Auto-Bäcklund Transformations and Exact Solutions for the Generalized Two-Dimensional Korteweg-de Vries-Burgers-type Equations and Burgers-type Equations

Biao Li[a,b], Yong Chen[a,b], and Hongqing Zhang[a,b]

[a] Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China
[b] Key Laboratory of Mathematics Mechanization, Chinese Academy of Sciences, Beijing 100080, China

Reprint requests to Dr. B. Li; E-mail: libiao@dlut.edu.cn.

Z. Naturforsch. **58a,** 464 – 472 (2003); received June 7, 2002

In this paper, based on the idea of the homogeneous balance method and with the help of *Mathematica*, we obtain a new auto-Bäcklund transformation for the generalized two-dimensional Korteweg-de Vries-Burgers-type equation and a new auto-Bäcklund transformation for the generalized two-dimensional Burgers-type equation by introducing two appropriate transformations. Then, based on these two auto-Bäcklund transformation, some exact solutions for these equations are derived. Some figures are given to show the properties of the solutions.

Key words: Auto-Bäcklund transformation; Homogeneous balance method; Two-dimensional Korteweg-de Vries-Burgers-type equation; Two-dimensional Burgers-type equation; Solitary-wave solution; *Mathematica*.

1. Introduction

The Bäcklund transformation (BT) of nonlinear partial differential equations (PDEs) plays an important role in soliton theory. It is an efficient method to obtain exact solutions of nonlinear PDEs. The nonlinear iterative principle of the BT converts the problem of solving nonlinear PDEs to purely algebraic calculations [1 – 5, 7, 8, 10]. In order to obtain the BT of the given nonlinear PDE, various methods have been presented. One of them is the homogenous balance (HB) method, which is a primary and concise method to seek for exact solutions of nonlinear PDEs [4 – 10]. In [4, 5], Fan extended the HB method to search for BTs and similarity reductions of nonlinear PDEs. So more exact solutions for these PDEs can be obtained by the improved HB method.

Now we briefly describe the HB method, for a given nonlinear PDE, say, in two variables,

$$H(u,u_x,u_t,u_{xx},\cdots)=0. \quad (1.1)$$

We seek for the BT of (1.1) in the form

$$u=\partial_x^m\partial_t^n f[w(x,t)]+\widetilde{u}, \quad (1.2)$$

where $w(x,t)$, $u=u(x,t)$, and $\widetilde{u}=\widetilde{u}(x,t)$ are undetermined functions and m, n are positive integers determined by balancing the highest derivative term with the nonlinear terms in (1.1) (see [4, 5] for details). However, we find that the constants m, n should not be restricted to positive integers. In order to apply the HB method to obtain a BT for a given PDE when m, n are not equal to positive integers, we have to seek for some proper transformations.

In this paper, we consider the generalized two-dimensional Korteweg-de Vries-Burgers-type (2D KdV-Burgers-type) equations

$$(u_t+au^pu_x+bu^{2p}u_x+ru_{xx}+\delta u_{xxx})_x+su_{yy}=0, \quad (1.3)$$

$$a,b,r,\delta,s,p=\text{const.}$$

and the generalized 2D Burgers-type equations, obtained if $b=\delta=0$,

$$(u_t+au^pu_x+ru_{xx})_x+su_{yy}=0, a,r,s,p=\text{const.} \quad (1.4)$$

Equations (1.3) and (1.4) include many important mathematical and physical equations which have been studied by many authors. To quote a few:

(I) In two-dimensional cases:

1. 2D KdV-Burgers equation:
   $(u_t+uu_x-\alpha u_{xx}+\beta u_{xxx})_x+\gamma u_{yy}=0$ [11 – 14].

2. the Kadomtsev-Petviashvili (KP) and the generalized KP equations:
   $(u_t + uu_x + u_{xxx})_x = u_{yy}$ [16].
   $(u_t + u^2u_x + u_{xxx})_x = u_{yy}$ [16].
   $(u_t + (u^{m+1})_x + u_{xxx})_x = \sigma^2 u_{yy} \quad (\sigma^2 = 1)$ [22].
   $(u_t + u^p u_x + u_{xxx})_x + \sigma^2 u_{yy} = 0$ [18, 19].

(II) In one-dimensional cases:

1. Burgers, KdV, mKdV, the combined KdV and mKdV equations, Burgers and KdV-Burgers equations (see e. g. [1, 2]):
   $u_t - 2uu_x - u_{xx} = 0.$
   $u_t + 6uu_x + u_{xxx} = 0.$
   $u_t + 6u^2u_x + u_{xxx} = 0.$
   $u_t + uu_x + u_{xx} = 0.$
   $u_t + auu_x + bu_{xx} + u_{xxx} = 0.$
2. Generalized KdV and KdV-Burgers like equations with higher order nonlinearity:
   $u_t + a(1+bu)uu_x + \delta u_{xxx} = 0,\ \ a,\delta \ge 0.$ [17].
   $u_t + a(1+bu^2)u^2u_x - \delta u_{xxx} = 0,\ \ a,\delta \ge 0$ [17].
   $u_t + u^p u_x + u_{xxx} = \alpha u_{xx}$ [21].
   $u_t + u^p u_x + u_{xxx} = 0$ [21].
   $u_t + \frac{1}{2}(p+1)(p+2)u^p u_x + u_{xxx} = 0$ [20].
   $u_t + au^p u_x + bu^{2p}u_x + \delta u_{xxx} = 0$ [15].
   $u_t + au^p u_x + bu^{2p}u_x + ru_{xx} + \delta u_{xxx} = 0$ [15].

In this paper, based on the idea of the HB method and with the help of a symbolic computation system as *Mathematica*, a new ABT for (1.3) and a new ABT for (1.4) are derived by use of two proper transformations. Then, based on these two BTs, several families of exact solutions for (1.3) and (1.4) are found.

This paper is organized as follows. In Sect. 2, we derive a BT for (1.3). In Sect. 3, based on this BT, a family of exact solutions for (1.3) is obtained. In Sect. 4, a new BT for (1.4) and a family of exact solutions for (1.4) are obtained. Concluding remarks are given in the last section.

2. Auto-Bäcklund Transformation for Eq. (1.3)

Let us consider the generalized 2D KdV-Burgers-type equation (1.3). Under the condition $b \neq 0$ and $\delta \neq 0$, according to the idea of the HB method [4–10], by balancing u_{xxxx} with $(u^{2p}u_x)_x$ in (1.3), we get the balance constants $m = 1/p$, $n = l = 0$. It is obvious that m may be an arbitrary constant. In order to apply the HB method in this case, we first make the transformation

$$u(x,y,t) = v^{1/p}(x,y,t). \tag{2.1}$$

Substituting (2.1) into (1.3) yields

$$\begin{aligned}&(1-6p+11p^2-6p^3)\delta v_x^4 + bp^3v^5v_{xx}\\ &\quad + p^2v^4[b(1+p)v_x^2 + apv_{xx}]\\ &\quad + p(1-3p+2p^2)vv_x^2(rv_x + 6\delta v_{xx})\\ &\quad - (-1+p)p^2v^2[v_tv_x + 3\delta v_{xx}^2\\ &\quad + v_x(3rv_{xx} + 4\delta v_{xxx}) + sv_y^2]\\ &\quad + p^2v^3[av_x^2 + p(v_{xt} + rv_{xxx} + \delta v_{xxxx} + sv_{yy})] = 0.\end{aligned} \tag{2.2}$$

Then by balancing v^3v_{xxxx} with v^5v_{xx} in (2.2), we get the balance constants $m = 1$, $n = l = 0$. Therefore we seek for the Bäcklund transformation of (2.2) in the form

$$v = f'w_x + \varphi. \tag{2.3}$$

Here and in the following context $' := \partial/\partial w$, $f^{(r)} = \partial^r/\partial w^r$, and $f = f(w)$, $w = w(x,y,t)$ and $\varphi = \varphi(x,y,t)$ are undetermined functions.

With the help of *Mathematica*, substituting (2.3) into (2.2) yields (because the formula is so long, just one part of it is shown here)

$$\begin{aligned}&[bp^2(1+p)(f')^4(f'')^2\\ &\quad + (1-6p+11p^2-6p^3)\delta(f'')^4 + bp^3(f')^5f^{(3)}\\ &\quad + 6p(1-3p+2p^2)\delta f'(f'')^2f^{(3)}\\ &\quad - (-1+p)p^2\delta(f')^2[3(f^{(3)})^2 + 4f''f^{(4)}]\\ &\quad + p^3\delta(f')^3f^{(5)}]w_x^8 + \cdots = 0.\end{aligned} \tag{2.4}$$

To simplify (2.4), we set the coefficient of w_x^8 to zero. Thus an ordinary differential equation for f is obtained:

$$\begin{aligned}&bp^2(1+p)(f')^4(f'')^2\\ &\quad + (1-6p+11p^2-6p^3)\delta(f'')^4 + bp^3(f')^5f^{(3)}\\ &\quad + 6p(1-3p+2p^2)\delta f'(f'')^2f^{(3)}\\ &\quad - (-1+p)p^2\delta(f')^2[3(f^{(3)})^2 + 4f''f^{(4)}]\\ &\quad + p^3\delta(f')^3f^{(5)} = 0.\end{aligned} \tag{2.5}$$

Solving (2.5) we obtain a solution

$$f=\pm\sqrt{-\frac{(1+p)(1+2p)\delta}{bp^2}}\log w. \tag{2.6}$$

Define $\beta=\pm\sqrt{-(1+p)(1+2p)\delta/bp^2}$ and substitute (2.6) into (2.4). Formula (2.4) can then be simplified to a linear polynomial of $\frac{1}{w}$. If we take the coefficients of $\frac{1}{w^i}$ $(i=0,\cdots,7)$ to be zero, we obtain a system of eight PDEs for $w(x,y,t)$ and $\varphi(x,y,t)$. Because the system is very complex, for simplicity we don't list them. We only give a simple program with some key *Mathematica* commands to show how to produce the system in the Appendix.

Under the conditions $p\neq 0,-1,-\frac{1}{2}$ and $b,\delta\neq 0$, from (2.1), (2.3) and (2.6), we obtain a desired ABT of (1.3):

$$u=\left[\pm\sqrt{-\frac{(1+p)(1+2p)\delta}{bp^2}}\,\frac{w_x}{w}+\varphi\right]^{\frac{1}{p}}, \tag{2.7}$$

where w satisfies the system of 8 PDEs produced by "Out [14]" in the Appendix, φ is a solution of (2.2).

3. Explicit Exact Solutions for Eq. (1.3)

Now we use the ABT consisting of (2.7) and the system of 8 PDEs produced by "Out [14]" in the Appendix to exploit some explicit exact solutions for (1.3). If we look for the solution of (2.2) with $\varphi=0$, then the system of 8 PDES reduces to only 4 equations,

$$\begin{aligned}&(1-6p+11p^2-6p^3)\delta w_{xx}^4\\&\quad+p(1-3p+2p^2)w_xw_{xx}^2(rw_{xx}+6\delta w_{xxx})\\&\quad-(-1+p)p^2w_x^2\left[w_{xt}w_{xx}+3\delta w_{xxx}^2\right.\\&\qquad\left.+w_{xx}(3rw_{xxx}+4\delta w_{xxxx})+sw_{xy}^2\right]\\&\quad+p^3w_x^3(w_{xxt}+rw_{xxxx}+\delta w_{xxxxx}+sw_{xyy})=0,\end{aligned} \tag{3.1}$$

$$\begin{aligned}&w_x^2\Big(2b(2-3p+p^2)\beta\delta w_{xx}^3\\&\quad+w_x\{bp^2\beta w_tw_{xx}+[3bpr\beta+a(1+3p+2p^2)\delta]w_{xx}^2\\&\qquad-2b(-6+p)p\beta\delta w_{xx}w_{xxx}+2bp^2s\beta w_{xy}w_y\}\\&\quad+pw_x^2\{bp(1+p)\beta w_{xt}\\&\qquad+[bp(3+p)r\beta+a(1+3p+2p^2)\delta]w_{xxx}\\&\qquad+bp\beta[(4+p)\delta w_{xxxx}+psw_{yy}]\}\Big)=0,\end{aligned} \tag{3.2}$$

$$\begin{aligned}&w_x^4\Big(p^2(1+p)^2(1+2p)\delta w_tw_x\\&\quad+pw_x\{[abp^3(2+3p)\beta^3+3(1+p)^3(1+2p)r\delta]w_{xx}\\&\qquad+[-b^2p^4\beta^4+2(1+p)^2(3+8p+4p^2)\delta^2]w_{xxx}\}\\&\quad-(1+p)\{[b^2p^4\beta^4\\&\qquad-3(2+8p+11p^2+7p^3+2p^4)\delta^2]w_{xx}^2\\&\qquad-p^2(1+3p+2p^2)s\delta w_y^2\}\Big)=0,\end{aligned} \tag{3.3}$$

$$\begin{aligned}&w_x^6[p(1+2p)\beta(r+pr-ap\beta)w_x\\&\quad+(2+7p+7p^2+2p^3)\beta\delta w_{xx}]=0.\end{aligned} \tag{3.4}$$

To solve (3.1) – (3.4), we assume that $w(x,y,t)$ has the form

$$w(x,y,t)=A+B\exp[k(x+cy-dt)], \tag{3.5}$$

where A, B, k, c, d are constants to be determined.

Substituting (3.5) into (3.1) – (3.4), we find that (3.5) satisfies (3.1) – (3.4) provided

$$k=-\frac{p(1+p)r-ap^2\beta}{(1+p)(2+p)\delta} \tag{3.6a}$$

and

$$\begin{aligned}d=&\{-a^2(1+2p)\delta+b[-(1+p)^2r^2+ap^2r\beta\\&+c^2(1+p)(2+p)^2s\delta]\}\{b(1+p)(2+p)^2\delta\}^{-1}\end{aligned} \tag{3.6b}$$

while A, B, c remain arbitrary constants.

From (2.7), (3.5) and (3.6) we can obtain a family of solutions of (1.3),

$$u=\left\{\frac{\beta kB\exp[k(x+cy-dt)]}{A+B\exp[k(x+cy-dt)]}\right\}^{\frac{1}{p}}, \tag{3.7}$$

where k and d satisfy (3.6), $\beta=\pm\sqrt{-(1+p)(1+2p)\delta/bp^2}$, A and B are arbitrary constants.

From (3.7), a family of kink-shaped solitary-wave solutions and a family of singular solitary-wave solutions for (1.3) are obtained as follows:

$$u_1=\left\{\frac{\beta k}{2}[1+\tanh\frac{k(x+cy-dt)}{2}]\right\}^{\frac{1}{p}},\ \text{if}\ A=B, \tag{3.8}$$

$$u_2=\left\{\frac{\beta k}{2}[1+\coth\frac{k(x+cy-dt)}{2}]\right\}^{\frac{1}{p}},\ \text{if}\ A=-B, \tag{3.9}$$

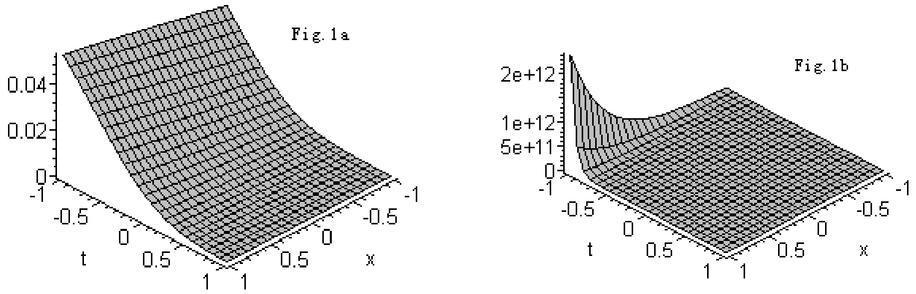

Fig. 1. Plots of solution (3.8) with $p=\frac{1}{6}$ in Fig. 1a ($p=-\frac{1}{6}$ in Fig. 1b), $\delta=-0.1$, $b=y=1$, $r=0.4$, $a=0.1$, $c=2$ and $s=3$.

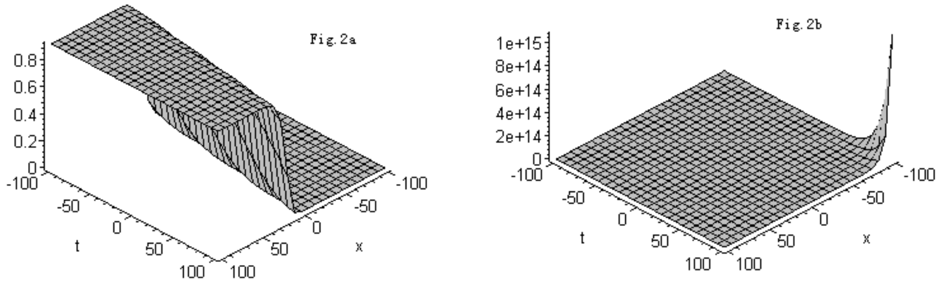

Fig. 2. Plots of solution (3.8) with $p=\frac{1}{3}$ in Fig. 1a ($p=-\frac{1}{3}$ in Fig. 1b), $\delta=-1$, $b=1$, $y=2$, $r=0.4$, $a=-1$, $c=1$ and $s=0.8$.

where k and d are determined by (3.6), $\beta=\pm\sqrt{-(1+p)(1+2p)\delta/bp^2}$, c is an arbitrary constant.

Remark 1: From our solutions (3.8), some known solutions of the KdV-Burgers-type equation in one- and two-dimensional cases can be recovered [11–22]. For example, if we set $c=s=0$, from (3.8) the equation

$$u_t+au^pu_x+bu^{2p}u_x+ru_{xx}+\delta u_{xxx}=0 \qquad (3.10)$$

has the kink-profile solitary-wave solutions

$$u=\left\{\frac{a(1+2p)\delta-bp\beta r}{2b(2+p)\delta}\cdot[1+\tanh[-\frac{p(1+p)r-ap^2\beta}{2(1+p)(2+p)\delta}(x-\lambda t)]]\right\}^{\frac{1}{p}}, \qquad (3.11)$$

where $\lambda=-a^2(1+2p)\delta+b[-(1+p)^2r^2+ap^2r\beta]\ /\ b(1+p)(2+p)^2\delta$, $\beta=\pm\sqrt{-(1+p)(1+2p)\delta/bp^2}$. It is not difficult to verify that the solutions (3.11) of (3.10) coincide with the solutions (4.7) and (4.8) in [15]. Therefore the solutions (4.7) and (4.8) given in [15] are special cases of our solutions (3.8). Plots of the solutions (3.8) and (3.9) with various parameters are given in Figs. 1–10. (*Note:* In Figs. 1–10 we take $\beta=\sqrt{-(1+p)(1+2p)\delta/bp^2}$. The plots with $\beta=-\sqrt{-(1+p)(1+2p)\delta/bp^2}$ and various parameters are similar to Figs. 1–10.)

4. ABT and Exact Solutions for Eq. (1.4)

We now consider the Burgers-type equation (1.4). Note that we do not obtain the BT for (1.4) from the BT derived in Section 2. But the method can be used.

Proceeding as before, balancing u_{xxx} with u^pu_{xx} in (1.4) yields $m=\frac{1}{p}$, $n=l=0$. Therefore we make the transformation

$$u(x,y,t)=v(x,y,t)^{\frac{1}{p}}. \qquad (4.1)$$

Then, substituting (4.1) into (1.4) yields

$$\begin{aligned}&(1-3p+2p^2)rv_x^3+ap^2v^3v_{xx}\\&\quad-(-1+p)pv[sv_y^2+v_x(v_t+3rv_{xx})]\\&\quad+pv^2(psv_{yy}+av_x^2+pv_{xt}+prv_{xxx}).\end{aligned} \qquad (4.2)$$

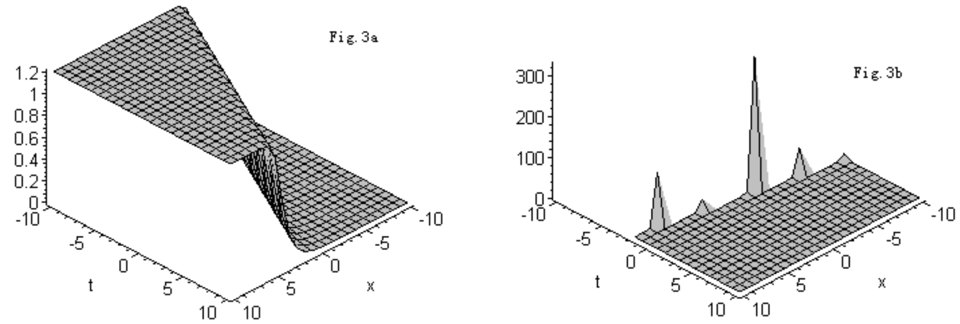

Fig. 3. Plots of solution (3.8) with $p=3$ in Fig. 3a ($p=-3$ in Fig. 3b), $\delta=-0.1$, $b=1$, $r=0.6$, $a=0.2$, $c=0.1$, $y=4$ and $s=1$.

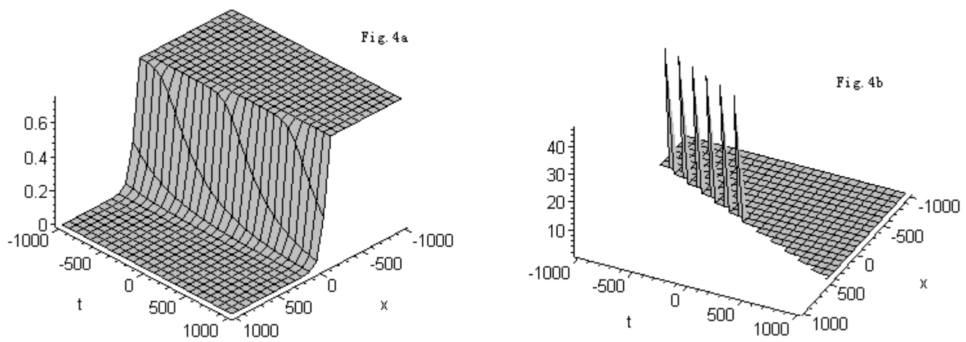

Fig. 4. Plots of solution (3.8) with $p=6$ in Fig. 4a ($p=-6$ in Fig. 4b), $\delta=-3.6$, $b=-5.5$, $r=6$, $a=-6$, $c=-0.1$, $y=40$ and $s=1$.

By balancing v^2v_{xxx} with v^3v_{xx} in (4.2), we get $m=1$, $n=l=0$. Therefore we assume that the BT of (4.2) has the form

$$v=f'(w)w_x+\varphi. \tag{4.3}$$

With the help of *Mathematica* again, substituting (4.3) into (4.2) yields (because the formula is so long, just one part of it is shown here)

$$\begin{aligned}&[(1-3p+2p^2)r(f'')^3+ap^2(f')^3f^{(3)}\\&-3(-1+p)prf'f''f^{(3)}+p(f')^2(a(f'')^2\\&+prf^{(4)})]w_x^6+\cdots=0.\end{aligned} \tag{4.4}$$

To simplify (4.3), we set the coefficient of w_x^6 to zero and obtian an ordinary differential equation for f:

$$\begin{aligned}&(1-3p+2p^2)r(f'')^3+ap^2(f')^3f^{(3)}\\&-3(-1+p)prf'f''f^{(3)}+p(f')^2(a(f'')^2\\&+prf^{(4)})=0.\end{aligned} \tag{4.5}$$

Solving (4.5) we obtain a solution

$$f=\frac{(1+p)r}{ap}\log w. \tag{4.6}$$

Define $\alpha=(1+p)r/ap$ and substitute (4.6) into (4.4). Formula (4.4) can be simplified to a linear polynomial of $1/w$, then setting the coefficients of $1/w^i$ ($i=0,\cdots,5$) to zero, we obtain a system of 6 PDEs for $w(x,y,t)$ and $\varphi(x,y,t)$. For simplicity, we don't list them. The reader can easily produce them by *Mathematica*.

From (4.1), (4.3) and (4.6), we obtain the desired Bäcklund transformation of (1.4)

$$u=\left[\frac{(1+p)r}{ap}\frac{w_x}{w}+\varphi\right]^{\frac{1}{p}}, \tag{4.7}$$

where w satisfies the system of 6 PDEs and φ is a solution of (4.2).

Now we use the Bäcklund transformation consisting of (4.7) and the system of 6 PDEs to exploit some ex-

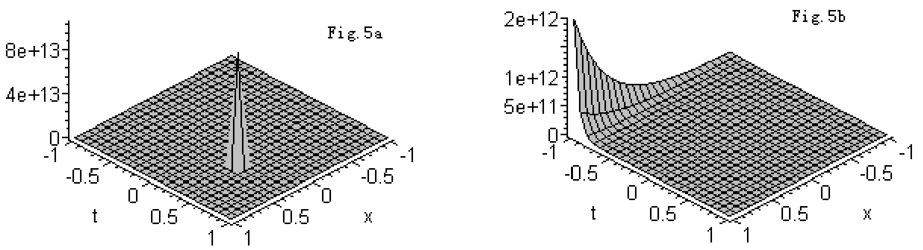

Fig. 5. Plots of solution (3.9) with $p=\frac{1}{6}$ in Fig. 7a ($p=-\frac{1}{6}$ in Fig. 7b), $\delta=-0.1$, $b=y=1$, $r=0.4$, $a=0.1$, $c=2$, $y=1$ and $s=3$.

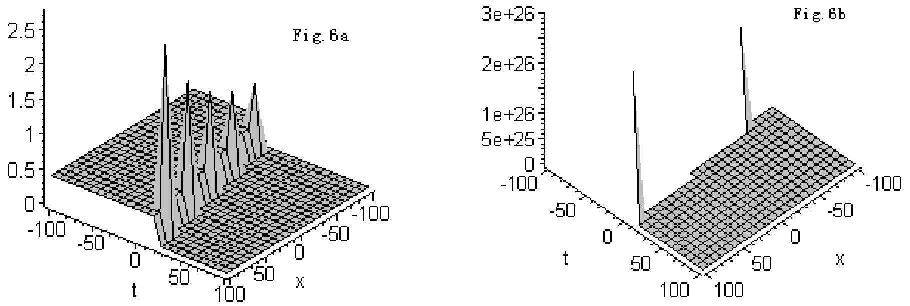

Fig. 6. Plots of solution (3.9) with $p=\frac{1}{3}$ in Fig. 6a ($p=-\frac{1}{3}$ in Fig. 6b), $\delta=-0.1$, $b=1$, $r=0.4$, $a=0.1$, $c=2$, $y=1$ and $s=3$.

plicit exact solutions for (1.4). If we consider the case $\varphi=0$, then the system of 6 PDEs reduces to

$$\begin{aligned}&(1-3p+2p^2)rw_{xx}^3-(-1+p)pw_x\\&\quad\cdot[w_{xx}(w_{xt}+3rw_{xxx})+sw_{xy}^2]\\&\quad+p^2w_x^2(w_{xxt}+rw_{xxxx}+sw_{xyy})=0,\end{aligned}\tag{4.8}$$

$$\begin{aligned}&w_x^2[pw_tw_{xx}+2rw_{xx}^2-prw_{xx}^2\\&\quad+2psw_{xy}w_y+pw_x((1+p)w_{xt}\\&\quad+2rw_{xxx}+psw_{yy})]=0,\end{aligned}\tag{4.9}$$

$$w_x^3(pw_tw_x+rw_xw_{xx}+psw_y^2)=0.\tag{4.10}$$

To solve (4.8) – (4.10), we again assume that w is of the form

$$w(x,y,t)=A+B\exp[k(x+cy-dt)],\tag{4.11}$$

where A, B, k, c, and d are constants to be determined. Then, substituting (4.11) into (4.8) – (4.10), we find that (4.11) satisfies (4.8)-(4.9), provided that A, B, c, d are arbitrary constants and

$$k=\frac{p(d-c^2s)}{r}.\tag{4.12}$$

Substituting (4.11) with (4.12) into (4.7), we can obtain a family of solutions for (1.4):

$$u=\left\{\frac{(1+p)(d-c^2s)}{a}\cdot\frac{B\exp[\frac{p(d-c^2s)}{r}(x+cy-dt)]}{A+B\exp[\frac{p(d-c^2s)}{r}(x+cy-dt)]}\right\}^{\frac{1}{p}}.\tag{4.13}$$

From (4.13), a family of kink-shaped solitary-wave solutions and a family of singular solitary-wave solutions for (1.3) are obtained:

$$u_1=\left\{\frac{(1+p)(d-c^2s)}{2a}\cdot\left[1+\tanh\left[\frac{p(d-c^2s)}{2r}(x+cy-dt)\right]\right]\right\}^{\frac{1}{p}},\tag{4.14}$$

if $\quad A=B$,

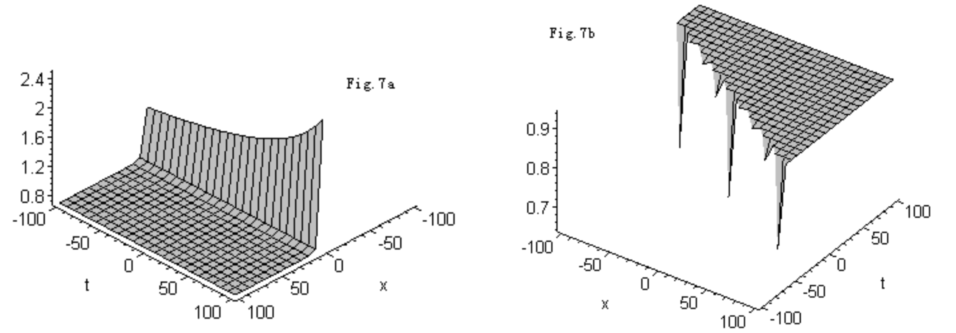

Fig. 7. Plots of solution (3.9) with $p = 3$ in Fig. 7a ($p = -3$ in Fig. 7b), $\delta = -9$, $b = 10$, $r = 0.6$, $a = -2$, $c = 0.1$, $y = 4$ and $d = 8$.

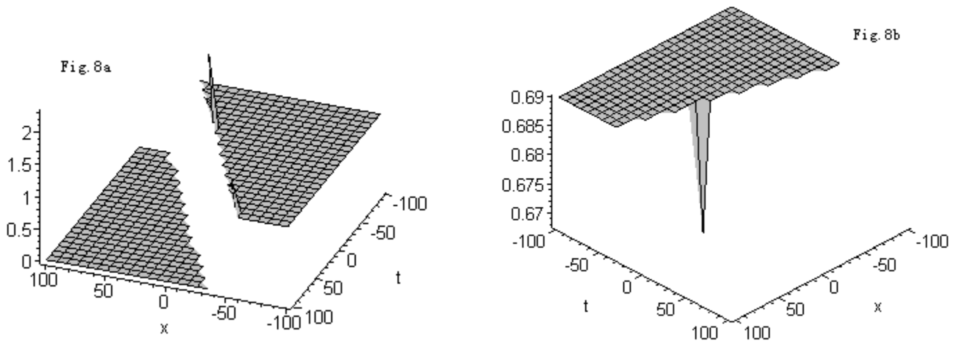

Fig. 8. Plots of solution (3.9) with $p = 4$ in Fig. 8a ($p = -4$ in Fig. 8b), $\delta = 1$, $b = -1$, $r = 0.4$, $a = c = y = 1$ and $s = 1$.

$$u_2 = \left\{ \frac{(1+p)(d-c^2 s)}{2a} \cdot [1+\coth[\frac{p(d-c^2 s)}{2r}(x+cy-dt)] \right\}^{\frac{1}{p}}, \quad \text{if} \quad A = -B, \tag{4.15}$$

where k, c, d are arbitrary constants.

Remark 2: 1. If we consider the case $\varphi = 0$, from (4.7) – (4.10), we obtain a BT of the Burgers equation $u_t - 2uu_x - u_{xx} = 0$ as follows:

$$u = \frac{w_x}{w}, \tag{4.16}$$

where $w = w(x,t)$ satisfies

$$w_t = w_{xx}. \tag{4.17}$$

It is well-known that this transformation is the Cole-Hopf transformation [1, 2].

2. From (4.14), the solutions of the Burgers equation $u_t - 2uu_x - u_{xx} = 0$ can be recovered

$$u = -\frac{d}{2}[1+\tanh[-\frac{d}{2}(x-dt)]]. \tag{4.18}$$

5. Concluding Remarks

In this paper, based on the idea of the HB method and using *Mathematica*, we derive an ABT for the generalized 2D KdV-Burgers-type equation and an ABT for the generalized 2D Burgers-type equation by introducing two proper transformations. Based on these BTs, several families of exact solutions for equations (1.3) and (1.4) are obtained. This method can also be applied to other PDEs. In addition, this method is also computerizable, which allows us to perform complicated and tedious algebraic calculations on a computer. But it is necessary to point out that from our solutions (3.7) – (3.9) obtained in this paper we can not deduce the solutions of KP-type equations, because when

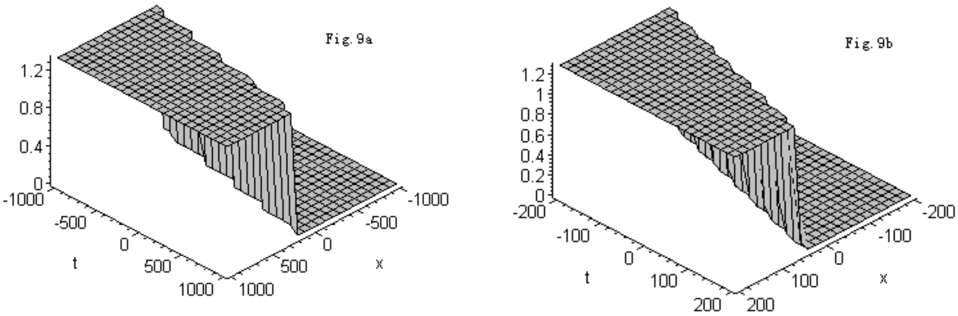

Fig. 9. Plots of solution (3.8) with $p=1$ in Fig. 9a ($p=2$ in Fig. 9b), $\delta=-1$, $b=1$, $r=4$, $a=-1$, $c=1$, $y=2$ and $s=0.8$.

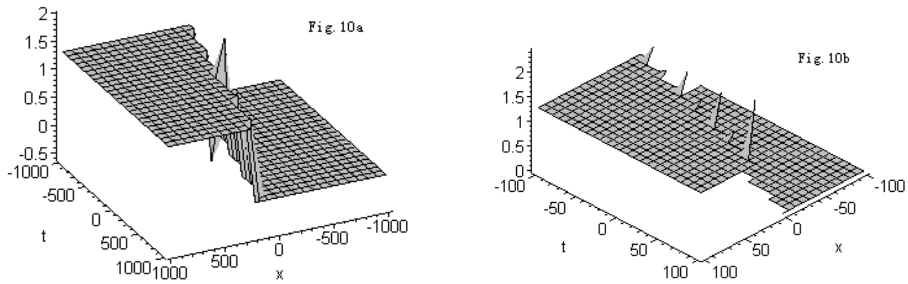

Fig. 10. Plots of solution (3.9) with $p=1$ in Fig. 10a ($p=2$ in Fig. 10b), $\delta=-1$, $b=1$, $r=4$, $a=-1$, $c=1$, $y=2$ and $s=0.8$.

$a=r=0$, the solutions (3.7)–(3.9) change into trivial zero solutions. We shall further study the characteristic and the solutions for these equations by other methods.

Acknowledgements

The authors would like to express their sincere thanks to the referee for his very helpful suggestions and language corrections. This work has been supported by the National Natural Science Foundation of China under the Grant No. 10072013, and the National Key Basic Research Development Project Program under the Grant No. G1998030600.

Appendix

A simple program produced the system of 8 PDEs in Section 2 with some Key *Mathematica* commands.

In[1]: = $u[x\_,y\_,t\_]=v[x,y,t]^{\frac{1}{p}}$
In[2]: = $p[1]$ = Simplify[Simplify[Simplify
$[D[(D[u[x,y,t],t]+a*u[x,y,t]^{p}*D[u[x,y,t],x]$
$+b*u[x,y,t]^{2*p}*D[u[x,y,t],x]$
$+r*D[u[x,y,t],\{x,2\}]+\delta*D[u[x,y,t],\{x,3\}]),x]$
$+s*D[u[x,y,t],y,2]/.(v[x,y,t]^{\frac{1}{p}})^{-1+p}$
$->v[x,y,t]^{1-\frac{1}{p}}/.(v[x,y,t]^{\frac{1}{p}})^{p}$
$->v[x,y,t]]/.(v[x,y,t]^{\frac{1}{p}})^{2*p}$
$->v[x,y,t]^{2}]*p^{4}*v[x,y,t]^{4-\frac{1}{p}}]$
Out[2] = {the left side of Eq. (2.2) in the paper}
In [3]:= $v[x\_,y\_,t\_]=$
$D[f[w[x,y,t]],w[x,y,t]]*D[w[x,y,t],x]+\varphi[x,y,t]$
In [4]:= $p[2]$ = Expand[Simplify[$p[1]$]]
In[5]:= $p[3]$ =
Simplify[Coefficient[$p[2],w^{(1,0,0)}[x,y,t]^{8}$]]
Out[5] = {the left side of Eq. (2.5) in the paper}
In [6]:= $f[w[x,y,t]]=\beta*\mathrm{Log}[w[x,y,t]]$
In [7]:= $f'[w[x,y,t]]=D[f[w[x,y,t]],w[x,y,t]]$
In [8]:= $f''[w[x,y,t]]=D[f'[w[x,y,t]],w[x,y,t]]$
In [9]:= $f^{(3)}[w[x,y,t]]=D[f''[w[x,y,t]],w[x,y,t]]$
In [10]:= $f^{(4)}[w[x,y,t]]=D[f^{(3)}[w[x,y,t]],w[x,y,t]]$
In [11]:= $f^{(5)}[w[x,y,t]]=D[f^{(4)}[w[x,y,t]],w[x,y,t]]$
In [12]:= $p[4]$ =Simplify[$p[3]$]
In [13]:= Solve[$p[4]==0,\beta$]
Out [13]= {the value of β}
In [14]:= $p[5]$ = Simplify[Simplify[CoefficientList
[Expand[$p[2]$], $\frac{1}{w[x,y,t]}$]]]
Out[14] = {the system of 8 PDEs in Section 2}.

[1] M. J. Ablowitz and P. A. Clarkson. Soliton, Nonlinear Evolution Equations and Inverse Scatting, Cambridge University Press, New York 1991.
[2] Gu CH etc. Soliton theory and its application (Zhejiang: Zhejiang Science and Technology Press, 1990).
[3] W. P. Hong and Y. D. Jung, Z. Naturforsch. **54a**, 549 (1990).
[4] E. G. Fan, Phys. Lett. A **265**, 353 (2000).
[5] E. G. Fan, Phys. Lett. A **294**, 26 (2002).
[6] M. L. Wang, Phys.Lett. A **213**, 279 (1996).
[7] M. L. Wang, Y. B. Zhou, and Z. B. Li, Phys. Lett. A **216**, 67 (1996).
[8] M. L. Wang and Y. M. Wang, Phys. Lett. A **287**, 211 (2001).
[9] E. G. Fan and H. Q. Zhang, Phys. Lett. A **246**, 403 (1998).
[10] Z. Y. Yan and H. Q. Zhang, Phys.Lett. A **285**, 355 (2001).
[11] W. X. Ma, J. Phys. A: Math.Gen. **26**, L17 (1993).
[12] E. J. Parkes, J. Phys. A: Math. Gen. **27**, L497 (1994).
[13] Z. B. Li and M. L. Wang, J. Phys. A: Math. Gen. **26**, 6027 (1993).
[14] E. Fan, J. Zhang, and Benny Y. C. Hon, Phys. Lett. A **291**, 376 (2001).
[15] W. G. Zhang, Q. S. Chang, and B. G. Jiang, Chaos, Solitons and Fractals **13**, 311 (2002).
[16] N. Joshi, J. A. Petersen, and L. M. Schubert, Stud. Appl. Math. **105**, 361 (2000).
[17] B. Dey, J. Phys. A: Math. Gen. **19**, L9 (1986).
[18] A. D. Bouard and J. C. Saut, SIAM J. Math. ANAL. **28**, 1064 (1997).
[19] X. P. Wang, M. J. Ablowitz, and H. Segur, Physica D **78**, 241 (1994).
[20] D. E. Pelinovsky, Roger H. J. Grimshav, Physica D **98**, 139 (1996).
[21] Robert L. Pego, P. Smereka, and M. I. Weinstein, Physica D **67**, 45 (1993).
[22] Y. Liu and X. P. Wang, Commun. Math. Phys. **183**, 253 (1997).