mu_2\right)\alpha^2 + a_2\left(\alpha^2 + \beta^2 - \lambda^2\right) = 0, \tag{12.3}$$

$$2a_1R^2\alpha^4 + (2a_1a_2 + 2\mu_1\mu_2b_1b_2)\alpha^2 = 0, \tag{12.4}$$

$$6a_2R^2\alpha^4 + \left(a_2^2 + \mu_1\mu_2b_2^2\right)\alpha^2 = 0, \tag{12.5}$$

$$b_1\mu_2R^2\alpha^4+2a_0b_1\alpha^2+b_1\bigl(\alpha^2+\beta^2-\lambda^2\bigr)=0, \tag{12.6}$$

$$5b_2\mu_2R^2\alpha^4+(2a_1b_1+2a_0b_2)\alpha^2+b_2\bigl(\alpha^2+\beta^2-\lambda^2\bigr)=0, \tag{12.7}$$

$$2b_1R^2\alpha^4+(2a_2b_1+2a_1b_2)\alpha^2=0, \tag{12.8}$$

$$6b_2R^2\alpha^4+2a_2b_2\alpha^2=0. \tag{12.9}$$

From which we have

Case 1

$$a_0=(-4\mu_2\pm 2)R^2\alpha^2,\qquad a_1=b_1=b_2=0,\qquad a_2=-6R^2\alpha^2,\qquad \lambda=\bigl[\bigl(\beta^2+\alpha^2\bigr)\pm 4R^2\alpha^2\bigr]^{1/2}.$$

Case 2

$$a_0=\frac{(-5\mu_2\pm 1)R^2\alpha^2}{2},\qquad a_1=b_1=0,\qquad a_2=-3R^2\alpha^2,\qquad b_2=\pm\frac{3R^2\alpha^2}{\sqrt{\mu_1\mu_2}},$$

$$\lambda=\bigl[\bigl(\beta^2+\alpha^2\bigr)\pm R^2\alpha^2\bigr]^{1/2}.$$

Therefore according to Step 5, eight families of explicit and exact travelling wave solutions, which contain solitary wave solutions, periodic wave solutions and new travelling wave solutions, are found as follows for (2)

$$u_1=(4\pm 2)R^2\alpha^2-6R^2\alpha^2\tanh^2\Bigl[R\bigl(\alpha x+\beta y-\bigl[\bigl(\beta^2+\alpha^2\bigr)\pm 4R^2\alpha^2\bigr]^{1/2}t\bigr)\Bigr],$$
$$u_2=(4\pm 2)R^2\alpha^2-6R^2\alpha^2\coth^2\Bigl[R\bigl(\alpha x+\beta y-\bigl[\bigl(\beta^2+\alpha^2\bigr)\pm 4R^2\alpha^2\bigr]^{1/2}t\bigr)\Bigr],$$
$$u_3=(-4\pm 2)R^2\alpha^2-6R^2\alpha^2\tan^2\Bigl[R\bigl(\alpha x+\beta y-\bigl[\bigl(\beta^2+\alpha^2\bigr)\pm 4R^2\alpha^2\bigr]^{1/2}t\bigr)\Bigr],$$
$$u_4=(-4\pm 2)R^2\alpha^2-6R^2\alpha^2\cot^2\Bigl[R\bigl(\alpha x+\beta y-\bigl[\bigl(\beta^2+\alpha^2\bigr)\pm 4R^2\alpha^2\bigr]^{1/2}t\bigr)\Bigr],$$
$$u_5=\frac{(5\pm 1)R^2\alpha^2}{2}-3R^2\alpha^2\Bigl\{\tanh^2\bigl[R(\alpha x+\beta y-\lambda t)\bigr]\mp i\tanh\bigl[R(\alpha x+\beta y-\lambda t)\bigr]\operatorname{sech}\bigl[R(\alpha x+\beta y-\lambda t)\bigr]\Bigr\},$$
$$u_6=\frac{(5\pm 1)R^2\alpha^2}{2}-3R^2\alpha^2\Bigl\{\coth^2\bigl[R(\alpha x+\beta y-\lambda t)\bigr]\mp \coth\bigl[R(\alpha x+\beta y-\lambda t)\bigr]\operatorname{csch}\bigl[R(\alpha x+\beta y-\lambda t)\bigr]\Bigr\},$$
$$u_7=\frac{(-5\pm 1)R^2\alpha^2}{2}-3R^2\alpha^2\Bigl\{\tan^2\bigl[R(\alpha x+\beta y-\lambda t)\bigr]\mp \tan\bigl[R(\alpha x+\beta y-\lambda t)\bigr]\sec\bigl[R(\alpha x+\beta y-\lambda t)\bigr]\Bigr\},$$
$$u_8=\frac{(-5\pm 1)R^2\alpha^2}{2}-3R^2\alpha^2\Bigl\{\cot^2\bigl[R(\alpha x+\beta y-\lambda t)\bigr]\mp \cot\bigl[R(\alpha x+\beta y-\lambda t)\bigr]\csc\bigl[R(\alpha x+\beta y-\lambda t)\bigr]\Bigr\},$$

where $\lambda=[(\beta^2+\alpha^2)\pm R^2\alpha^2]^{1/2}$.

Remark 1. It is easily seen that u_1, u_2, u_3, u_4 are just the solution (22) and (23) by Senthilvelan [16]. But to our knowledge, the obtained solutions of (13), u_5, u_6, u_7, u_8 were not found before. M. Chen obtained many exact solution of various Boussinesq systems by the method presented in [21], due to our more generalized transformation than the ansatz in [21], so by our method we can recover the solutions in [21].

3.2. (3 + 1)-*dimensional KP equation*

Let us now consider the (3 + 1)-dimensional KP equation

$$u_{xt} - 6u_x^2 + 6uu_{xx} - u_{xxxx} - u_{yy} - u_{zz} = 0. \tag{13}$$

According to the same as the above-mentioned steps, we firstly make the following formal travelling wave transformation:

$$u(x, y, t) = u(\xi), \quad \xi = \alpha x + \beta y + \gamma z - \lambda t, \tag{14}$$

where $\alpha, \beta, \gamma, \lambda$ are constants to be determined.

Substituting (14) into (13) gives rise to

$$\alpha^4 u'' - 3\alpha^2 u^2 + \left(\alpha^2 + \gamma^2 + \lambda\alpha\right)u = 0. \tag{15}$$

We assume that (15) has the solution in the form

$$u = a_0 + a_1\omega + b_1\sqrt{\mu_1\left(1 + \mu_2\omega^2\right)} + a_2\omega^2 + b_2\omega\sqrt{\mu_1\left(1 + \mu_2\omega^2\right)} \tag{16}$$

and $\omega = \omega(\xi)$ satisfying Eq. (5), where a_0, a_1, a_2, b_1, b_2 are constants to be determined later.

Substituting (16) into (15) along with (5), we can obtain a system of over-determined algebraic polynomials

$$2a_2R^2\alpha^4 + \left(a_0^2 + b_1^2\mu_1\right)\left(-3\alpha^2\right) + a_0\left(\alpha\lambda + \gamma^2 + \beta^2\right) = 0, \tag{17.1}$$

$$2a_1\mu_2R^2\alpha^4 + (2a_0a_1 + 2b_1b_2\mu_1)\left(-3\alpha^2\right) + a_1\left(\alpha\lambda + \gamma^2 + \beta^2\right) = 0, \tag{17.2}$$

$$8a_2\mu_2R^2\alpha^4 + \left(a_1^2 + 2a_0a_2 + b_2^2\mu_1 + b_1^2\mu_1\mu_2\right)\left(-3\alpha^2 2\right) + a_2\left(\alpha\lambda + \gamma^2 + \beta^2\right) = 0, \tag{17.3}$$

$$2a_1R^2\alpha^4 + (2a_1a_2 + 2b_1b_2\mu_1\mu_2)\left(-3\alpha^2\right) = 0, \tag{17.4}$$

$$6a_2R^2\alpha^4 + \left(a_2^2 + b_2^2\mu_2\mu_1\right)\left(-3\alpha^2\right) = 0, \tag{17.5}$$

$$b_1\mu_2R^2\alpha^4 - 6a_0b_1\alpha^2 + b_1\left(\alpha\lambda + \gamma^2 + \beta^2\right) = 0, \tag{17.6}$$

$$5b_2\mu_2R^2\alpha^4 + (2a_1b_1 + 2a_0b_2)\left(-3\alpha^2\right) + b_2\left(\alpha\lambda + \gamma^2 + \beta^2\right) = 0, \tag{17.7}$$

$$2b_1R^2\alpha^4 + (2a_2b_1 + 2a_1b_2)\alpha^2 = 0, \tag{17.8}$$

$$6b_2R^2\alpha^4 + 2a_2b_2\left(-3\alpha^2\right) = 0, \tag{17.9}$$

from which we can obtain

Case 1

$$a_0 = \frac{4\mu_2R^2\alpha^2 \pm 2R^2\alpha^2}{3}, \qquad a_1 = b_1 = 0, \qquad a_2 = 2R^2\alpha^2, \qquad \lambda = \frac{-(\beta^2 + \gamma^2) \pm 4R^2\alpha^4}{\alpha}.$$

Case 2

$$a_2 = R^2\alpha^2, \qquad a_1 = b_1 = 0, \qquad b_2 = \pm\frac{R^2\alpha^2}{\sqrt{\mu_1\mu_2}}, \qquad a_0 = \frac{5\mu_2 R^2\alpha^2 \pm R^2\alpha^2}{6},$$

$$\lambda = \frac{-(\beta^2+\gamma^2) \pm R^2\alpha^4}{\alpha}.$$

Thus we can find eight families of explicit and exact travelling wave solutions, which contain solitary wave solutions, periodic wave solutions and new travelling wave solutions, are found as follows for (13)

$$u_1 = \frac{-4R^2\alpha^2 \pm 2R^2\alpha^2}{3} + 2R^2\alpha^2 \tanh^2\left[R\left(\alpha x + \beta y + \gamma z - \frac{-(\beta^2+\gamma^2) \pm 4R^2\alpha^4}{\alpha}t\right)\right],$$

$$u_2 = \frac{-4R^2\alpha^2 \pm 2R^2\alpha^2}{3} + 2R^2\alpha^2 \coth^2\left[R\left(\alpha x + \beta y + \gamma z - \frac{-(\beta^2+\gamma^2) \pm 4R^2\alpha^4}{\alpha}t\right)\right],$$

$$u_3 = \frac{4R^2\alpha^2 \pm 2R^2\alpha^2}{3} + 2R^2\alpha^2 \tan^2\left[R\left(\alpha x + \beta y + \gamma z - \frac{-(\beta^2+\gamma^2) \pm 4R^2\alpha^4}{\alpha}t\right)\right],$$

$$u_4 = \frac{4R^2\alpha^2 \pm 2R^2\alpha^2}{3} + 2R^2\alpha^2 \cot^2\left[R\left(\alpha x + \beta y + \gamma z - \frac{-(\beta^2+\gamma^2) \pm 4R^2\alpha^4}{\alpha}t\right)\right],$$

$$u_5 = \frac{-5R^2\alpha^2 \pm R^2\alpha^2}{6} + R^2\alpha^2\Big[\tanh^2\big[R(\alpha x + \beta y + \gamma z - \lambda t)\big] \pm i\tanh\big[R(\alpha x + \beta y + \gamma z - \lambda t)\big] \times \operatorname{sech}\big[R(\alpha x + \beta y + \gamma z - \lambda t)\big]\Big],$$

$$u_6 = \frac{-5R^2\alpha^2 \pm R^2\alpha^2}{6} + R^2\alpha^2\Big[\coth^2\big[R(\alpha x + \beta y + \gamma z - \lambda t)\big] \pm \coth\big[R(\alpha x + \beta y + \gamma z - \lambda t)\big] \times \operatorname{csch}\big[R(\alpha x + \beta y + \gamma z - \lambda t)\big]\Big],$$

$$u_7 = \frac{5R^2\alpha^2 \pm R^2\alpha^2}{6} + R^2\alpha^2\Big[\tan^2\big[R(\alpha x + \beta y + \gamma z - \lambda t)\big] \pm \tan\big[R(\alpha x + \beta y + \gamma z - \lambda t)\big] \times \sec\big[R(\alpha x + \beta y + \gamma z - \lambda t)\big]\Big],$$

$$u_8 = \frac{5R^2\alpha^2 \pm R^2\alpha^2}{6} + R^2\alpha^2\Big[\cot^2\big[R(\alpha x + \beta y + \gamma z - \lambda t)\big] \pm \cot\big[R(\alpha x + \beta y + \gamma z - \lambda t)\big] \times \csc\big[R(\alpha x + \beta y + \gamma z - \lambda t)\big]\Big],$$

where $\lambda = \frac{-(\beta^2+\gamma^2) \pm R^2\alpha^4}{\alpha}$.

Remark 2. It is easily seen that u_1, u_2, u_3, u_4 are just the solution (28) and (29) by Senthilvelan [16]. But to our knowledge, the obtained solutions of (13), u_5, u_6, u_7, u_8 were not found before.

4. Conclusions

In summary, based on the well-known Riccati equation, many new types of exact solutions for both $(2+1)$-dimensional Boussinesq equation and $(3+1)$-dimensional KP equation have been derived by a generalized transformation. These solutions contain the known ones [16]. Seven kinds of them are singular soliton solutions. Such solutions develop a singularity at a finite point, i.e., for any fixed $t = t_0$, there exist x_0 at which these solutions blow up. There is much current interest in the formation of so-called "hot spots" or "blow up" of solutions. It appears that these singular solutions will model this physical phenomena. The method can be also easy to be extended to other NEEs and is sufficient to seek more new solitary wave solutions of NEEs. It not only uses a more generalized transformation to produce a overdetermined system of nonlinear algebraic equation but also can look for more solutions. In addition, this method is also computerizable, which allow us to perform complicated and tedious algebraic calculation on a computer.

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