Explicit exact solutions for some nonlinear partial differential equations with nonlinear terms of any order

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In this paper, by introducing some appropriate transformation and with the help of symbolic computation, we study exact travelling wave solutions for the high-order modified Boussinesq equation, a single nonlinear reaction-diffusion equation and a generalized nonlinear Schrödinger equation with nonlinear terms of any order by use of the extended-tanh method. Thus, some new exact travelling-wave solutions, which contain kink-shaped solitons, bell-shaped solitons, periodic solutions, combined formal solitons, rational solutions and singular solitons for these equations, are obtained.

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1 Introduction

In recent years, nonlinear evolution equations (NEEs) have attracted considerable attention, partly due to their occurrence in many fields of science, in physics as well as in chemistry or biology, partly due to the interesting feature and rich variety of properties of their solutions. In addition, importantly, due to the availability of computer symbolic system like *Mathematica* or *Maple*, which allow us to perform some complicated and tedious algebraic calculations on a computer, as well as help us to find new exact solutions of nonlinear partial differential equations (NPDEs). At the same time, various powerful methods, such as inverse scattering method, Bäcklund transformation, Darboux transformation, Cole–Hopf transformation, Painlevé method, Hirota method (see e.g. Refs. [1,2]), tanh method [3,4], homogeneous balance method [5–8], have been utilized to explore different kinds of solutions of various physical models described by NPDEs. Particularly, some general Ansätze have been proposed in order to obtain new formal solutions for given NPDEs.

Recently, Fan presented an extended-tanh method [9], which is a very effectively straightforward method for constructing travelling-wave solutions of NPDEs. Using the method, exact travelling wave solutions for a large variety of NPDEs are obtained [10–16]. The motivation of this paper is to utilize the extended-tanh method by means of some appropriate transformation to explore some exact solutions for

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the following three NPDEs with nonlinear terms of any order:

$$u_{tt} + \alpha u_{xxt} + \beta u_{xxxx} + \gamma (u^n)_{xx} = 0, \tag{1.1}$$

$$u_t = (au^p u_x)_x - bu + qu^{p+1}, \tag{1.2}$$

$$\mathrm{i}\alpha\varphi_t + \gamma\varphi_{tt} + \delta\varphi_{xx} + \rho\varphi + \kappa|\varphi|^{2p}\varphi = 0. \tag{1.3}$$

The first equation is called the high-order modified Boussinesq equation (HMBE) with the damping term u_{xxt} [17–21], where α, β, γ are costants. The subscripts in Eq. (1.1) refer to the partial derivatives of $u(x,t)$ with respect to time t and space x. This equation appears in several domains of mathematics and physics. When $\alpha = 0$, $\beta = -1$, $\gamma = 1$ and $n = 3$, Eq. (1.1) becomes the modified Boussinesq equation [17–19]

$$u_{tt} - u_{xxxx} + (u^3)_{xx} = 0, \tag{1.4}$$

which was presented in the famous Fermi–Pasta–Ulam problem. Equation (1.4) is used to investigate the behavior of systems which are primarily linear but a nonlinearity is introduced as a perturbation. It also arises in other physical applications. In fact, the damping term u_{xxt} has effect on the Fermi–Pasta–Ulma problems. This is shown in the nonlinear wave equation with the damping term [19,20]

$$u_{tt} + \alpha u_{xxt} + \beta u_{xxxx} + \gamma (u_x^n)_x = 0, \quad (n \geq 2). \tag{1.5}$$

In [21], three types of symmetry reductions of Eq. (1.4) were derived and it was shown that the equation is unintegrable. Recently, three types of symmetry reductions of Eq. (1.1) and solitary-wave solutions for Eq. (1.1) with $n = 3$ were obtained [22].

In [4], Khater et al. introduced a single nonlinear reaction-diffusion equation

$$u_t = (au^\delta u_x)_x - bu + qu^p, \tag{1.6}$$

where u describes a population density and b, q are constant coefficients, with $q > 0$ for explosive cases, p is a positive quantity, $p > 1$ and au^δ with a being a constant coefficient and δ is a positive quantity. By use of the tanh method, Khater et al. found a stationary periodic solutions of Eq. (1.6) with $\delta = p - 1$. In this paper, we consider Eq. (1.6) with p, q being arbitrary constants and $\delta = p - 1$, i.e., Eq. (1.2).

In [23], Arai considered the following nonlinear partial differential equation for a C^N-valued function

$$\varphi(x,t) = (\varphi_1(x,t), \cdots, \varphi_N(x,t)$$

on the two-dimensional space-time $R^2 = \{(x,t)|x,t \in R\}(N \geq 1)$:

$$\mathrm{i}\alpha\varphi_t + \beta\varphi_x + \gamma\varphi_{tt} + \varphi_{xx} + \rho\varphi + \kappa|\varphi^{2p}|\varphi = 0\,, \tag{1.7}$$

where α, β, γ, ρ, $\kappa \in C(\kappa \neq 0)$, $p \in R\backslash\{0\}$ (not necessarily an integer) are constants and $|\varphi(x,t)| := \sqrt{\sum_{n=1}^N |\varphi_n(x,t)|^2}$. Equation (1.7) unifies N-component nonlinear Schrödinger and Klein–Gordon equations on R^2. Arai presented some exact solutions of Eq. (1.7) in the following cases: (i) $N = 1$, $p \in R\backslash\{0, -1\}$ arbitrary; (ii) $N = 2$, $p = 1$; (iii) $N = 3$, $p = 1$. Recently, using $\mathrm{sec_q}$-$\mathrm{tanh_q}$ method, Liu et al. considered Eq. (1.7) and found some new solitary-wave solutions [24]. In this paper, in order to be convenient to discuss on R^2, we only consider the following case: $\beta = 0$, $N = 1$, $p \in R\backslash\{0, -1\}$ and the coefficient of term φ_{xx} is taken as an arbitrary constant δ, cf. Eq. (1.3).

The plan of this paper is as follows. In Section 2, we briefly describe the extended-tanh method. In Section 3, we apply the extended-tanh method to Eq. (1.1) and bring out some solutions. In Section 4, some solutions of Eq. (1.2) are found. In Sect. 5, we derived some solutions of Eq. (1.3). Conclusions will be finally presented.

2 The extended-tanh method

In this section, the extended-tanh method proposed by Fan [9–16] will be described. For a given nonlinear evolution equations, say, in two variables, x, t,

$$F(u, u_t, u_x, u_{xt}, u_{tt}, ...) = 0, \tag{2.1}$$

where $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, etc., we seek the following formal travelling-wave solutions:

$$u(x,t) = u(\xi), \qquad \xi = x - \lambda t, \tag{2.2}$$

where λ is a constant to be determined later. Then Eq. (2.1) reduces to a nonlinear ordinary differential equation (ODE) under (2.2)

$$G(u, u', u'', u''' \cdots) = 0, \tag{2.3}$$

where prime denotes a derivative, i.e. $\mathrm{d}/\mathrm{d}\xi$. The next crucial step is to express the solution of the resulting ODE by the following more general Ansätze

$$u(\xi) = \sum_{i=1}^{m} \omega^{i-1}(\xi) \left[A_i \omega(\xi) + B_i \sqrt{R + \omega^2(\xi)}\right] + A_0 \tag{2.4}$$

and the new variable $\omega = \omega(\xi)$ satisfying

$$\omega' - (R + \omega^2) = \frac{\mathrm{d}\omega}{\mathrm{d}\xi} - (R + \omega^2) = 0, \tag{2.5}$$

where A_0, A_i, B_i $(i = 1, 2, ..., m)$ and R are constants to be determined later, and m is an positive integer. However, when we balance the highest order partial derivative term and the nonlinear term in Eq. (2.3), we find that the constant m need not be a positive integer. In order to apply the extended-tanh method described in [9–16] when m is equal to a fraction or a negative integer, we make the following transformation:

(1) when $m = q/p$ (where $m = q/p$ is an irreducible fraction), we let

$$u(\xi) = \varphi^{q/p}(\xi), \tag{2.6}$$

then substitute (2.6) into Eq. (2.3) and return to determine the value of m by balancing the highest order partial derivative term and the nonlinear term in the new Eq. (2.3),

(2) when m is a negative integer, we let

$$u(\xi) = \varphi^{m}(\xi), \tag{2.7}$$

then we substitute (2.7) into Eq. (2.3) and return to determine the value of m once again.

In general, the constant m can be changed into a positive integer by means of the above appropriate transformation. Otherwise we have to seek other proper transformation.

We summarize the extended-tanh method as follows:

Step 1. Determine the values of m in (2.4) by balancing the highest order partial derivative terms and the nonlinear terms in Eq. (2.3).

(i) If m is a positive integer, then *Step 2*;

(ii) If $m = q/p$, we make the transformation (2.6) and then return to the beginning of *Step 1*;

(iii) If m is a negative integer, we make the transformation (2.7) and then return to the beginning of *Step 1*.

Step 2. With the aid of *Mathematica*, substituting (2.4) with the condition (2.5) into Eq. (2.3), yields a system of algebraic equations wrt $\omega^r(R+\omega^2)^{s/2}$ $(s = 0, 1;\ r = 0, 1, 2, \cdots)$.

Step 3. Collect all the terms with the same power in $\omega^r(R+\omega^2)^{s/2}$ $(s = 0, 1;\ r = 0, 1, 2, \cdots)$. Set the coefficients of the terms $\omega^r(R+\omega^2)^{s/2}$ $(s = 0, 1;\ r = 0, 1, 2, \cdots)$ to zero to get a over-determined system of nonlinear algebraic equations wrt the unknown variables $\lambda, R, A_0, A_i, B_i$ $(i = 1, 2, \cdots, m)$.

Step 4. With the aid of *Mathematica*, solving the above over-determined system of nonlinear algebraic equations, obtained in *Step 3*, yields the values of $\lambda, R, A_0, A_i, B_i$ $(i = 1, 2, \cdots, m)$.

Step 5. It is well known that the general solutions of Eq. (2.5) are

(1) when $R < 0$,

$$\omega(\xi) = -\sqrt{-R}\tanh(\sqrt{-R}\xi), \qquad \omega(\xi) = -\sqrt{-R}\coth(\sqrt{-R}\xi)\,, \tag{2.8}$$

(2) when $R = 0$,

$$\omega(\xi) = -\frac{1}{\xi}\,, \tag{2.9}$$

(3) when $R > 0$,

$$\omega(\xi) = \sqrt{R}\tan(\sqrt{R}\xi)\,, \qquad \omega(\xi) = -\sqrt{R}\cot(\sqrt{R}\xi)\,. \tag{2.10}$$

Thus according to Eqs. (2.2), (2.4), (2.6) or (2.7)–(2.10) and the conclusions in *Step 4*, we can obtain many travelling-wave solutions of Eq. (2.1).

3 The high-order modified Boussinesq equation

Let us consider the HMBE [17–22], i.e., Eq. (1.1). Firstly we take the following travelling-wave transformation

$$u(x,t) = v(\xi), \quad \xi = x - \lambda t, \tag{3.1}$$

where λ is a constant to be determined. Thus HMBE becomes

$$\lambda^2 v'' - \alpha\lambda v^{(3)} + \beta v^{(4)} + \gamma(v^n)'' = 0. \tag{3.2}$$

Integrating the above equation twice with regard to ξ, we obtain

$$\beta v'' - \alpha\lambda v' + \lambda^2 v + \gamma v^n = 0\,, \tag{3.3}$$

with the integration constants taken to be zero. According to *Step 1* in Section 2, balancing v'' with v^n in Eq. (3.3), we get $m = 2/(n-1)$. Therefore we make the following transformation

$$v(\xi) = \phi^{2/(n-1)}(\xi), \tag{3.4}$$

then substituting (3.4) into Eq.(3.3) reads

$$\beta[2(3-n)\phi'^2 + 2(n-1)\phi\phi''] - 2\alpha\lambda(n-1)\phi\phi' + (n-1)^2(\lambda^2\phi^2 + \gamma\phi^4) = 0. \tag{3.5}$$

Now balancing ϕ'^2 (or $\phi\phi''$) with ϕ^4, we find $m = 1$. So we can assume that

$$\phi(\xi) = A_0 + A_1\omega + B_1\sqrt{R+\omega^2} \tag{3.6}$$

and $\omega = \omega(\xi)$ satisfies (2.5), where A_0, A_1, B_1 are constants to be determined later.

With the aid of *Mathematica*, substituting (3.6) into (3.5) together with (2.5) and collecting all terms with the same power in $\omega^r(R+\omega^2)^{s/2}$ $(s = 0, 1;\ r = 0, 1, 2, 3, 4)$, yields a system of equations wrt $\omega^r(R+\omega^2)^{s/2}$. Setting the coefficients of $\omega^r(R+\omega^2)^{s/2}$ $(s = 0, 1;\ \ r = 0, 1, 2, 3, 4)$ in the obtained system of equations to zero, we can deduce the following set of over-determined algebraic polynomials wrt the unknowns A_0, A_1, B_1, R and λ:

$$\begin{aligned} &-2{A_1}^2(-3+n)R^2\beta - 2A_0A_1(-1+n)R\alpha\lambda \\ &+(-1+n)({B_1}^4(-1+n)R^2\gamma + {A_0}^2(-1+n)({A_0}^2\gamma+\lambda^2) \\ &+{B_1}^2R(2R\beta + (-1+n)(6{A_0}^2\gamma+\lambda^2))) = 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &2B_1(-1+n)(2{A_0}^3(-1+n)\gamma - A_1R\alpha\lambda \\ &+A_0(R(\beta+2{B_1}^2(-1+n)\gamma) + (-1+n)\lambda^2)) = 0, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &(-1+n)(2{A_0}^3A_1(-1+n)\gamma - ({A_1}^2+{B_1}^2)R\alpha\lambda \\ &+A_0A_1(2R(\beta+3{B_1}^2(-1+n)\gamma) + (-1+n)\lambda^2)) = 0, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &2B_1(-A_0(-1+n)\alpha\lambda + A_1(R((3+n)\beta \\ &+2{B_1}^2(-1+n)^2\gamma) + (-1+n)^2(6{A_0}^2\gamma+\lambda^2))) = 0, \end{aligned} \tag{3.10}$$

$$-2A_0A_1(-1+n)\alpha\lambda + {A_1}^2(2R(4\beta+3{B_1}^2(-1+n)^2\gamma) + (-1+n)^2(6{A_0}^2\gamma+\lambda^2)) + {B_1}^2(6{A_0}^2\gamma+2{B_1}^2R\gamma+\lambda^2 + 2n(2R\beta-6{A_0}^2\gamma-2{B_1}^2R\gamma-\lambda^2)+n^2(6{A_0}^2\gamma+2{B_1}^2R\gamma+\lambda^2)) = 0, \quad (3.11)$$

$$4B_1(-1+n)(A_0(\beta+(3{A_1}^2+{B_1}^2)(-1+n)\gamma)-A_1\alpha\lambda) = 0, \quad (3.12)$$

$$2(-1+n)(2A_0A_1(\beta+({A_1}^2+3{B_1}^2)(-1+n)\gamma)-({A_1}^2+{B_1}^2)\alpha\lambda) = 0, \quad (3.13)$$

$$4A_1B_1((1+n)\beta+({A_1}^2+{B_1}^2)(-1+n)^2\gamma) = 0, \quad (3.14)$$

$$ {A_1}^4(-1+n)^2\gamma + {B_1}^2(2(1+n)\beta+{B_1}^2(-1+n)^2\gamma) + 2{A_1}^2((1+n)\beta+3{B_1}^2(-1+n)^2\gamma) = 0. \quad (3.15)$$

Solving Eqs. (3.7)–(3.15) by means of *Mathematica*, we get the following solutions:
Case 1.

$$A_0 = \pm\sqrt{-\frac{\lambda^2}{4\gamma}}, \quad A_1 = \pm\frac{2(n+1)\alpha}{(n-1)(n+3)}\sqrt{-\frac{1}{\gamma}}, \quad B_1 = 0,$$

$$R = -\frac{(n-1)^2(n+3)^2\lambda^2}{16(n+1)^2\alpha^2}, \quad \beta = \frac{2(n+1)\alpha^2}{(n+3)^2}. \quad (3.16)$$

Case 2.

$$A_0 = \pm\sqrt{\frac{-\lambda^2}{4\gamma}}, \quad A_1 = \pm B_1 = \pm\frac{(n+1)\alpha}{(n-1)(3+n)}\sqrt{-\frac{1}{\gamma}},$$

$$R = -\frac{(n-1)^2(n+3)^2\lambda^2}{4(n+1)^2\alpha^2}, \quad \beta = \frac{2(n+1)\alpha^2}{(n+3)^2}. \quad (3.17)$$

Case 3.

$$\alpha = A_0 = A_1 = 0, \quad B_1^2 = -\frac{2(1+n)\beta}{(-1+n)^2\gamma}, \quad R = \frac{(-1+n)^2\lambda^2}{4\beta}. \quad (3.18)$$

Case 4.

$$\alpha = A_0 = B_1 = 0, \quad n = 3, \quad A_1^2 = -\frac{2\beta}{\gamma}, \quad R = -\frac{\lambda^2}{2\beta}. \quad (3.19)$$

Since $R<0$, from the Eqs. (2.8), (3.1), (3.4), (3.6), and (3.16),(3.17), when $\gamma<0$ and $\beta = 2(n+1)\alpha^2/(n+3)^2$, the HMBE have the following solutions:

$$u_1 = \left\{\pm\sqrt{-\frac{\lambda^2}{4\gamma}}\left(1\pm\tanh\left[\frac{(n-1)(n+3)\lambda}{4(n+1)\alpha}(x-\lambda t)\right]\right)\right\}^{2/(n-1)}, \quad (3.20)$$

$$u_2 = \left\{\pm\sqrt{-\frac{\lambda^2}{4\gamma}}\left(1\pm\coth\left[\frac{(n-1)(n+3)\lambda}{4(n+1)\alpha}(x-\lambda t)\right]\right)\right\}^{2/(n-1)}, \quad (3.21)$$

$$u_3 = \left\{ A_0 \left(1 \pm \tanh\left[\sqrt{-R}(x-\lambda t)\right] \pm \mathrm{i}\,\mathrm{sech}\left[\sqrt{-R}(x-\lambda t)\right]\right) \right\}^{2/(n-1)}, \tag{3.22}$$

$$u_4 = \left\{ A_0 \left(1 \pm \coth\left[\sqrt{-R}(x-\lambda t)\right] \pm \mathrm{csch}\left[\sqrt{-R}(x-\lambda t)\right]\right) \right\}^{2/(n-1)}, \tag{3.23}$$

where

$$A_0 = \pm\sqrt{\frac{-\lambda^2}{4\gamma}}, \quad R = -\frac{(n-1)^2(n+3)^2\lambda^2}{4(n+1)^2\alpha^2}.$$

From (3.18), the solutions of the equation $u_{tt} + \beta u_{xxxx} + \gamma(u^n)_{xx} = 0$ are as follows:

$$u_5 = \left\{ \pm\sqrt{-\frac{(1+n)\lambda^2}{2\gamma}}\,\mathrm{sech}\left[\sqrt{-\frac{(-1+n)^2\lambda^2}{4\beta}}(x-\lambda t)\right] \right\}^{2/(n-1)}, \quad \beta < 0, \tag{3.24}$$

$$u_6 = \left\{ \pm\sqrt{\frac{(1+n)\lambda^2}{2\gamma}}\,\mathrm{csch}\left[\sqrt{-\frac{(-1+n)^2\lambda^2}{4\beta}}(x-\lambda t)\right] \right\}^{2/(n-1)}, \quad \beta < 0, \tag{3.25}$$

$$u_7 = \left\{ \pm\sqrt{\frac{(1+n)\lambda^2}{2\gamma}}\,\sec\left[\sqrt{\frac{(-1+n)^2\lambda^2}{4\beta}}(x-\lambda t)\right] \right\}^{2/(n-1)}, \quad \beta > 0, \tag{3.26}$$

$$u_8 = \left\{ \pm\sqrt{\frac{(1+n)\lambda^2}{2\gamma}}\,\csc\left[\sqrt{\frac{(-1+n)^2\lambda^2}{4\beta}}(x-\lambda t)\right] \right\}^{2/(n-1)}, \quad \beta > 0. \tag{3.27}$$

From (3.19), the equation $u_{tt} + \beta u_{xxxx} + \gamma(u^3)_{xx} = 0$ has also the following solutions:

$$u_9 = \pm\sqrt{-\frac{\lambda^2}{\gamma}}\tanh\left[\sqrt{\frac{\lambda^2}{2\beta}}(x-\lambda t)\right], \quad \beta > 0, \tag{3.28}$$

$$u_{10} = \pm\sqrt{-\frac{\lambda^2}{\gamma}}\coth\left[\sqrt{\frac{\lambda^2}{2\beta}}(x-\lambda t)\right], \quad \beta > 0, \tag{3.29}$$

$$u_{11} = \pm\sqrt{\frac{\lambda^2}{\gamma}}\tan\left[\sqrt{-\frac{\lambda^2}{2\beta}}(x-\lambda t)\right], \quad \beta < 0, \tag{3.30}$$

$$u_{12} = \pm\sqrt{\frac{\lambda^2}{\gamma}}\cot\left[\sqrt{-\frac{\lambda^2}{2\beta}}(x-\lambda t)\right]. \quad \beta < 0. \tag{3.31}$$

4 A single nonlinear reaction-diffusion equation

In this section we consider the single nonlinear reaction-diffusion equation (1.2) [4]. Let $u(x,t) = v(\xi)$, $\xi = x - \lambda t$, then Eq. (1.2) reduces to

$$a(v^p v')' + \lambda v' - bv + qv^{p+1} = 0. \tag{4.1}$$

Balancing between $(v^p v')'$ and v' yields $m = -1/p$, which need not be a positive integer. Let $v = \phi^{-1/p}$, then Eq. (4.1) becomes

$$-ap\phi\phi'' + a(1+2p)\phi'^2 + p\phi^2(pq - \lambda\phi') - bp^2\phi^3 = 0. \tag{4.2}$$

Now balancing terms $\phi\phi''$ (or ϕ'^2) and ϕ^3 gives the balancing number $m = 2$. In this case, we can assume that

$$\phi = A_0 + A_1\omega + A_2\omega^2 + B_1\sqrt{R+\omega^2} + B_2\omega\sqrt{R+\omega^2}, \tag{4.3}$$

where $\omega = \omega(\xi)$ satisfies (2.5) and A_0, A_1, A_2, B_1, B_2 are arbitrary constants.

Substituting (4.3) into (4.2) together with (2.5) and collecting all terms with the same power in $\omega^r(R+\omega^2)^{s/2}$ $(s = 0,1;\ r = 0,1,\cdots,7)$, yields a system of equations wrt $\omega^r(R+\omega^2)^{s/2}$. Setting the coefficients of $\omega^r(R+\omega^2)^{s/2}$ $(s = 0,1;\ r = 0,1,2,\cdots,7)$ in the obtained system of equations equal to zero, we can deduce a set of over-determined algebraic polynomials equations wrt the unknowns A_0, A_1, A_2, B_1, B_2, R, and λ.

With the help of *Mathematica*, solving the above equations we obtain the following four sets of solutions:

Case 1.

$$A_0 = \frac{q}{2b},\quad A_1 = \pm\sqrt{-\frac{a(1+p)q}{b^2p^2}},$$
$$B_1 = A_2 = B_2 = 0,\quad R = \frac{p^2q}{4a(1+p)},\quad \lambda = \pm\sqrt{-\frac{ab^2}{q(1+p)}}. \tag{4.4}$$

Case 2.

$$A_0 = \frac{q}{2b},\quad A_1 = \pm B_1 = \pm\sqrt{-\frac{a(1+p)q}{4b^2p^2}},$$
$$A_2 = B_2 = 0,\quad R = \frac{p^2q}{a(1+p)},\quad \lambda = \pm\sqrt{-\frac{ab^2}{q(1+p)}}. \tag{4.5}$$

Case 3.

$$\lambda = A_1 = B_1 = B_2 = 0,\ A_0 = \frac{q(2+p)}{2b(1+p)},\ A_2 = \frac{2a(2+p)}{bp^2},\ R = \frac{p^2q}{4a(1+p)}. \tag{4.6}$$

Case 4.

$$\lambda = A_1 = B_1 = 0,\quad A_0 = \frac{q(2+p)}{b(1+p)},\quad A_2 = \pm B_2 = \frac{a(2+p)}{bp^2},\quad R = \frac{p^2q}{a(1+p)}. \tag{4.7}$$

From (4.4)–(4.7), when $R < 0$, i.e., $a(1+p)q < 0$, Eq.(1.2) has the following solutions. In order to compare our solutions with the solutions in Ref. [4], in Case III below the periodic solutions are also listed.

Case I.

$$u_{11} = \left\{ \frac{q}{2b} \left[1 \pm \tanh \left[\sqrt{-\frac{p^2 q}{4a(1+p)}} \left(x \pm \sqrt{-\frac{ab^2}{q(1+p)}} t \right) \right] \right] \right\}^{-1/p}, \tag{4.8}$$

$$u_{12} = \left\{ \frac{q}{2b} \left[1 \pm \coth \left[\sqrt{-\frac{p^2 q}{4a(1+p)}} \left(x \pm \sqrt{-\frac{ab^2}{q(1+p)}} t \right) \right] \right] \right\}^{-1/p}. \tag{4.9}$$

Case II.

$$u_{21} = \left\{ \frac{q}{2b} \left(1 \pm \tanh \left[\sqrt{-R}(x - \lambda t) \right] \pm \mathrm{i}\, \mathrm{sech} \left[\sqrt{-R}(x - \lambda t) \right] \right) \right\}^{-1/p}, \tag{4.10}$$

$$u_{22} = \left\{ \frac{q}{2b} \left(1 \pm \coth \left[\sqrt{-R}(x - \lambda t) \right] \pm \mathrm{csch} \left[\sqrt{-R}(x - \lambda t) \right] \right) \right\}^{-1/p}, \tag{4.11}$$

where

$$R = \frac{p^2 q}{a(1+p)}, \quad \lambda = \pm \sqrt{-\frac{ab^2}{q(1+p)}}.$$

Case III.

$$u_{31} = \left\{ \frac{q(2+p)}{2b(1+p)} \left[1 - \tanh^2 \left(\frac{p}{2} \sqrt{-\frac{q}{a(1+p)}} x \right) \right] \right\}^{-1/p}, \tag{4.12}$$

$$u_{32} = \left\{ \frac{q(2+p)}{2b(1+p)} \left[1 - \coth^2 \left(\frac{p}{2} \sqrt{-\frac{q}{a(1+p)}} x \right) \right] \right\}^{-1/p}, \tag{4.13}$$

$$u_{33} = \left\{ \frac{q(2+p)}{2b(1+p)} \left[1 + \tan^2 \left(\frac{p}{2} \sqrt{\frac{q}{a(1+p)}} x \right) \right] \right\}^{-1/p}, \quad aq(1+p) > 0, \tag{4.14}$$

$$u_{34} = \left\{ \frac{q(2+p)}{2b(1+p)} \left[1 + \cot^2 \left(\frac{p}{2} \sqrt{\frac{q}{a(1+p)}} x \right) \right] \right\}^{-1/p}, \quad aq(1+p) > 0. \tag{4.15}$$

Case IV.

$$u_{41} = \left[\frac{q(2+p)}{b(1+P)} \left(1 - \tanh^2 \xi \pm \mathrm{i} \tanh \xi \, \mathrm{sech}\, \xi \right) \right]^{-1/p}, \tag{4.16}$$

$$u_{42} = \left[\frac{q(2+p)}{b(1+P)} \left(1 - \coth^2 \xi \pm \coth \xi \, \mathrm{csch}\, \xi \right) \right]^{-1/p}, \tag{4.17}$$

where

$$\xi = \sqrt{-\frac{p^2 q}{a(1+p)}}x, \quad aq(1+p) < 0.$$

From (4.14), when setting $p = p - 1$, the solution u_{33} is changed into

$$u'_{33} = \left\{ \frac{2bp}{q(p+1)} \cos^2\left(\frac{p-1}{2}\sqrt{\frac{q}{ap}}x \right) \right\}^{1/(p-1)}, \tag{4.18}$$

which is the same as the solution (24) in Ref. [4]. Therefore the solution of Eq. (1.2) in [4] is a special case of our obtained solutions.

5 Generalized nonlinear Schrödinger equation

In this section we shall consider the generalized nonlinear Schrödinger equation (GNLSE) — (1.3) [23–24]. Let $\varphi(x,t) = v(\xi)\exp(\mathrm{i}\eta)$, $\xi = x - \lambda t$, $\eta = Mx - Nt$, then GNLSE (1.3) reduces to

$$(\delta+\gamma\lambda^2)v'' + i(2\lambda N\gamma + 2\delta M - \alpha\lambda)v' + (N\alpha - \gamma N^2 - \delta M^2 + \rho)v + \kappa v^{2p+1} = 0. \tag{5.1}$$

Under the condition

$$2\lambda N\gamma + 2\delta M - \alpha\lambda = 0, \tag{5.2}$$

Eq. (5.1) can be rewritten as follows:

$$(\delta + \gamma\lambda^2)v'' + (N\alpha - \gamma N^2 - \delta M^2 + \rho)v + \kappa v^{2p+1} = 0. \tag{5.3}$$

Balancing between v'' and v^{2p+1} yields $m = 1/p$, which is not a positive integer. Let $v = \phi^{1/p}$, then Eq. (5.3) becomes

$$(\delta + \gamma\lambda^2)[p\phi\phi'' - (-1+p)\phi'^2] + p^2[(N\alpha - \gamma N^2 - \delta M^2 + \rho)\phi^2 + \kappa\phi^4]. \tag{5.4}$$

Now balancing terms $\phi\phi''$ (or ϕ'^2) and ϕ^4 gives the balancing number $m = 1$. In this case, we can assume that

$$\phi = A_0 + A_1\omega + B_1\sqrt{R + \omega^2} \tag{5.5}$$

and $\omega = \omega(\xi)$ satisfies (2.5), where A_0, A_1, B_1 are arbitrary constants.

Proceeding as before, solving the corresponding set of over-determined algebraic polynomial equations, we obtain the following two solutions:

Case 1.

$$A_0 = A_1 = 0, \quad B_1 = \pm\sqrt{-\frac{(1+p)(\delta+\gamma\lambda^2)}{p^2\kappa}}, \quad R = \frac{p^2(N\alpha - \gamma N^2 - \delta M^2 + \rho)}{\delta + \gamma\lambda^2}. \tag{5.6}$$

Case 2.

$$R = A_0 = 0, \quad A_1 = \pm B_1 = \pm\sqrt{-\frac{(1+p)(\delta+\gamma\lambda^2)}{4p^2\kappa}}, \quad \rho = \gamma N^2 + \delta M^2. \tag{5.7}$$

From (5.2), we have

$$\lambda = \frac{2\delta M}{\alpha - 2N\gamma}. \tag{5.8}$$

Therefore GNLSE (1.3) has the following solutions:

Case 1.

$$u_1 = \left\{B_1\sqrt{R}\,\mathrm{sech}\left[\sqrt{-R}(x-\lambda t)\right]\right\}^{1/p}\exp[\mathrm{i}(Mx-Nt)], \quad R<0, \tag{5.9}$$

$$u_2 = \left\{B_1\sqrt{-R}\,\mathrm{csch}\left[\sqrt{-R}(x-\lambda t)\right]\right\}^{1/p}\exp[\mathrm{i}(Mx-Nt)], \quad R<0, \tag{5.10}$$

$$u_3 = \left\{B_1\sqrt{R}\,\sec\left[\sqrt{R}(x-\lambda t)\right]\right\}^{1/p}\exp[\mathrm{i}(Mx-Nt)], \quad R>0, \tag{5.11}$$

$$u_4 = \left\{B_1\sqrt{R}\,\csc\left[\sqrt{R}(x-\lambda t)\right]\right\}^{1/p}\exp[\mathrm{i}(Mx-Nt)], \quad R>0, \tag{5.12}$$

where

$$B_1 = \pm\sqrt{-\frac{(1+p)(\delta+\gamma\lambda^2)}{p^2\kappa}}, \ R = \frac{p^2(N\alpha-\gamma N^2-\delta M^2+\rho)}{\delta+\gamma\lambda^2}, \ \lambda = \frac{2\delta M}{\alpha-2N\gamma},$$

and M, N are arbitrary constants.
Case 2. From (5.7), when $\rho = \gamma N^2 + \delta M^2$, the following solutions for GNLSE (1.3) are obtained:

$$u = \left\{\pm\sqrt{-\frac{(1+p)(\delta+\gamma\lambda^2)}{p^2\kappa}}\,\frac{1}{x-\lambda t}\right\}^{1/p}\exp[\mathrm{i}(Mx-Nt)], \tag{5.13}$$

where $\lambda = 2\delta M/(\alpha - 2N\gamma)$ and M, N are arbitrary constants.

From (5.9), the solutions of GNLSE (see, e.g., Ref. [2])

$$\mathrm{i}q_t - q_{xx} - 2|q|^2 q = 0 \tag{5.14}$$

can be recovered as follows:

$$q = \pm\sqrt{N+M^2}\,\mathrm{sech}\left[\sqrt{N+M^2}(x+2Mt)\right]\exp[\mathrm{i}(Mx-Nt)],$$

where M, N are arbitrary constants and $N + M^2 \geq 0$.

6 Conclusions

In this paper, by introducing some appropriate transformation, we utilize the extended-tanh method to study the travelling-wave solutions for the HMB equation, a single NRD equation and a GNLS equation with nonlinear terms of any order with the help of symbolic computation *Mathematica*. As a result, many types of exact travelling-wave solutions of these equations, which contain soliton solutions, rational solutions, periodic solutions and singular soliton solutions, are obtained. The method can also be easily extended to treat other NPDEs and is sufficient to seek more solitary wave solutions and other formal solutions of given NPDEs. In addition, this method is also computerizable, which allows us to perform complicated and tedious symbolic algebraic calculation on a computer.

Very recently, a new unified algebraic method, which is more general than extended-tanh method, for constructing multiple travelling-wave solutions of NPDEs has been presented [25,26]. Using it, we can obtain not only solitary wave solutions but also Jacobi doubly periodic wave solutions for a large variety of NPDEs. In the future, we shall study solitary wave solutions and Jacobi doubly periodic wave solutions for some NPDEs with nonlinear terms of any order using this new method.

Appendix: Simple *Mathematica* verifying program

By use of the following simple *Mathematica* program, which is implemented in symbolic computation system *Mathematica 4.0*, we can verify the validity of our results. We take Eq. (1.2) as an example.

In[1]: =

$$u[\xi\_] = \phi[\xi]^{-1/p}$$

In[2]: =

$$P_1 = \mathrm{Simplify}[\mathrm{Simplify}[D[a\ u[\xi]^p\ D[u[\xi],\xi],\xi]+ \lambda\ D[u[\xi],\xi] - b\ u[\xi] + q\ u[\xi]^{p+1}]/.(\phi[\xi]^{-1/p})^p -> \phi[\xi]^{-1}] * p^2 * \phi[\xi]^{3+1/p}$$

Out[2]={the left side of Eq. (4.2)}

In[3]:=

$$\phi[\xi\_] = A0 + A1\ \omega[\xi] + A2\ \omega[\xi]^2 + B1\ \sqrt{R+\omega[\xi]^2} + B2\ \omega[\xi]\ \sqrt{R+\omega[\xi]^2}$$

In[4]:=

$$\omega'[\xi] = R + \omega[\xi]^2$$

In[5]:=

$$\omega''[\xi] = D[\omega'[\xi],\xi]$$

In[6]:=

$$P2 = \mathrm{Simplify}[\mathrm{CoefficientList}[\mathrm{Expand}[P1], \{\omega[\xi], \sqrt{R+\omega[\xi]^2}\}]]$$

Out[6]={the left side of the corresponding equations omitted in Section 4.}

In[7]:=

$$P41 = \mathrm{Simplify}[\mathrm{Simplify}[P2/.\{B1->0, A2->0, B2->0\}/.A1->\tfrac{a}{p\lambda} /.A0->\tfrac{q}{2b}/.R->\tfrac{p^2q}{4a+4ap}]/.\lambda^2->\tfrac{-ab^2}{q+pq}/.\lambda^4->(\tfrac{-ab^2}{q+pq})^2]$$

Out[7]={{0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}}

This shows that the values of A_0, A_1, A_2, B_1, B_2 in (4.4) are a set of solutions of the corresponding equations. The other cases can be similarly verified.

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