Nonlocal symmetry constraints and exact interaction solutions of the (2+1) dimensional modified generalized long dispersive wave equation

Junchao Chen
Shanghai Key Laboratory of Trustworthy Computing,East China Normal University, Shanghai, 200062, People's Republic of China

Department of Mathematics, The University of Texas-Pan American, Edinburg, TX 78541, USA

Yong Chen*
Shanghai Key Laboratory of Trustworthy Computing,East China Normal University, Shanghai, 200062, People's Republic of China
ychen@sei.ecnu.edu.cn

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In this paper, nonlocal symmetry of the (2+1) dimensional modified generalized long dispersive wave system and its applications are investigated. The nonlocal symmetry related to the eigenfunctions in Lax pairs is derived, and infinitely many nonlocal symmetries are obtained. By introducing three potentials, the prolongation is found to localize the given nonlocal symmetry. Various finite- and infinite-dimensional integrable models are constructed by using the nonlocal symmetry constraint method. Moreover, applying the general Lie symmetry approach to the enlarged system, the finite symmetry transformation and similarity reductions are computed to give novel exact interaction solutions. In particular, the explicit soliton-cnoidal wave solution is obtained for the modified generalized long dispersive wave system, and it can be reduced to the two-dark-soliton solution in one special case.

Keywords: (2+1) dimensional modified generalized long dispersive equation; nonlocal symmetry; localization; nonlocal symmetry constraint; exact interaction solutions.

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1. Introduction

The (2+1) dimensional Long Dispersive Wave (LDW) equation takes the form

$$\begin{aligned}&\lambda q_t + q_{xx} - 2qV = 0,\\ &\lambda r_t - r_{xx} + 2rV = 0,\\ &V_y - (qr)_x = 0 \Rightarrow V = \int_{-\infty}^{y} (qr)_x dy'.\end{aligned} \tag{1.1}$$

When $\lambda = i$ and $r = q^*$, it is identical to the so-called the simplest (2+1) dimensional integrable equation proposed by Fokas [8]. In particular, the reduction $x = y$ leads the equation presented by Fokas to the nonlinear Schrödinger equation [43].

*Corresponding author.

The real version of (1.1) has been introduced by Chakravarty, Kent and Newman [5] as a particular reduction of the self-dual Yang-Mills field equation while the complex version of (1.1) has been discussed by Maccari [32] using the asymptotically exact reduction method with the Kadomtsev-Petviashili equation as a starting point. The Painlevé property for the equation (1.1) has been studied by Radha and Lakshmanan [35] for the real version and by Porsezian [34] for the complex case. In Ref. [35], the bilinear method was applied to obtain some soliton and dromion solutions.

More recently, Cerveró and Estévez [6] have shown that Eq.(1.1) is nothing but the modified version of the Generalized Long Dispersive Wave (GLDW) equation which has been thoroughly studied by Boiti, León and Pempinelli [3]. Since a Miura transformation between Eq.(1.1) and the GLDW equation was directly found by using the singularity analysis in Ref. [6], Eq. (1.1) was called as the modified GLDW (MGLDW) equation. In a subsequent paper, Estévez [7] constructed the Darboux transformation of MGLDW equation (1.1) by using Painlevé analysis and the singular manifold method.

Recently, nonlocal symmetry study have attracted a lot of attention and some effective techniques to find nonlocal symmetries have been proposed and developed [1, 2, 4, 9–29, 33, 36–42]. Nonlocal symmetries for partial differential equations were first studied rigorously from a geometric viewpoint by Vinogradov and Krasil'shchik [21]. By introducing the notion of coverings in the category of differential equations [22], they constructed the general theory of nonlocal symmetries and nonlocal conservation laws for differential equations. The Krasil'shchik-Vinogradov theory is usually viewed as the most complete and satisfactory theory of nonlocal symmetries [4, 10, 11, 16–18, 36–38]. In integrable system, one can obtain nonlocal symmetry by inverse recursion operators, the Möbious (conformal) invariant form, Darboux transformation, Bäcklund transformation and so on [2, 4, 9–29, 36–42]. In particular, these nonlocal symmetries were commonly related to Lax pairs (pseudopotential and Bäcklund transformation) [2,4,9–29,36–42]. In this case, Lou [19, 27] and Reyes [10, 11, 16] have presented the method how to generate an infinite number of nonlocal symmetries starting from a parameter-dependent one without using recursion operators.

Moreover, after the derivation of nonlocal symmetry, it is necessary to inquire whether nonlocal symmetries can be transformed to local ones. The general localization approach was also provided in the Krasil'shchik-Vinogradov's work [21]. Starting from nonlocal symmetries related to pseudopotential, Galas [9] showed the first non-trivial instances of more explicit localization process for several integrable equations (KdV, Harry Dym, and AKNS). He further considered how to use these local symmetries to obtain the corresponding finite symmetry transformations and special solutions. Other interesting results on the localization of nonlocal symmetries were given in Refs. [11, 15, 20, 23, 24, 29, 37–42]. Considering the application for nonlocal symmetries of the Camassa-Holm (CH) equation, the associated CH equation and the modified CH Equation, the localization method was re-taken by Reyes in his work [4, 10, 11, 16, 17, 36].

In the process of localization, one need to extend the space of independent variables by adding news auxiliary variables and then obtained a enlarged system embedding the original equation. Searching for the general Lie symmetry of this enlarge system, one can get 'localized' nonlocal symmetries of the augmented system. The full classifications of 'localized' nonlocal symmetries for several integrable equations appeared in [4, 11, 16–18, 42] by computing the corresponding Lie algebra. To use nonlocal symmetries to generate exact solutions is of great interest: the finite symmetry transformation [4,9,15,16,18,20,24,29,37–40,42], which allows one to acquire new solutions from old ones; and similarity reduction [15, 20, 29, 40, 42], in which the group invariant solution is deduced directly.

On the other hand, symmetry constraint method is one of the most powerful tools to give out new integrable models from known ones. By using general symmetry constraints, one only obtain the lower dimensional integrable models from higher ones. In nonlocal symmetries case, the higher dimensional integrable models can also be obtained. For instance, by introducing some inner parameters, Lou *et al.* [28] extended the usual (1+1)-dimensional AKNS system to a (2+1)-dimensional case, which includes the Davey-Stewartson and asymmetric Nizhnik-Novikov-Veselov systems. Other examples (Broer-Kaup system, sine-Gordon) were considered in Ref. [26, 28, 29]. Thus the nonlocal symmetry constraint method is a natural extension of the general one.

In the present paper, we focus on the nonlocal symmetries of the MGLDW system and their applications. A class of nonlocal symmetry related to the eigenfunctions in Lax pairs is derived, and infinitely many nonlocal symmetries are obtained. The prolongation of the new nonlocal symmetries is found after extending the MGLDW system to an auxiliary system with three dependent variables. Then, various finite- and infinite-dimensional integrable models are constructed by means of the nonlocal symmetry constraint method. Moreover, applying the general Lie symmetry approach to the enlarged system, the finite symmetry transformation and similarity reductions are computed to give novel exact solutions of the MGLDW system. It is worthy to mention that the explicit soliton-cnoidal wave solution is found for the MGLDW system, and it can degenerate to the two-dark-soliton solution in one special case.

The organization of this paper is as follows. In Section 2, the nonlocal symmetry is derived for the MGLDW system and infinitely many nonlocal symmetries are obtained. The nonlocal symmetry is extended to be equivalent to the Lie point symmetry of some auxiliary prolonged system. In Section 3, various integrable systems are constructed by means of the nonlocal symmetry constraint method. Section 4 is devoted to finding some new exact solutions by using the finite symmetry transformations and similar reductions of the prolonged system. The last section contains a summary and discussion.

2. Nonlocal symmetry and its localization

2.1. *Lax pair of the MGLDW equation*

In Ref. [7], in order to write down Eq.(1.1) as a system for just one field, one may set $V = -m_x$, then Eq.(1.1) becomes

$$\begin{aligned} &q_t + q_{xx} + 2qm_x = 0, \\ &r_t - r_{xx} - 2rm_x = 0, \\ &m_y + qr = 0, \end{aligned} \tag{2.1}$$

where time is rescaled in the form $t \to \lambda t$.

Furthermore, considering the following transformation

$$q = -\sqrt{m_y}\exp(-\int \frac{n_y}{2m_y}dx), \quad r = \sqrt{m_y}\exp(\int \frac{n_y}{2m_y}dx), \tag{2.2}$$

Eq.(2.1) can be written as one field form

$$m_t - n_x = 0, \tag{2.3}$$

$$m_y^2(n_{yt} - m_{xxxy}) + m_{xy}(n_y^2 - m_{xy}^2) + 2m_y(m_{xy}m_{xxy} - n_y n_{xy}) - 4m_y^3 m_{xx} = 0. \tag{2.4}$$

It has been shown in Ref. [6] that there exist a Mirua transformation between Eqs.(2.3)-(2.4) and the GLDW equation proposed by Boiti *et al.* [3]:

$$\theta_{ty} + (\eta_{xy} + 2\theta\theta_y)_x = 0, \;\; \eta_{ty} + (\theta_{xy} + 2\theta\eta_y)_x = 0. \tag{2.5}$$

Starting from the known Lax pair of the GDLW equation, the lax pairs of the MGLW system (2.1) and one field form (2.3)-(2.4) were constructed directly via the explicit Mirua transformation.

The MGLDW system (2.3)-(2.4) possesses the Lax pair [6, 7]

$$\psi_t - \psi_{xx} - 2m_x\psi = 0, \tag{2.6}$$

$$2m_y\psi_{xy} - (m_{xy} + n_y)\psi_y + 2m_y^2\psi = 0, \tag{2.7}$$

and the adjoint Lax pair

$$\phi_t + \phi_{xx} + 2m_x\phi = 0, \tag{2.8}$$

$$2m_y\phi_{xy} - (m_{xy} - n_y)\phi_y + 2m_y^2\phi = 0. \tag{2.9}$$

2.2. *Nonlocal symmetry*

A symmetry $\sigma \equiv (\sigma^m, \sigma^n)$ of the MGLDW system is defined as a solution of its linearized equations

$$\sigma^m - \sigma^n = 0, \tag{2.10}$$

$$\begin{aligned} &4\sigma^m_{xx}m_y^3 + (12m_{xx}\sigma^m_y + \sigma^m_{xxxy} - \sigma^n_{yt})m_y^2 - \sigma^m_{xy}n_y^2 + (3m_{xy}\sigma^m_{xy} - 2m_{xxy}\sigma^m_y)m_{xy} \\ &\quad +2[m_{xxxy}\sigma^m_y - (m_{xy}\sigma^m_{xy})_x - n_{yt}\sigma^m_y + (n_y\sigma^n_y)_x]m_y + 2(n_{xy}\sigma^m_y - m_{xy}\sigma^n_y)n_y = 0. \end{aligned} \tag{2.11}$$

That is to say, the MGLDW system (2.3)-(2.4) is form invariant under the following transformations

$$m \to m + \varepsilon\sigma^m, \;\; n \to n + \varepsilon\sigma^n,$$

with an infinitesimal parameter ε.

Proposition 1. *The MGLDW system* (2.3)-(2.4) *has a nonlocal symmetry given by*

$$\sigma \equiv (\sigma^m, \sigma^n) = (\psi\phi, \;\; \phi\psi_x - \psi\phi_x), \tag{2.12}$$

where ψ and ϕ satisfy the Lax pair (2.6)-(2.7) *and the adjoint one* (2.8)-(2.9).

Proof. By direct calculation.

Following the method in Ref. [10, 11, 16, 19, 27], one can get an infinite number of nonlocal symmetries for the MGLDW system (2.3)-(2.4). To this end, we consider (2.6) and (2.8) in Lax pairs as

$$\psi_t - \psi_{xx} - 2m_x\psi - \lambda\psi = 0, \tag{2.13}$$

$$\phi_t + \phi_{xx} + 2m_x\phi - \lambda\phi = 0, \tag{2.14}$$

where the eigenfunctions ψ and ϕ in Lax pairs are λ dependent, and the fields m and n are λ independent. Then, the nonlocal symmetry for the MGLDW system (2.3)-(2.4) has the form

$$\sigma^m = \mathrm{e}^{-2\lambda t}\psi\phi, \;\; \sigma^n = \mathrm{e}^{-2\lambda t}(\phi\psi_x - \psi\phi_x). \tag{2.15}$$

Let $\psi = \sum_{k=0}^{\infty} \psi[k]\lambda^k$ and $\phi = \sum_{k=0}^{\infty} \phi[k]\lambda^k$, one has

$$\sum_{k=0}^{\infty} \psi_t[k]\lambda^k - \sum_{k=0}^{\infty} \psi_{xx}[k]\lambda^k - \sum_{k=0}^{\infty} 2m_x\psi[k]\lambda^k - \sum_{k=0}^{\infty} \psi[k]\lambda^{k+1} = 0, \tag{2.16}$$

$$\sum_{k=0}^{\infty} \phi_t[k]\lambda^k + \sum_{k=0}^{\infty} \phi_{xx}[k]\lambda^k + \sum_{k=0}^{\infty} 2m_x\phi[k]\lambda^k - \sum_{k=0}^{\infty} \phi[k]\lambda^{k+1} = 0. \tag{2.17}$$

Equating the coefficients of λ^k of (2.16) and (2.17) yields the following recursion relation

$$\psi_t[0] - \psi_{xx}[0] - 2m_x\psi[0] = 0, \tag{2.18}$$

$$\phi_t[0] + \phi_{xx}[0] + 2m_x\phi[0] = 0, \tag{2.19}$$

$$(\partial_t - \partial_{xx} - 2m_x)\psi[k] = \psi[k-1], \tag{2.20}$$

$$(\partial_t + \partial_{xx} + 2m_x)\phi[k] = \phi[k-1]. \tag{2.21}$$

By setting $L_1 = \partial_t - \partial_{xx} - 2m_x$ and $L_2 = \partial_t + \partial_{xx} + 2m_x$, Eqs. (2.20) and (2.21) can be solved recursively in terms of $\psi[0]$ and $\phi[0]$:

$$\psi[k] = L_1^{-k}\psi[0], \tag{2.22}$$

$$\phi[k] = L_2^{-k}\phi[0], \tag{2.23}$$

where $\psi[0]$ and $\phi[0]$ satisfy (2.18) and (2.19).

Substituting (2.22) and (2.23) into (2.15), we can obtain

$$\sigma^m = \mathrm{e}^{-2\lambda t} \sum_{k=0}^{\infty} \sum_{j=0}^{k} L_1^{-j}\psi[0] \cdot L_2^{j-k}\phi[0]\lambda^k, \tag{2.24}$$

$$\sigma^n = \mathrm{e}^{-2\lambda t} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \left\{ \partial_x(L_1^{-j}\psi[0]) \cdot L_2^{j-k}\phi[0] - L_1^{-j}\psi[0] \cdot \partial_x(L_2^{j-k}\phi[0]) \right\} \lambda^k. \tag{2.25}$$

Further, expanding $\mathrm{e}^{-2\lambda t}$ as a series in λ leads to

$$\sigma^m = \sum_{l=0}^{\infty} \sum_{k=0}^{l} \left[\sum_{j=0}^{k} L_1^{-j}\psi[0] \cdot L_2^{j-k}\phi[0] \right] \cdot \left[\frac{(-2t)^{l-k}}{(l-k)!} \right] \lambda^l, \tag{2.26}$$

$$\sigma^n = \sum_{l=0}^{\infty} \sum_{k=0}^{l} \left[\sum_{j=0}^{k} \left\{ \partial_x(L_1^{-j}\psi[0]) \cdot L_2^{j-k}\phi[0] - L_1^{-j}\psi[0] \cdot \partial_x(L_2^{j-k}\phi[0]) \right\} \right] \cdot \left[\frac{(-2t)^{l-k}}{(l-k)!} \right] \lambda^l. \tag{2.27}$$

Finally, the set of the nonlocal symmetries can be obtained

$$\sigma_l^m = \sum_{k=0}^{l} \left[\sum_{j=0}^{k} L_1^{-j}\psi[0] \cdot L_2^{j-k}\phi[0] \right] \cdot \left[\frac{(-2t)^{l-k}}{(l-k)!} \right], \tag{2.28}$$

$$\sigma_l^n = \sum_{k=0}^{l} \left[\sum_{j=0}^{k} \left\{ \partial_x(L_1^{-j}\psi[0]) \cdot L_2^{j-k}\phi[0] - L_1^{-j}\psi[0] \cdot \partial_x(L_2^{j-k}\phi[0]) \right\} \right] \cdot \left[\frac{(-2t)^{l-k}}{(l-k)!} \right], \tag{2.29}$$

where $l = 0, 1, 2, 3, \cdots$ and $\psi[0], \phi[0]$ satisfy (2.18) and (2.19).

2.3. *Localization of the nonlocal symmetry*

As we know, the general Lie point symmetries can be applied to construct explicit solutions for differential equations, whereas the similar calculations seem to be invalid for nonlocal symmetries. So it is anticipated to turn the nonlocal symmetries into local ones, especially into Lie point symmetries. Following this idea [4, 9–11, 15–17, 20, 23, 24, 29, 36–42], one may extend the original system to a closed prolonged system by introducing some additional dependent variables.

In order to localize the nonlocal symmetry (2.12), we introduce $\psi_1 \equiv \psi_1(x,y,t)$ and $\phi_1 \equiv \phi_1(x,y,t)$ by

$$\psi_x = \psi_1, \;\; \phi_x = \phi_1, \tag{2.30}$$

which leads the symmetry (2.12) to

$$\sigma^m = \psi\phi, \;\; \sigma^n = \phi\psi_1 - \psi\phi_1. \tag{2.31}$$

For the local symmetries of the variables ψ, ϕ, ψ_1 and ϕ_1, we need to introduce another prolonged potential $p \equiv p(x,y,t)$ with

$$p_x = \psi\phi, \;\; p_y = -\frac{\psi_y\phi_y}{m_y}, \;\; p_t = \psi_x\phi - \phi_x\psi, \tag{2.32}$$

which means that the conditions $p_{xt} = p_{tx}$, $p_{yt} = p_{ty}$ and $p_{xy} = p_{yx}$ are satisfied identically. Then the inclusion of ψ_1, ϕ_1 and p yields

$$\sigma^{\psi} = -p\psi, \;\; \sigma^{\phi} = -p\phi, \;\; \sigma^{\psi_1} = -p\psi_1 - \psi^2\phi, \;\; \sigma^{\phi_1} = -p\phi_1 - \phi^2\psi, \;\; \sigma^p = -p^2, \tag{2.33}$$

where $\sigma^{\psi}, \sigma^{\phi}, \sigma^{\psi_1}, \sigma^{\phi_1}$ and σ^p denote the symmetries of $\psi, \phi, \psi_1, \phi_1$ and p, respectively.

Finally, the prolongation is closed after covering dependent variables $m, n, \psi, \phi, \psi_1, \phi_1$ and p for the nonlocal symmetry (2.12) with the vector form

$$\begin{aligned} V = {} & \psi\phi\frac{\partial}{\partial m} + (\phi\psi_1 - \psi\phi_1)\frac{\partial}{\partial n} - p\psi\frac{\partial}{\partial \psi} - p\phi\frac{\partial}{\partial \phi} - (p\psi_1 + \psi^2\phi)\frac{\partial}{\partial \psi_1} \\ & -(p\phi_1 + \phi^2\psi)\frac{\partial}{\partial \phi_1} - p^2\frac{\partial}{\partial p}. \end{aligned} \tag{2.34}$$

It is necessary to point out that if we consider the differential equation for the introduced variable p from above localized procedure, the resulting differential equation is nothing but the Schwartz form of the MGLDW system (2.1):

$$\left(\frac{f_t}{f_x}\right)_{ty} - \frac{1}{2}\left(\frac{f_t^2}{f_x^2}\right)_{xy} + 2\sigma\left(\frac{f_t}{f_x}\right)_{xxy} + \left(\frac{f_{xxx}}{f_x} - \frac{3}{2}\frac{f_{xx}^2}{f_x^2}\right)_{xy} = 0, \;\; \sigma^2 = 1, \tag{2.35}$$

The condition (2.32), that the variable p need to satisfy, coincides with the relation between the singular manifolds and eigenfunctions (Eqs.(3.10)-(3.12) in Ref. [7]). The fact shows us that Darboux transformation is associated with the Möbious (conformal) transformation [20, 25].

3. Integrable models from nonlocal symmetry constraints

For a higher dimensional model, its symmetry can be used to reduce the original model to its lower form. For instance, in Ref. [6], the authors applied the symmetry constraint conditions

$$n_t = m_t = 0,\ \ n_x - n_y = m_x - m_y = 0 \tag{3.1}$$

to reduce the MGLW system (2.3)-(2.4) to the AKNS equation in (1+1) dimensions and to the nonlocal Boussineq equation, respectively.

In order to get more integrable models from the given integrable equation, one has to use the nonlocal symmetry constraints as in [26, 28, 29]. From **Proposition 1**, we can easily obtain a non-trivial nonlocal symmetry

$$\sigma_N \equiv (\sigma_N^m, \sigma_N^n) = \left(\sum_{i=1}^{N} a_i \psi_i \phi_i,\ \sum_{i=1}^{N} a_i (\phi_i \psi_{ix} - \psi_i \phi_{ix}) \right), \tag{3.2}$$

where a_i, $i = 1, 2, ...N$ are constants and $\{\psi_i, \phi_i\}$ are independent solutions the Lax pairs (2.6)-(2.9).

3.1. *Finite-dimensional integrable systems*

(1) Let us consider the nonlocal symmetry constraints:

$$m_x = \sum_{i=1}^{N} a_i \psi_i \phi_i,\ \ n_x = \sum_{i=1}^{N} a_i (\phi_i \psi_{ix} - \psi_i \phi_{ix}). \tag{3.3}$$

Substituting the constraint condition (3.3) to the (2.6) and (2.8), we have the usual 2N-component AKNS system:

$$\psi_{it} - \psi_{ixx} - 2\sum_{j=1}^{N} a_j \psi_j \phi_j \psi_i = 0, \tag{3.4}$$

$$\phi_{it} + \phi_{ixx} + 2\sum_{j=1}^{N} a_j \psi_j \phi_j \phi_i = 0, \tag{3.5}$$

Specifically, when $N = 1$, $\psi_1 = \psi$, $\phi_1 = \phi$, $a_1 = -1$, system (3.3) and (3.4) is the nonlinear Schrödinger equation.

Substituting the constraint condition (3.3) to the (2.7) and (2.9), another part of the Lax pair becomes the (1+1) dimensional 2N-component integrable system:

$$\psi_{ixy} - \frac{\sum_{j=1}^{N} a_j (\psi_j \phi_j - \partial_x^{-1} \psi_j \phi_{jx})_y}{\sum_{j=1}^{N} a_j \partial_x^{-1} (\psi_j \phi_j)_y} \psi_{iy} + \sum_{j=1}^{N} a_j \partial_x^{-1} (\psi_j \phi_j)_y \psi_i = 0, \tag{3.6}$$

$$\phi_{ixy} - \frac{\sum_{j=1}^{N} a_j (\partial_x^{-1} \psi_j \phi_{jx})_y}{\sum_{j=1}^{N} a_j \partial_x^{-1} (\psi_j \phi_j)_y} \phi_{iy} + \sum_{j=1}^{N} a_j \partial_x^{-1} (\psi_j \phi_j)_y \phi_i = 0. \tag{3.7}$$

To obtain the simplest form of (3.6)-(3.7), we set $N = 1$, $\psi_1 = \psi$, $\phi_1 = \phi$, $a_1 = 1$. In this special case, system (3.6)-(3.7) is reduced to

$$\psi_{xy} - \frac{m_{xy}+n_y}{2m_y}\psi_y + m_y\psi = 0, \tag{3.8}$$

$$\phi_{xy} - \frac{m_{xy}-n_y}{2m_y}\phi_y + m_y\phi = 0, \tag{3.9}$$

$$m = \partial_x^{-1}\psi\phi, \;\; n = \partial_x^{-1}(\phi\psi_x - \psi\phi_x) \tag{3.10}$$

(2) In this case, we use

$$m_y = \sum_{i=1}^{N} a_i\psi_i\phi_i, \;\; n_y = \sum_{i=1}^{N} a_i(\phi_i\psi_{ix} - \psi_i\phi_{ix}), \tag{3.11}$$

as the nonlocal symmetry constraint condition of the MGLW equation. Substituting (3.11) to (2.6) and (2.8) yields the generalized 2N-component (2+1) dimensional AKNS extension:

$$\psi_{it} - \psi_{ixx} - 2\sum_{i=1}^{N} a_j\partial_y^{-1}(\psi_j\phi_j)_x\psi_i = 0, \tag{3.12}$$

$$\phi_{it} + \phi_{ixx} + 2\sum_{i=1}^{N} a_j\partial_y^{-1}(\psi_j\phi_j)_x\phi_i = 0. \tag{3.13}$$

When taking $N = 1$, $\psi_1 = \psi$, $\phi_1 = \phi$, $a_1 = -\frac{1}{2}$, one can get the asymmetric DS (ADS) system [28]:

$$\psi_t - \psi_{xx} + \psi\partial_y^{-1}(\psi\phi)_x = 0, \tag{3.14}$$

$$\phi_t + \phi_{xx} - \phi\partial_y^{-1}(\psi\phi)_x = 0, \tag{3.15}$$

Substituting the nonlocal symmetry constraints (3.11) to (2.6) and (2.8), another part of the Lax pair becomes the (1+1) dimensional 2N-component integrable system:

$$\psi_{ixy} - \frac{\sum_{j=1}^{N} a_j\psi_{jx}\phi_j}{\sum_{j=1}^{N} a_j\psi_j\phi_j}\psi_{iy} + \psi_i\sum_{j=1}^{N} a_j\psi_j\phi_j = 0, \tag{3.16}$$

$$\phi_{ixy} + \frac{\sum_{j=1}^{N} a_j\psi_j\phi_{jx}}{\sum_{j=1}^{N} a_j\psi_j\phi_j}\phi_{iy} + \phi_i\sum_{j=1}^{N} a_j\psi_j\phi_j = 0. \tag{3.17}$$

The simplest form of (3.16)-(3.17) is

$$\left(\frac{\psi_y}{\psi}\right)_x + \psi\phi = 0, \;\; \left(\frac{\phi_y}{\phi}\right)_x + \phi\psi = 0, \tag{3.18}$$

for $N = 1$, $\psi_1 = \psi$, $\phi_1 = \phi$ and $a_1 = 1$. The equation (3.18) becomes a coupled Liouville equation [30]:

$$H_{xy} + \exp(H+K) = 0, \;\; K_{xy} + \exp(H+K) = 0, \tag{3.19}$$

under the transformations $\psi = \exp(H)$ and $\phi = \exp(K)$.

3.2. *Infinite-dimensional integrable systems*

From the known nonlocal symmetries, one can also obtain some higher dimensional integrable systems. We consider the fields $\{m,n\}$ in the MGLW equation not only the functions of the explicit $\{x,y,t\}$ but also the functions of the inner space variables $\{z,z_1,z_2,\ldots\}$. Therefore, some special types of integrable models in any dimensional case can be derived.

The MGLW system (2.3)-(2.4) is invariant under the inner parameter (z) translation. So

$$m_z=\sum_{i=1}^{N}a_i\psi_i\phi_i,\ \ n_z=\sum_{i=1}^{N}a_i(\phi_i\psi_{ix}-\psi_i\phi_{ix}), \tag{3.20}$$

can be viewed as a new symmetry constraint condition. Acting (3.20) on the parts of Lax pairs (2.6) and (2.8) yields the integrable models equivalent to (3.12)-(3.13) only with the variable z instead of y. Then we can extend the generalized 2N-component AKNS system to the higher dimensional case.

Similar to the finite dimensional case, considering (3.20) with the other parts of Lax pairs (2.7) and (2.9) results in the following 2N-component integrable models:

$$\psi_{ixy}-\frac{\sum_{j=1}^{N}a_j\partial_z^{-1}(\psi_{jx}\phi_j)_y}{\sum_{j=1}^{N}a_j\partial_z^{-1}(\psi_j\phi_j)_y}\psi_{iy}+\sum_{j=1}^{N}a_j\partial_z^{-1}(\psi_j\phi_j)_y\psi_i=0, \tag{3.21}$$

$$\phi_{ixy}+\frac{\sum_{j=1}^{N}a_j\partial_z^{-1}(\psi_j\phi_{jx})_y}{\sum_{j=1}^{N}a_j\partial_z^{-1}(\psi_j\phi_j)_y}\phi_{iy}+\sum_{j=1}^{N}a_j\partial_z^{-1}(\psi_j\phi_j)_y\phi_i=0. \tag{3.22}$$

When $N=M=1$, $\psi_1=\psi$, $\phi_1=\phi$ and $a_1=1$, equation system (3.21)-(3.22) is reduced to the simplest form:

$$\psi_{xy}\partial_z^{-1}(\psi\phi)_y-\partial_z^{-1}(\psi_x\phi)_y\psi_y+[\partial_z^{-1}(\psi\phi)_y]^2\psi=0, \tag{3.23}$$

$$\phi_{xy}\partial_z^{-1}(\psi\phi)_y+\partial_z^{-1}(\psi\phi_x)_y\phi_y+[\partial_z^{-1}(\psi\phi)_y]^2\phi=0. \tag{3.24}$$

Here, we simply use the inner parameter translation symmetry (3.20) as the constraint condition to get the higher dimensional integrable model (3.21)-(3.22). Compared with the lower dimensional integrable models, we believe that the higher dimensional one also have many nice integrable properties but its completely integrability need to study in the further work.

4. Exact solutions from nonlocal symmetry

After making the nonlocal symmetry (2.12) be equivalent to Lie point symmetry (2.34) of the related prolonged system successfully, the exact solutions can be constructed by Lie group theory in two aspects.

4.1. *Finite symmetry transformation*

For the obtained local symmetry in (2.34), it is natural to seek the corresponding finite transformation [4, 9, 15, 16, 18, 20, 24, 29, 37–40, 42], . By solving the initial value problem,

$$\begin{aligned}
&\frac{d\hat{m}}{d\varepsilon}=\hat{\psi}\hat{\phi},\ \frac{d\hat{n}}{d\varepsilon}=\hat{\phi}\hat{\psi}_1-\hat{\psi}\hat{\phi}_1,\ \frac{d\hat{\psi}}{d\varepsilon}=-\hat{p}\hat{\psi},\ \frac{d\hat{\phi}}{d\varepsilon}=-\hat{p}\hat{\phi},\\
&\frac{d\hat{\psi}_1}{d\varepsilon}=-\hat{p}\hat{\psi}_1-\hat{\psi}^2\hat{\phi},\ \frac{d\hat{\phi}_1}{d\varepsilon}=-\hat{p}\hat{\phi}_1-\hat{\phi}^2\hat{\psi},\ \frac{d\hat{p}}{d\varepsilon}=-\hat{p}^2,\\
&\hat{m}|_{\varepsilon=0}=m,\ \hat{n}|_{\varepsilon=0}=n,\ \hat{\psi}|_{\varepsilon=0}=\psi,\ \hat{\phi}|_{\varepsilon=0}=\phi,\ \hat{\psi}_1|_{\varepsilon=0}=\psi_1,\ \hat{\phi}_1|_{\varepsilon=0}=\phi_1,\ \hat{p}|_{\varepsilon=0}=p,
\end{aligned}\tag{4.1}$$

one can arrive at the symmetry group transformation theorem as follows:

Theorem 1 If $\{m,n,\psi,\phi,\psi_1,\phi_1,p\}$ is the solution of the enlarged system (2.3)-(2.9), (2.30) and (2.32), so is $\{\hat{m},\hat{n},\hat{\psi},\hat{\phi},\hat{\psi}_1,\hat{\phi}_1,\hat{p}\}$, where

$$\begin{aligned}
&\hat{m}=m+\frac{\varepsilon\psi\phi}{1+\varepsilon p},\ \hat{n}=n+\frac{\varepsilon(\psi_1\phi-\psi\phi_1)}{1+\varepsilon p},\ \hat{\psi}=\frac{\psi}{1+\varepsilon p},\ \hat{\phi}=\frac{\phi}{1+\varepsilon p},\\
&\hat{\psi}_1=\frac{\psi_1}{1+\varepsilon p}-\frac{\varepsilon\phi\psi^2}{(1+\varepsilon p)^2},\ \hat{\phi}_1=\frac{\phi_1}{1+\varepsilon p}-\frac{\varepsilon\psi\phi^2}{(1+\varepsilon p)^2},\ \hat{p}=\frac{p}{1+\varepsilon p},
\end{aligned}\tag{4.2}$$

with ε is an arbitrary group parameter.

For the original MGLDW system (2.3)-(2.4), the finite symmetry transformation can generate a new solution from old one. So we call the transformation (4.2) in **Theorem 1** as the Darboux-like transformation, which is distinct from the Darboux transformation provided in Ref. [7]. Actually, the transformation (4.2) is equivalent to the so-called Levi transformation [31].

As an example, considering the trivial solution $m=c_1y$ and $n=c_2$ of the MGLDW system (2.3)-(2.4), we can obtain the special solutions for the introduced dependent variables:

$$\begin{aligned}
&\psi=\exp(k_1x-\frac{c_1}{k_1}y+k_1^2t+c_{10}),\ \phi=\exp(k_2x-\frac{c_1}{k_2}y-k_2^2t+c_{20}),\\
&\psi_1=k_1\exp(k_1x-\frac{c_1}{k_1}y+k_1^2t+c_{10}),\ \phi_2=k_2\exp(k_2x-\frac{c_1}{k_2}y-k_2^2t+c_{20}),\\
&p=\frac{1}{k_1+k_2}\exp[(k_1+k_2)x-\frac{c_1(k_1+k_2)}{k_1k_2}y+(k_1^2-k_2^2)t+c_{10}+c_{20}].
\end{aligned}\tag{4.3}$$

Substituting (4.3) into (4.2) yields the non-trivial solution of the MGLDW system (2.3)-(2.4):

$$m=c_1y+\frac{\varepsilon(k_1+k_2)\exp(\xi)}{k_1+k_2+\varepsilon\exp(\xi)},\ n=c_2+\frac{\varepsilon(k_1^2-k_2^2)\exp(\xi)}{k_1+k_2+\varepsilon\exp(\xi)},\tag{4.4}$$

with $\xi=(k_1+k_2)x-\frac{c_1(k_1+k_2)}{k_1k_2}y+(k_1^2-k_2^2)t+c_{10}+c_{20}$.

4.2. *Similarity reductions*

One of the main purposes for calculating symmetries of a differential equation is to use them for obtaining symmetry reductions and finding exact solutions. In this subsection, we will employ the classical Lie point symmetry method to study the whole prolonged system (2.3)-(2.9), (2.30) and (2.32) instead of the MGLDW equations (2.3)-(2.4).

Accordingly, we consider the one-parameter Lie group of infinitesimal transformation as follows:

$$\begin{aligned}&x \to x+\varepsilon X,\ \ y \to y+\varepsilon Y,\ \ t \to t+\varepsilon T,\ \ m \to m+\varepsilon M,\ \ n \to n+\varepsilon N,\\&\psi \to \psi+\varepsilon\Psi,\ \ \phi \to \phi+\varepsilon\Phi,\ \ \psi_1 \to \psi_1+\varepsilon\Psi_1,\ \ \phi_1 \to \phi_1+\varepsilon\Phi_1,\ \ p \to p+\varepsilon P,\end{aligned} \tag{4.5}$$

with a small parameter ε. The vector field associated with the above group of transformations can be written as

$$V = X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+T\frac{\partial}{\partial t}+M\frac{\partial}{\partial m}+N\frac{\partial}{\partial n}+\psi\frac{\partial}{\partial \psi}+\phi\frac{\partial}{\partial \phi}+\psi_1\frac{\partial}{\partial \psi_1}+\phi_1\frac{\partial}{\partial \phi_1}+p\frac{\partial}{\partial p}. \tag{4.6}$$

Then the invariance of system (2.3)-(2.9), (2.30) and (2.32) under transformation (4.5) leads to the expressions for the functions:

$$\begin{aligned}
&X=\frac{1}{2}f_{1t}x+f_2,\ \ Y=g,\ \ T=f_1,\\
&M=-\frac{1}{2}f_{1t}m-\frac{1}{48}f_{1ttt}x^3-\frac{1}{8}f_{2tt}x^2+f_{3t}x+f_4+c_1\psi\phi,\\
&N=f_{1t}n-\frac{1}{2}(f_{1tt}x+2f_{2t})m-\frac{1}{192}f_{1tttt}x^4-\frac{1}{24}f_{2ttt}x^3+\frac{1}{2}f_{3tt}x^2+f_{4t}x+f_5+c_1(\psi_1\phi-\phi_1\psi),\\
&\Psi=-\Big(\frac{1}{8}f_{1tt}x^2+\frac{1}{2}f_{2t}x+\frac{1}{4}f_{1t}-2f_3-c_2\Big)\psi-c_1p\psi,\\
&\Phi=\Big(\frac{1}{8}f_{1tt}x^2+\frac{1}{2}f_{2t}x-\frac{1}{4}f_{1t}-2f_3+c_3\Big)\phi-c_1p\phi,\\
&\Psi_1=-\Big(\frac{1}{8}f_{1tt}x^2+\frac{1}{2}f_{2t}x+\frac{3}{4}f_{1t}-2f_3-c_2\Big)\psi_1-\Big(\frac{1}{4}f_{1tt}x+\frac{1}{2}f_{2t}\Big)\psi-c_1(p\psi_1+\phi\psi^2),\\
&\Phi_1=\Big(\frac{1}{8}f_{1tt}x^2+\frac{1}{2}f_{2t}x-\frac{3}{4}f_{1t}-2f_3+c_3\Big)\phi_1+\Big(\frac{1}{4}f_{1tt}x+\frac{1}{2}f_{2t}\Big)\phi-c_1(p\phi_1+\psi\phi^2),\\
&P=-c_1p^2+(c_2+c_3)p+c_4,
\end{aligned} \tag{4.7}$$

where $g \equiv g(y)$ is an arbitrary function of y, $f_1 \equiv f_1(t)$, $f_2 \equiv f_2(t)$, $f_3 \equiv f_3(t)$, $f_4 \equiv f_4(t)$ and $f_5 \equiv f_5(t)$ are arbitrary functions of t and c_1, c_2, c_3 and c_4 are arbitrary constants. Especially, when $g = f_1 = f_2 = f_3 = f_4 = f_5 = c_2 = c_3 = c_4 = 0$, the obtained symmetry is just Eq.(2.34), and when $c_1 = c_2 = c_3 = c_4 = 0$, the related symmetry is only the general Lie point symmetry of the original MGLDW equations (2.3)-(2.4).

From (4.7), the Lie algebra of infinitesimal symmetries is spanned by the four vector fields

$$\begin{aligned}
V_1 &= \psi\phi\frac{\partial}{\partial m}+(\phi\psi_1-\psi\phi_1)\frac{\partial}{\partial n}-p\psi\frac{\partial}{\partial \psi}-p\phi\frac{\partial}{\partial \phi}-(p\psi_1+\psi^2\phi)\frac{\partial}{\partial \psi_1}\\
&\quad-(p\phi_1+\phi^2\psi)\frac{\partial}{\partial \phi_1}-p^2\frac{\partial}{\partial p},\\
V_2 &= \psi\frac{\partial}{\partial \psi}+\psi_1\frac{\partial}{\partial \psi_1}+p\frac{\partial}{\partial p},\\
V_3 &= \phi\frac{\partial}{\partial \phi}+\phi_1\frac{\partial}{\partial \phi_1}+p\frac{\partial}{\partial p},\\
V_4 &= \frac{\partial}{\partial p},
\end{aligned} \tag{4.8}$$

and the infinite-dimensional subalgebra

$$
\begin{aligned}
V_5(f_1) &= (-\frac{1}{2}f_{1t}m - \frac{1}{48}f_{1ttt}x^3)\frac{\partial}{\partial m} + (f_{1t}n - \frac{1}{2}f_{1tt}xm - \frac{1}{192}f_{1tttt}x^4)\frac{\partial}{\partial n} \\
&\quad -(\frac{1}{8}f_{1tt}x^2 + \frac{1}{4}f_{1t})\psi\frac{\partial}{\partial \psi} + (\frac{1}{8}f_{1tt}x^2 - \frac{1}{4}f_{1t})\phi\frac{\partial}{\partial \phi} \\
&\quad -[(\frac{1}{8}f_{1tt}x^2 + \frac{3}{4}f_{1t})\psi_1 + \frac{1}{4}f_{1tt}x\psi]\frac{\partial}{\partial \psi_1} + [(\frac{1}{8}f_{1tt}x^2 - \frac{3}{4}f_{1t})\phi_1 + \frac{1}{4}f_{1tt}x\phi]\frac{\partial}{\partial \phi_1}, \\
V_6(f_2) &= -\frac{1}{8}f_{2tt}x^2\frac{\partial}{\partial m} - (f_{2t}m + \frac{1}{24}f_{2ttt}x^3)\frac{\partial}{\partial n} - \frac{1}{2}f_{2t}x\psi\frac{\partial}{\partial \psi} + \frac{1}{2}f_{2t}x\phi\frac{\partial}{\partial \phi} \\
&\quad -(\frac{1}{2}f_{2t}x\psi_1 + \frac{1}{2}f_{2t}\psi)\frac{\partial}{\partial \psi_1} + (\frac{1}{2}f_{2t}x\phi_1 + \frac{1}{2}f_{2t}\phi)\frac{\partial}{\partial \phi_1}, \\
V_7(f_3) &= f_{3t}x\frac{\partial}{\partial m} + \frac{1}{2}f_{3tt}x^2\frac{\partial}{\partial n} + 2f_3\psi\frac{\partial}{\partial \psi} - 2f_3\phi\frac{\partial}{\partial \phi} + 2f_3\psi_1\frac{\partial}{\partial \psi_1} - 2f_3\phi_1\frac{\partial}{\partial \phi_1}, \\
V_8(f_4) &= f_4\frac{\partial}{\partial m} + f_{4t}x\frac{\partial}{\partial n}, \\
V_9(f_5) &= f_5\frac{\partial}{\partial n}, \\
V_{10}(g) &= g\frac{\partial}{\partial y}.
\end{aligned}
\tag{4.9}
$$

The commutation relations between these vector fields is given by the following table, the entry in row i and column j representing $[V_i, V_j]$:

$[V_i,V_j]$	V_1	V_2	V_3	V_4	$V_5(f_1)$	$V_6(f_2)$	$V_7(f_3)$	$V_8(f_4)$	$V_9(f_5)$	$V_{10}(g)$
V_1	0	$-V_1$	$-V_1$	V_2+V_3	0	0	0	0	0	0
V_2		0	0	$-V_4$	0	0	0	0	0	0
V_3			0	$-V_4$	0	0	0	0	0	0
V_4				0	0	0	0	0	0	0
$V_5(\tilde{f}_1)$					$V_5(\hat{f}_1)$	$V_6(\hat{f}_2)$	$V_7(\hat{f}_3)$	$V_8(\hat{f}_4)$	$V_9(\hat{f}_5)$	0
$V_6(\tilde{f}_2)$						$V_7(\hat{f}_6)$	$V_8(\hat{f}_7)$	$V_9(\hat{f}_8)$	0	0
$V_7(\tilde{f}_3)$							0	0	0	0
$V_8(\tilde{f}_4)$								0	0	0
$V_9(\tilde{f}_5)$									0	0
$V_{10}(\tilde{g})$										$V_{10}(\hat{g})$

where

$$
\hat{f}_1 = f_1\tilde{f}_{1t} - \tilde{f}_1 f_{1t}, \;\; \hat{f}_2 = \tilde{f}_1 f_{2t} - \frac{1}{2}f_2\tilde{f}_{1t}, \;\; \hat{f}_3 = \tilde{f}_1 f_{3t}, \;\; \hat{f}_4 = \tilde{f}_1 f_{4t} + \frac{1}{2}f_4\tilde{f}_{1t}, \;\; \hat{f}_5 = (\tilde{f}_1 f_5)_t,
$$

$$
\hat{f}_6 = \frac{1}{4}(\tilde{f}_2 f_{2t} - f_2\tilde{f}_{2t}), \;\; \hat{f}_7 = \tilde{f}_2 f_{3t}, \;\; \hat{f}_8 = (\tilde{f}_2 f_4)_t, \;\; \hat{g} = g\tilde{g}_y - \tilde{g}g_y.
$$

To find symmetry reductions, i.e., to find group invariant solutions, one need to solve the characteristic equations:

$$
\frac{dx}{X} = \frac{dy}{Y} = \frac{dt}{T} = \frac{dm}{M} = \frac{dn}{N} = \frac{d\psi}{\Psi} = \frac{d\phi}{\Phi} = \frac{d\psi_1}{\Psi_1} = \frac{d\phi_1}{\Phi_1} = \frac{dp}{P}. \tag{4.10}
$$

In the following, only two nontrivial cases are considered in detail to obtain several novel exact solutions.

Case 1 Soliton-Cnoidal Waves Solution

For simplicity, we let $f_1 = f_3 = f_4 = f_5 = 0, f_2 = 1$ and $g = k$, and redefine $\Delta^2 = (c_2+c_3)^2 + 4c_1c_4$ and $\Omega_0 = c_2 - c_3$. By solving (4.10), the group invariant solutions have the forms

$$\begin{aligned}
m &= M + \frac{2c_1}{\Delta}\Psi_0\Phi_0\tanh[\frac{1}{2}\Delta(x+P)],\\
n &= N + \frac{2c_1}{\Delta}[\exp(\frac{\Omega_0}{2}P)\Phi_0\Psi_1 - \exp(-\frac{\Omega_0}{2}P)\Psi_0\Phi_1]\tanh[\frac{1}{2}\Delta(x+P)],\\
\psi &= \exp(\frac{\Omega_0}{2}x)\Psi_0\mathrm{sech}[\frac{1}{2}\Delta(x+P)],\\
\phi &= \exp(-\frac{\Omega_0}{2}x)\Phi_0\mathrm{sech}[\frac{1}{2}\Delta(x+P)],\\
\psi_1 &= \frac{2c_1}{\Delta}\exp[\frac{\Omega_0}{2}x - \frac{1}{2}\Delta(x+P)]\Psi_0^2\Phi_0\mathrm{sech}^2[\frac{1}{2}\Delta(x+P)]\\
&\quad +\exp[\frac{\Omega_0}{2}(x+P)]\Psi_1\mathrm{sech}[\frac{1}{2}\Delta(x+P)],\\
\phi_1 &= \frac{2c_1}{\Delta}\exp[-\frac{\Omega_0}{2}x - \frac{1}{2}\Delta(x+P)]\Phi_0^2\Psi_0\mathrm{sech}^2[\frac{1}{2}\Delta(x+P)]\\
&\quad +\exp[-\frac{\Omega_0}{2}(x+P)]\Phi_1\mathrm{sech}[\frac{1}{2}\Delta(x+P)],\\
p &= \frac{c_2+c_3}{2c_1} + \frac{\Delta}{2c_1}\tanh[\frac{1}{2}\Delta(x+P)],
\end{aligned} \tag{4.11}$$

where $M \equiv M(\xi,\eta), N \equiv N(\xi,\eta), \Psi_0 \equiv \Psi_0(\xi,\eta), \Phi_0 \equiv \Phi_0(\xi,\eta), \Psi_1 \equiv \Psi_1(\xi,\eta), \Phi_1 \equiv \Phi_1(\xi,\eta)$, $P \equiv P(\xi,\eta)$ and the similarity variables $\xi = y - kx$ and $\eta = t$.

Substituting Eq.(4.11) into the enlarged system (2.3)-(2.9), (2.30) and (2.32) yields

$$\begin{aligned}
\Psi_0 &= \frac{\Delta}{2}\sqrt{\frac{1-kP_\xi}{c_1}},\ \ \Phi_0 = \frac{\Delta}{2}\sqrt{\frac{1-kP_\xi}{c_1}},\\
\Psi_1 &= \frac{\Delta}{4c_1}\sqrt{\frac{c_1}{1-kP_\xi}}\exp(-\frac{\Omega_0}{2}P)[k^2P_{\xi\xi} - \Delta(1-kP_\xi)^2 - \Omega_0(1-kP_\xi)],\\
\Phi_1 &= \frac{\Delta}{4c_1}\sqrt{\frac{c_1}{1-kP_\xi}}\exp(\frac{\Omega_0}{2}P)[k^2P_{\xi\xi} - \Delta(1-kP_\xi)^2 - \Omega_0(1-kP_\xi)],\\
M_\xi &= \frac{\Omega_0^2}{8k} + \frac{(1-kP_\xi)^2\Delta^2}{8k} + \frac{k^2}{8(1-kP_\xi)^2}(2kP_{\xi\xi\xi}P_\xi - kP_{\xi\xi}^2 - 2P_{\xi\xi\xi}),\\
M_\eta &= -kN_\xi = \frac{k\Omega_0\Delta^2}{4}(1-kP_\xi)P_\xi + \frac{\Omega_0k^3P_{\xi\xi}^2}{4(1-kP_\xi)P_\xi},
\end{aligned} \tag{4.12}$$

with

$$P_\eta - \Omega_0(1-kP_\xi) = 0, \tag{4.13}$$

$$2k^3P_\xi(kP_\xi - 1)P_{\xi\xi\xi} - k^3(3kP_\xi - 2)P_{\xi\xi}^2 + (1-kP_\xi)^2P_\xi[\Omega_0^2 - \Delta^2(1-k^2P_\xi^2)] = 0. \tag{4.14}$$

From Eqs.(4.13) and (4.14), it is straightforward to derive

$$P = \frac{\xi}{k} + \int P_2(\chi)d\chi,\ \ \chi = \eta - \frac{\xi}{k\Omega_0}, \tag{4.15}$$

where $P_2 \equiv P_2(\chi)$ satisfies

$$\left(\frac{dP_2}{d\chi}\right)^2 = a_1P_2 + a_2P_2^2 + a_3P_2^3 + \Delta^2P_2^4, \tag{4.16}$$

with

$$a_1 = a_3\Omega_0^2 + \Delta^2\Omega_0^3 - \Omega_0^5, \;\; a_2 = -2a_3\Omega_0 + 2\Delta^2\Omega_0^2 + \Omega_0^4. \tag{4.17}$$

We take one solution of Eq.(4.16) given by Jacobi elliptic function

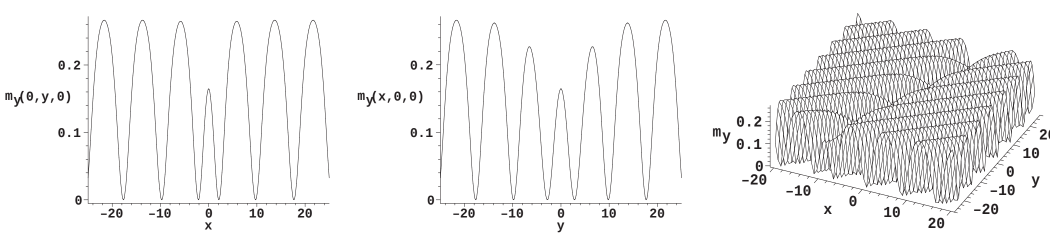
$$P_2 = \frac{1}{b_0 + b_2\mathrm{sn}^2(b_2\chi, l)}, \tag{4.18}$$

which leads to the soliton-cnoidal waves solution of the MGLDW system (2.3)-(2.4):

$$\begin{aligned} m_y = {} & \frac{\Omega_0^2}{8k} + \frac{\Delta^2\mathrm{sech}^2[\frac{\Delta}{2k}(y+kX)]}{4k\Omega_0[b_0+b_1\mathrm{sn}^2(b_2Y,l)]} + \frac{2\Delta^2\tanh^2[\frac{\Delta}{2k}(y+kX)] - \Delta^2}{8k\Omega_0^2[b_0+b_1\mathrm{sn}^2(b_2Y,l)]^2} \\ & + \frac{b_1b_2\Delta\mathrm{sn}(b_2Y,l)\mathrm{cn}(b_2Y,l)\mathrm{dn}(b_2Y,l)\tanh[\frac{\Delta}{2k}(y+kX)]}{k\Omega_0^2[b_0+b_1\mathrm{sn}^2(b_2Y,l)]^2} \\ & - \frac{4b_1b_2^2[(b_1+3b_0l^2+b_1l^2)\mathrm{sn}^4(b_2Y,l) - 2(b_1+b_0l^2+b_0)\mathrm{sn}^2(b_2Y,l)+b_0]}{8k\Omega_0^2[b_0+b_1\mathrm{sn}^2(b_2Y,l)]^2}, \end{aligned} \tag{4.19}$$

where

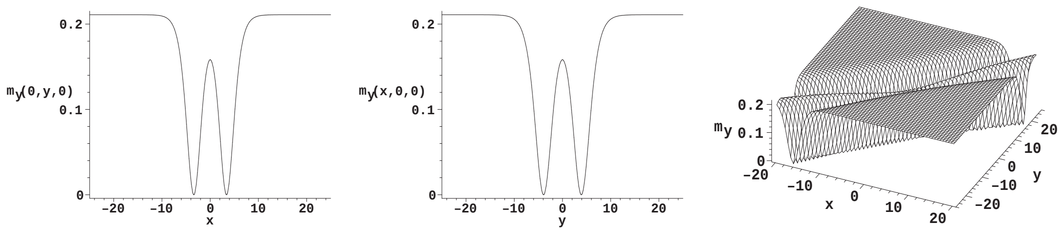
$$X = \int_{T_0}^{Y} \frac{1}{b_0 + b_2\mathrm{sn}^2(b_2T,l)}dT, \;\; Y = \frac{kx-y}{k\Omega_0} + t,$$
$$\mu = b_1(1+l^2)\Omega_0 + l^2, \;\; b_0 = \Omega_0^{-1}, \;\; b_2 = \frac{\Omega_0^2\sqrt{\mu b_1\Omega_0}}{2\mu}, \;\; c_4 = -\frac{c_2^2+c_3^2}{2c_1} - \frac{b_1^2\Omega_0^4}{4c_1\mu}.$$



(a) (b) (c)

Fig. 1. The wave propagation plots of the MGLDW system given by Eq.(4.19), with the parameters $T_0 = 0, c_2 = b_1 = k = 1, c_3 = 1.5$ and $l = 0.9$. (a) The wave propagation pattern of the wave along x axis at $t = 0$ and $y = 0$; (b) The wave propagation pattern of the wave along t axis at $t = 0$ and $x = 0$; (c) The two-dimensional perspective view of the corresponding solution .

The solitons and the cnoidal periodic waves are two types of typical excitations in nonlinear systems. However, it is quite difficult to find the interaction solutions between solitons and cnoidal periodic waves, especially the explicit exact expression. From Eq.(4.19), one can see that the final form of the exact solution which contain the Jacobi elliptic functions and the hyperbolic functions,



(a) (b) (c)

Fig. 2. The wave propagation plots of the MGLDW system given by Eq.(4.19), with the parameters $T_0 = 0, c_2 = b_1 = k = 1, c_3 = 7/4$ and $l = 1$. (a) The wave propagation pattern of the wave along x axis at $t = 0$ and $y = 0$; (b) The wave propagation pattern of the wave along t axis at $t = 0$ and $x = 0$; (c) The two-dimensional perspective view of the corresponding solution .

represents the interaction between the soliton and the cnoidal periodic wave. The simulation of soliton-cnoidal waves solution (4.19) is illustrated in Fig.1 and Fig.2 at two different choices of the arbitrary parameters. In Fig.1, when the value of the Jacobi elliptic function's module $l \neq 1$, the dark soliton propagates on the cnoidal periodic wave background. When the value of module $l = 1$, the Jacobi elliptic functions are reduced to the hyperbolic functions, thus the feature of the two-dark-soliton is exhibited distinctly in Fig.2.

Case 2 Non-travelling Wave Solution

As an another example, we let $f_3 = f_4 = f_5 = 0, f_1 = k_1, f_2 = k_2$ and $g = h_y^{-1}$ with $h \equiv h(y)$, and redefine $\Delta^2 = (c_2 + c_3)^2 + 4c_1c_4$ and $\Omega_0 = c_2 - c_3$. By solving (4.10), the group invariant solutions have the forms

$$
\begin{aligned}
m &= M + \frac{2c_1}{\Delta}\Psi_0\Phi_0 \tanh[\frac{1}{2}\Delta(h+P)],\\
n &= N + \frac{2c_1}{\Delta}[\Phi_0\Psi_1 - \Psi_0\Phi_1]\tanh[\frac{1}{2}\Delta(h+P)],\\
\psi &= \exp[\frac{\Omega_0}{2}(h+P)]\Psi_0 \mathrm{sech}[\frac{1}{2}\Delta(h+P)],\\
\phi &= \exp[-\frac{\Omega_0}{2}(h+P)]\Phi_0 \mathrm{sech}[\frac{1}{2}\Delta(h+P)],\\
\psi_1 &= \frac{2c_1}{\Delta}\exp[\frac{\Omega_0-\Delta}{2}(h+P)]\Psi_0^2\Phi_0 \mathrm{sech}^2[\frac{1}{2}\Delta(h+P)]\\
&\quad + \exp[\frac{\Omega_0}{2}(h+P)]\Psi_1 \mathrm{sech}[\frac{1}{2}\Delta(h+P)],\\
\phi_1 &= \frac{2c_1}{\Delta}\exp[-\frac{\Omega_0+\Delta}{2}(h+P)]\Phi_0^2\Psi_0 \mathrm{sech}^2[\frac{1}{2}\Delta(h+P)]\\
&\quad + \exp[-\frac{\Omega_0}{2}(h+P)]\Phi_1 \mathrm{sech}[\frac{1}{2}\Delta(h+P)],\\
p &= \frac{c_2+c_3}{2c_1} + \frac{\Delta}{2c_1}\tanh[\frac{1}{2}\Delta(h+P)],
\end{aligned}
\tag{4.20}
$$

where $M \equiv M(\xi,\eta), N \equiv N(\xi,\eta), \Psi_0 \equiv \Psi_0(\xi,\eta), \Phi_0 \equiv \Phi_0(\xi,\eta), \Psi_1 \equiv \Psi_1(\xi,\eta), \Phi_1 \equiv \Phi_1(\xi,\eta)$, $P \equiv P(\xi,\eta)$ and the similarity variables $\xi = x - k_2h$ and $\eta = t - k_1h$.

Substituting Eq.(4.20) into the enlarged system (2.3)-(2.9), (2.30) and (2.32) yields

$$P = \omega\xi + \omega^2\Omega_0\eta,\ \ \Psi_0 = \Phi_0 = \frac{\Delta}{2}\sqrt{\frac{\omega}{c_1}},\ \ \Psi_1 = -\Phi_1 = \frac{\Delta(\Omega_0-\Delta)\Omega^2}{4c_1\sqrt{\omega/c_1}},$$
$$M = \frac{\Omega_0^2-\Delta^2}{8k_1}[k_1\omega^2\xi + \omega(2k_1\Omega_0\omega^2 + k_2\omega - 2)\eta] + \omega_0, \tag{4.21}$$

which leads to the non-travelling wave solution of the MGLDW system (2.3)-(2.4):

$$m_y = -\frac{1}{4}\omega(k_1\Omega_0\omega^2 + k_2\omega - 1)h_y[\Omega_0^2 - \Delta^2\tanh^2(\frac{1}{2}\Delta\Lambda)], \tag{4.22}$$

with $\Lambda = \omega x + \Omega_0\omega^2 t - (k_1\Omega_0\omega^2 + k_2\omega - 1)h$. By choosing the arbitrary function $h(y)$ as the Jacobi elliptic function, the simulation of the non-travelling waves solution (4.22) is shown in Fig.3. When the value of the Jacobi elliptic function's module $l \neq 1$, the periodic dromions are displayed in two different cases. When $l = 1$, the periodic behaviours degenerate to one dromion and $2+1$ dromion.

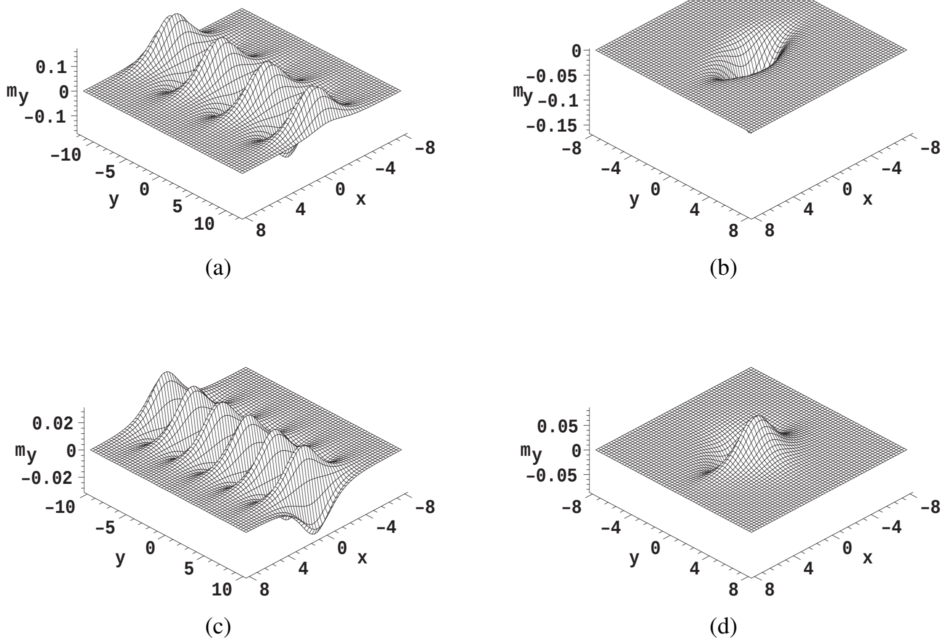


Fig. 3. The wave propagation plots of the MGLDW system given by Eq.(4.22), with the parameters $c_2 = k_2 = 2, c_3 = k_1 = \omega = \Delta = 1$ and $h = \frac{1}{3}[a_0\mathrm{sn}(y,l) + b_0\mathrm{dn}(y,l)]$: (a) $b_0 = 0, l = 0.6$; (b) $b_0 = 0, l = 1$; (c) $a_0 = 0, l = 0.6$; (d) $a_0 = 0, l = 1$.

5. Discussion and Summary

In this paper, we investigate the nonlocal symmetry of the (2+1) dimensional modified generalized long dispersive wave system and its application. First, the nonlocal symmetry related to the eigenfunctions in Lax pairs is derived, and infinitely many nonlocal symmetries are obtained directly with the help of the parameter in Lax pairs. Then, by introducing three potentials, the nonlocal symmetry is successfully localized to some local ones. The original MGLDW system also be extended to one prolonged system. In this procedure, it is shown that the nonlocal symmetry is closely related to the Möbious transformation invariance of the Schwartz form.

Starting from the known nonlocal symmetry, various types of integrable models are obtained through the nonlocal symmetry constraints approach. Using the independent variable translation symmetry and the nonlocal symmetry on the Lax pairs of the MGLDW system, the usual (1+1)-dimensional AKNS system, multi-component asymmetric DS system and multi-component coupled Liouville model are constructed. By introducing some inner parameters, we embed the MGLDW system in higher dimensional case. Considering the inner parameter translation symmetry and the nonlocal symmetry, some infinite-dimensional integrable systems are given.

For the prolonged system, the general finite symmetry transformations and the corresponding similarity reductions are considered by applying the Lie point symmetry method. The explicit finite symmetry transformation is also viewed as the Darboux-like transformation, which can generate new solutions from old ones. The similarity reductions lead to the novel exact interaction solution among solitons and periodic cnoidal waves, and some non-travelling wave solution. In particular, the soliton-cnoidal wave solution may degenerate to the usual two-soliton solution. In fact, it is very difficult to obtain these types of solutions from the original DT by solving the spectral problem directly. The reason is that to solve the spectral problem with the seed solution being non-constant is not usually easy work. Therefore, with the aid of the nonlocal symmetry, an alternative way is provided to construct some new solutions for the integrable models.

To search for nonlocal symmetries of integrable systems and then to apply them to obtain new results are both of considerable interest. Using the nonlocal symmetry constraints, one can easily get various types of integrable models. The other integrability for these multi-component models need to be further considered. Moreover, localization is viewed as a very important step to extend applicability of nonlocal symmetry. However, since the prolongation does not close generally, there is not a universal way to estimate what kind of nonlocal symmetries can be localized to the Lie point symmetries of some related prolonged system. More details on the results of this paper, especially on the applications of soliton-cnoidal wave solutions, are worthy of further study.

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