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Multi-component generalizations of the Hirota–Satsuma coupled KdV equation

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ABSTRACT

In this paper, we consider multi-component generalizations of the Hirota–Satsuma coupled Korteweg–de Vries (KdV) equation. By introducing a Lax pair, we present a matrix generalization of the Hirota–Satsuma coupled KdV equation, which is shown to be reduced to a vector Hirota–Satsuma coupled KdV equation. By using Hirota's bilinear method, we find a few soliton solutions to the vector Hirota–Satsuma coupled KdV equation in a symmetric case. Finally, in this symmetric case, we give a multi-soliton solution expressed by pfaffians and prove it by pfaffian techniques.

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1. Introduction

In Ref. [1], Hirota and Satsuma proposed a coupled Korteweg–de Vries (KdV) equation

$$u_t = \frac{1}{2}(u_{xxx} - 6uu_x) + 6\phi\phi_x, \tag{1}$$

$$\phi_t = -\phi_{xxx} + 3u\phi_x, \tag{2}$$

which describes an interaction of two long waves with different dispersion relations. Soon after, they found that Eqs. (1)–(2), often referred to as the Hirota–Satsuma coupled KdV equation, is a special case of a more generalized coupled system [2]

$$u_t = \frac{1}{4}u_{xxx} + 3uu_x + 3(-\phi^2 + w)_x, \tag{3}$$

$$\phi_t = -\frac{1}{2}\phi_{xxx} - 3u\phi_x, \tag{4}$$

$$w_t = -\frac{1}{2}w_{xxx} - 3uw_x, \tag{5}$$

which can be obtained from the 4-reduction of the KP hierarchy.

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By introducing a 4×4 spectral problem with three potentials, Wu et al. [3] derived a hierarchy of nonlinear evolution equations. One typical equation in this hierarchy is a generalized Hirota–Satsuma coupled KdV equation

$$u_t = \frac{1}{2}(u_{xxx} - 6uu_x) + 3(vw)_x, \tag{6}$$

$$v_t = -v_{xxx} + 3uv_x, \tag{7}$$

$$w_t = -w_{xxx} + 3uw_x. \tag{8}$$

For Eqs. (6)–(8), there are three nontrivial reductions: (i) $w = v$ yields the Hirota–Satsuma coupled KdV equations (1) and (2); (ii) $w = v^*$ (complex conjugate) leads to a complex coupled KdV equation

$$u_t = \frac{1}{2}(u_{xxx} - 6uu_x) + 3(|v|^2)_x, \tag{9}$$

$$v_t = -v_{xxx} + 3uv_x, \tag{10}$$

(iii) $w = 1$ reduces to another coupled KdV equation [2,4,5]

$$u_t = \frac{1}{2}(u_{xxx} - 6uu_x) + 3v_x, \tag{11}$$

$$v_t = -v_{xxx} + 3uv_x. \tag{12}$$

A number of research have been done for the Hirota–Satsuma coupled KdV equations (1)–(2) and its generalized version (6)–(8). Eqs. (1)–(2) are found as examples arising from the Drinfeld–Sokolov hierarchy [4,6] and have been studied by various approaches such as the bilinear method [1,7], Lax pair [8–10], Bäcklund transformation [11], Darboux transformation [12–14], Painlevé property [9,10] and infinitely many symmetries and conservation laws [15]. Regarding Eqs. (6)–(8), soliton, periodic and other types of solutions were constructed by various methods [16–24].

In the present paper, we consider a multi-component generalization of the Hirota–Satsuma coupled KdV equation

$$u_t + \frac{1}{2}u_{xxx} + 3uu_x = \frac{3}{2}\sum_{1\le j,k\le N} c_{jk}\left(\phi_j\phi_k\right)_x + 3\sum_{1\le j\le N} c_j\phi_{j,x}, \tag{13}$$

$$\phi_{i,t} - \phi_{i,xxx} - 3u\phi_{i,x} = 0, \quad i = 1, 2, \dots, N. \tag{14}$$

Note that if $c_{jj} = c_j = 0$ and $c_{jk} = c_{kj}$, Eqs. (13)–(14) are reduced to the following coupled KdV equation

$$u_t + \frac{1}{2}u_{xxx} + 3uu_x = 3\sum_{1\le j<k\le N} c_{jk}\left(\phi_j\phi_k\right)_x, \tag{15}$$

$$\phi_{i,t} - \phi_{i,xxx} - 3u\phi_{i,x} = 0, \quad i = 1, 2, \dots, N, \tag{16}$$

which is exactly Eqs. (6)–(8) for $N = 2$ under appropriate scaling transformations.

On the other hand, if $c_{jk} = 0$ ($j \neq k$), Eq. (13)–(14) can be viewed as a multi-component generalization of Eqs. (3)–(5)

$$u_t + \frac{1}{2}u_{xxx} + 3uu_x = 3\sum_{1\le j\le N}\left(c_{jj}\phi_j\phi_{j,x} + c_j\phi_{j,x}\right), \tag{17}$$

$$\phi_{i,t} - \phi_{i,xxx} - 3u\phi_{i,x} = 0, \quad i = 1, 2, \dots, N. \tag{18}$$

The rest of the paper is organized as follows. In Section 2, we propose a matrix generalization of the Hirota–Satsuma coupled KdV equation and show its integrability by introducing a Lax pair. The matrix equation is then reduced to a multi-component Hirota–Satsuma coupled KdV equations (13)–(14). In Section 3, we present the bilinear form of Eq. (15)–(16) and find a few soliton solutions for $N = 3$ by Hirota's direct method. In Section 4, the multi-soliton solution for Eqs. (15) and (16) is expressed by pfaffians and is proved by pfaffian techniques. The present paper is concluded by Section 5.

2. Lax pair

Let us introduce a set of auxiliary linear equations

$$\Psi_x = U\Psi, \qquad \Psi_t = V\Psi,$$

$$U = \begin{pmatrix} \mathbf{0} & \mathbf{I}_0 \\ A & \mathbf{0} \end{pmatrix}, \qquad V = \begin{pmatrix} B & C \\ D & -B \end{pmatrix}, \tag{19}$$

with

$$A = \begin{pmatrix} -(u-\lambda)\mathbf{I}_1 & Q \\ R & -(u+\lambda)\mathbf{I}_2 \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{1}{2}u_x\mathbf{I}_1 & Q_x \\ R_x & \frac{1}{2}u_x\mathbf{I}_2 \end{pmatrix},$$

$$C = \begin{pmatrix} -(u+2\lambda)\mathbf{I}_1 & -2Q \\ -2R & -(u-2\lambda)\mathbf{I}_2 \end{pmatrix},$$

$$D = \begin{pmatrix} \left(\frac{1}{2}u_{xx} + u^2 + \lambda u - 2\lambda^2\right)\mathbf{I}_1 - 2QR & Q_{xx} + uQ \\ R_{xx} + uR & \left(\frac{1}{2}u_{xx} + u^2 - \lambda u - 2\lambda^2\right)\mathbf{I}_2 - 2RQ \end{pmatrix},$$

where $\mathbf{0}$ represents the $(l+s) \times (l+s)$ zero matrix, $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_0$ are $l \times l$, $s \times s$ and $(l+s) \times (l+s)$ identity matrices respectively, Q is a $l \times s$ matrix, R is a $s \times l$ matrix and Ψ is a $2(l+s)$-component vector.

From the zero curvature equation $U_t - V_x + [U, V] = 0$, one can obtain a set of matrix equations

$$\tilde{U}_{1,t} + \frac{1}{2}\tilde{U}_{1,xxx} + 3\tilde{U}_1\tilde{U}_{1,x} = 3(QR)_x, \qquad \tilde{U}_{2,t} + \frac{1}{2}\tilde{U}_{2,xxx} + 3\tilde{U}_2\tilde{U}_{2,x} = 3(RQ)_x, \tag{20}$$

$$Q_t - Q_{xxx} - 3\tilde{U}_1 Q_x = 0, \qquad R_t - R_{xxx} - 3\tilde{U}_2 R_x = 0, \tag{21}$$

where $\tilde{U}_1$ and $\tilde{U}_2$ are matrices defined by $\tilde{U}_1 \equiv u\mathbf{I}_1$ and $\tilde{U}_2 \equiv u\mathbf{I}_2$.

In order to reduce the matrix equations (20)–(21) to a vector equation, we recursively define $2^{n-1} \times 2^{n-1}$ matrices $Q^{(n)}$ and $R^{(n)}$ as follows

$$Q^{(1)} = q_1, \qquad R^{(1)} = r_1,$$

$$Q^{(n+1)} = \begin{pmatrix} Q^{(n)} & q_{n+1}\mathbf{I}_{2^{n-1}} \\ -r_{n+1}\mathbf{I}_{2^{n-1}} & R^{(n)} \end{pmatrix}, \qquad R^{(n+1)} = \begin{pmatrix} R^{(n)} & -q_{n+1}\mathbf{I}_{2^{n-1}} \\ r_{n+1}\mathbf{I}_{2^{n-1}} & Q^{(n)} \end{pmatrix}. \tag{22}$$

Here $\mathbf{I}_{2^{n-1}}$ is a $2^{n-1} \times 2^{n-1}$ identity matrix. Note that $Q^{(n)}$ and $R^{(n)}$ defined above satisfy $Q^{(n)}R^{(n)} = R^{(n)}Q^{(n)} = \sum_{i=1}^{n} q_i r_i \mathbf{I}_{2^{n-1}}$. By substituting $Q^{(n)}$ and $R^{(n)}$ for Q and R into the matrix equations (20)–(21), one can obtain

$$u_t + \frac{1}{2}u_{xxx} + 3uu_x = 3(\mathbf{q}^T\mathbf{r})_x, \tag{23}$$

$$\mathbf{q}_t - \mathbf{q}_{xxx} - 3u\mathbf{q}_x = 0, \qquad \mathbf{r}_t - \mathbf{r}_{xxx} - 3u\mathbf{r}_x = 0, \tag{24}$$

with the vectors $\mathbf{q}$ and $\mathbf{r}$ given by $\mathbf{q} = (q_1, q_2, \ldots, q_n)^T$, $\mathbf{r} = (r_1, r_2, \ldots, r_n)^T$, where ^{T} denotes the transposition.

In what follows, we choose $n = N$ and

$$\mathbf{q} = \Phi, \qquad \mathbf{r} = G\Phi + G_0, \tag{25}$$

where $\Phi = (\phi_1, \phi_2, \ldots, \phi_N)^T$, G is a $N \times N$ constant matrix and G_0 is a $1 \times N$ constant matrix

$$G = \frac{1}{2}\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ c_{21} & c_{22} & \cdots & c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1} & c_{N2} & \cdots & c_{NN} \end{pmatrix}, \qquad G_0 = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix},$$

and the coupled system (23)–(24) is reduced to the multi-component Hirota–Satsuma coupled KdV equations (13)–(14).

3. Bilinear form and a few soliton solutions

In this section, we seek for one- and two-soliton solutions to (15)–(16) for $N = 3$ through Hirota's direct method. By the dependent variable transformations

$$u = 2(\ln f)_{xx}, \qquad \phi_i = \frac{g_i}{f}, \tag{26}$$

Eqs. (15)–(16) are cast into the following bilinear form:

$$\left(D_x D_t + \frac{1}{2}D_x^4\right) f \cdot f = 3 \sum_{1 \le j < k \le N} c_{jk} g_j g_k, \tag{27}$$

$$(D_t - D_x^3) g_i \cdot f = 0, \quad i = 1, 2, \ldots, N. \tag{28}$$

Let $N = 3$. Using Hirota's direct method [25], we deduce the solution which has one soliton for ϕ_i, $i = 1, 2, 3$ as follows

$$g_1 = \exp(\eta_1) + \alpha_{12}\alpha_{13}\alpha_{23}\beta_{23}\exp(\eta_1+\eta_2+\eta_3),$$
$$g_2 = \exp(\eta_2) - \alpha_{12}\alpha_{13}\alpha_{23}\beta_{13}\exp(\eta_1+\eta_2+\eta_3),$$
$$g_3 = \exp(\eta_3) + \alpha_{12}\alpha_{13}\alpha_{23}\beta_{12}\exp(\eta_1+\eta_2+\eta_3),$$
$$f = 1 + \alpha_{12}\beta_{12}\exp(\eta_1+\eta_2) + \alpha_{13}\beta_{13}\exp(\eta_1+\eta_3) + \alpha_{23}\beta_{23}\exp(\eta_2+\eta_3),$$

with

$$\eta_\mu = p_\mu x + p_\mu^3 t + \eta_\mu^0, \qquad \alpha_{\mu\nu} = \frac{p_\mu - p_\nu}{p_\mu + p_\nu}, \qquad \beta_{\mu\nu} = \frac{c_{\mu\nu}}{p_\mu^4 - p_\nu^4},$$

where p_μ and η_μ^0 are arbitrary parameters for $\mu, \nu = 1, 2, 3$.

The solution which has one soliton for ϕ_1 and ϕ_2 and two soliton for ϕ_3 is given by

$$g_1 = \exp(\eta_1) + \alpha_{12}\alpha_{13}\alpha_{23}\beta_{23}\exp(\eta_1+\eta_2+\eta_3) + \alpha_{12}\alpha_{14}\alpha_{24}\beta_{24}\exp(\eta_1+\eta_2+\eta_4),$$
$$g_2 = \exp(\eta_2) - \alpha_{12}\alpha_{23}\alpha_{13}\beta_{13}\exp(\eta_1+\eta_2+\eta_3) - \alpha_{12}\alpha_{24}\alpha_{14}\beta_{14}\exp(\eta_1+\eta_2+\eta_4),$$
$$\begin{aligned} g_3 = {} & \exp(\eta_3) + \exp(\eta_4) + \alpha_{12}\alpha_{13}\alpha_{23}\beta_{12}\exp(\eta_1+\eta_2+\eta_3) + \alpha_{12}\alpha_{14}\alpha_{24}\beta_{12}\exp(\eta_1+\eta_2+\eta_4) \\ & + \frac{c_{13}(p_3-p_4)^2(p_3^2+p_4^2)}{(p_1+p_3)^2(p_1+p_4)^2(p_1^2+p_3^2)(p_1^2+p_4^2)}\exp(\eta_1+\eta_3+\eta_4) \\ & + \frac{c_{23}(p_3-p_4)^2(p_3^2+p_4^2)}{(p_2+p_3)^2(p_2+p_4)^2(p_2^2+p_3^2)(p_2^2+p_4^2)}\exp(\eta_2+\eta_3+\eta_4), \end{aligned}$$
$$\begin{aligned} f = {} & 1 + \alpha_{12}\beta_{12}\exp(\eta_1+\eta_2) + \alpha_{13}\beta_{13}\exp(\eta_1+\eta_3) + \alpha_{14}\beta_{14}\exp(\eta_1+\eta_4) \\ & + \alpha_{23}\beta_{23}\exp(\eta_2+\eta_3) + \alpha_{24}\beta_{24}\exp(\eta_2+\eta_4) \\ & + \frac{c_{13}c_{23}(p_1-p_2)^2(p_3-p_4)^2(p_1^2+p_2^2)(p_3^2+p_4^2)\exp(\eta_1+\eta_2+\eta_3+\eta_4)}{(p_1+p_3)^2(p_1+p_4)^2(p_2+p_3)^2(p_2+p_4)^2(p_1^2+p_3^2)(p_1^2+p_4^2)(p_2^2+p_3^2)(p_2^2+p_4^2)}, \end{aligned}$$

with

$$\eta_\mu = p_\mu x + p_\mu^3 t + \eta_\mu^0, \qquad \alpha_{\mu\nu} = \frac{p_\mu - p_\nu}{p_\mu + p_\nu}, \qquad \beta_{\mu\nu} = \frac{c_{\mu\nu}}{p_\mu^4 - p_\nu^4}, \qquad c_{\mu 4} \equiv c_{\mu 3},$$

where p_μ and η_μ^0 are arbitrary parameters for $\mu, \nu = 1, 2, 3, 4$.

4. Pfaffian representation for a multi-soliton solution

Similar to a coupled modified KdV equation and a vector potential KdV equation [26,27], we find out that the multi-soliton solution to the multi-component Hirota–Satsuma coupled KdV equations (15)–(16) can be expressed by pfaffians.

We assume that component ϕ_i contains M_i soliton(s), and define $L = M_1 + M_2 + \cdots + M_N$. We further define the elements of the pfaffians as follows

$$\mathrm{pf}(d_n, a_\mu) \equiv \frac{\partial^n}{\partial x^n}\exp(\eta_\mu) = p_\mu^n \exp(\eta_\mu), \qquad \mathrm{pf}(a_\mu, a_\nu) = \frac{p_\mu - p_\nu}{p_\mu + p_\nu}\exp(\eta_\mu + \eta_\nu),$$
$$\mathrm{pf}(a_\mu, b_\nu) = \begin{cases} 1, & \text{if } \mu = \nu, \\ 0, & \text{if } \mu \neq \nu, \end{cases} \qquad \mathrm{pf}(b_\mu, \beta_i) = \begin{cases} 1, & \text{if } b_\mu \in B_i, \\ 0, & \text{if } b_\mu \notin B_i, \end{cases}$$
$$\mathrm{pf}(b_\mu, b_\nu) = -\frac{c_{jk}}{p_\mu^4 - p_\nu^4}\big(b_\mu \in B_j,\ \ b_\nu \in B_k\big), \qquad \mathrm{pf}(\text{otherwise}) \equiv 0,$$

where $\eta_\mu = p_\mu x + p_\mu^3 t + \eta_\mu^0$, the elements in sets B_i are chosen out of $\{b_1, b_2, \dots b_L\}$ for $i = 1, 2, \dots N$ to satisfy the following conditions,

$$\begin{cases} M_i = \text{number of elements in the set } B_i, \\ B_i \cap B_j = \varnothing, \\ \displaystyle\bigcup_{i=1}^{N} B_i = \{b_1, b_2, \dots b_L\}. \end{cases}$$

It is found that the multi-soliton solution for Eq. (15)–(16) is expressed by the following pfaffians

$$g_i = \mathrm{pf}(d_0, a_1, a_2, \dots, a_L, b_1, b_2, \dots, b_L, \beta_i), \tag{29}$$
$$f = \mathrm{pf}(a_1, a_2, \dots, a_L, b_1, b_2, \dots, b_L), \tag{30}$$

for $i = 1, 2, \dots N$.

Proof. From the definition of the functions g_i and f, using the same procedure as in [26,27], we can derive the following pfaffian's rules:

$$\frac{\partial f}{\partial x} = \mathrm{pf}(d_0, d_1, \bullet), \qquad \frac{\partial^2 f}{\partial x^2} = \mathrm{pf}(d_0, d_2, \bullet), \tag{31}$$

$$\frac{\partial^3 f}{\partial x^3} = \mathrm{pf}(d_0, d_3, \bullet) + \mathrm{pf}(d_1, d_2, \bullet), \qquad \frac{\partial^4 f}{\partial x^4} = \mathrm{pf}(d_0, d_4, \bullet) + 2\mathrm{pf}(d_1, d_3, \bullet), \tag{32}$$

$$\frac{\partial f}{\partial t} = \mathrm{pf}(d_0, d_3, \bullet) - 2\mathrm{pf}(d_1, d_2, \bullet), \qquad \frac{\partial^2 f}{\partial x \partial t} = \mathrm{pf}(d_0, d_4, \bullet) - \mathrm{pf}(d_1, d_3, \bullet). \tag{33}$$

and

$$\frac{\partial g_i}{\partial x} = \mathrm{pf}(d_1, \bullet, \beta_i), \qquad \frac{\partial^2 g_i}{\partial x^2} = \mathrm{pf}(d_2, \bullet, \beta_i), \qquad \frac{\partial^3 g_i}{\partial x^3} = \mathrm{pf}(d_3, \bullet, \beta_i) + \mathrm{pf}(d_0, d_1, d_2, \bullet, \beta_i), \tag{34}$$

$$\frac{\partial g_i}{\partial t} = \mathrm{pf}(d_3, \bullet, \beta_i) - 2\mathrm{pf}(d_0, d_1, d_2, \bullet, \beta_i), \tag{35}$$

where $(\bullet) = (a_1, a_2, \ldots, a_L, b_1, b_2, \ldots, b_L)$.

Now we proceed to the proof of the first bilinear equation (15). Note that its r.h.s can be written as

$$\begin{aligned}
3\sum_{1\le j<k\le N} c_{jk} g_j g_k &= \frac{3}{2}\sum_{j,k=1}^{N} c_{jk} g_j g_k = \frac{3}{2}\sum_{j,k=1}^{N} c_{jk}\mathrm{pf}(d_0, \bullet, \beta_j)\mathrm{pf}(d_0, \bullet, \beta_k) \\
&= \frac{3}{2}\sum_{j,k=1}^{N} c_{jk} \sum_{\mu,\nu=1}^{L} (-1)^{\mu+\nu}\mathrm{pf}(b_\mu, \beta_j)\mathrm{pf}(b_\nu, \beta_k)\mathrm{pf}(d_0, \ldots, \hat{b}_\mu, \ldots)\mathrm{pf}(d_0, \ldots, \hat{b}_\nu, \ldots) \\
&= \frac{3}{2}\sum_{\mu,\nu=1}^{L} (-1)^{\mu+\nu} \sum_{j,k=1}^{N} c_{jk}\mathrm{pf}(b_\mu, \beta_j)\mathrm{pf}(b_\nu, \beta_k)\mathrm{pf}(d_0, \ldots, \hat{b}_\mu, \ldots)\mathrm{pf}(d_0, \ldots, \hat{b}_\nu, \ldots) \\
&= -\frac{3}{2}\sum_{\mu,\nu=1}^{L} (-1)^{\mu+\nu}(p_\mu^4 - p_\nu^4)\mathrm{pf}(b_\mu, b_\nu)\mathrm{pf}(d_0, \ldots, \hat{b}_\mu, \ldots)\mathrm{pf}(d_0, \ldots, \hat{b}_\nu, \ldots) \\
&= -\frac{3}{2}\sum_{\mu,\nu=1}^{L} (-1)^{\mu+\nu}p_\mu^4\mathrm{pf}(b_\mu, b_\nu)\mathrm{pf}(d_0, \ldots, \hat{b}_\mu, \ldots)\mathrm{pf}(d_0, \ldots, \hat{b}_\nu, \ldots) \\
&\quad -\frac{3}{2}\sum_{\nu,\mu=1}^{L} (-1)^{\mu+\nu}p_\nu^4\mathrm{pf}(b_\nu, b_\mu)\mathrm{pf}(d_0, \ldots, \hat{b}_\mu, \ldots)\mathrm{pf}(d_0, \ldots, \hat{b}_\nu, \ldots) \\
&= -3\sum_{\mu,\nu=1}^{L} (-1)^{\mu+\nu}p_\mu^4\mathrm{pf}(b_\mu, b_\nu)\mathrm{pf}(d_0, \ldots, \hat{b}_\mu, \ldots)\mathrm{pf}(d_0, \ldots, \hat{b}_\nu, \ldots),
\end{aligned} \tag{36}$$

where $(\ldots, \hat{a}_\mu, \ldots) = (a_1, a_2, \ldots, \hat{a}_\mu, \ldots, a_L, b_1, b_2, \ldots, b_L)$ and $(\ldots, \hat{b}_\mu, \ldots) = (a_1, a_2, \ldots, a_L, b_1, b_2, \ldots, \hat{b}_\mu, \ldots, b_L)$.

Furthermore, the expansion of the vanishing pfaffian $\mathrm{pf}(d_0, \bullet, b_\mu)$ on b_μ yields

$$\sum_{\nu=1}^{L}(-1)^{\mu+\nu}\mathrm{pf}(b_\mu, b_\nu)\mathrm{pf}(d_0, \ldots, \hat{b}_\nu, \ldots) = (-1)^L\mathrm{pf}(d_0, \ldots, \hat{a}_\mu, \ldots). \tag{37}$$

A substitution of (37) into (36) results in

$$3\sum_{1\le j\le k\le N} c_{jk} g_j g_k = 3(-1)^{L+1}\sum_{\mu=1}^{L} p_\mu^4\mathrm{pf}(d_0, \ldots, \hat{a}_\mu, \ldots)\mathrm{pf}(d_0, \ldots, \hat{b}_\mu, \ldots). \tag{38}$$

On the other hand, we can derive the following formulas by using (31)–(33),

$$\begin{aligned}
&\left(\frac{\partial^4 f}{\partial x^4} + 2\frac{\partial^2 f}{\partial x \partial t}\right)\cdot 0 - \left(4\frac{\partial^3 f}{\partial x^3} + 2\frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial x} + 3\left(\frac{\partial^2 f}{\partial x^2}\right)^2 \\
&\quad = 3\mathrm{pf}(d_0, d_4, \bullet)\mathrm{pf}(d_0, d_0, \bullet) - 6\mathrm{pf}(d_0, d_3, \bullet)\mathrm{pf}(d_0, d_1, \bullet) + 3\mathrm{pf}(d_0, d_2, \bullet)\mathrm{pf}(d_0, d_2, \bullet)
\end{aligned}$$

$$
\begin{aligned}
&= 3\left\{\sum_{\mu=1}^{L}(-1)^{\mu}\mathrm{pf}(d_4,a_\mu)\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\right\}\times\left\{\sum_{\nu=1}^{L}(-1)^{\nu}\mathrm{pf}(d_0,a_\nu)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)\right\}\\
&\quad -6\left\{\sum_{\mu=1}^{L}(-1)^{\mu}\mathrm{pf}(d_3,a_\mu)\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\right\}\times\left\{\sum_{\nu=1}^{L}(-1)^{\nu}\mathrm{pf}(d_1,a_\nu)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)\right\}\\
&\quad +3\left\{\sum_{\mu=1}^{L}(-1)^{\mu}\mathrm{pf}(d_2,a_\mu)\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\right\}\times\left\{\sum_{\nu=1}^{L}(-1)^{\nu}\mathrm{pf}(d_2,a_\nu)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)\right\}\\
&= 3\sum_{\mu,\nu=1}^{L}(-1)^{\mu+\nu}\left[\mathrm{pf}(d_4,a_\mu)\mathrm{pf}(d_0,a_\nu)-2\mathrm{pf}(d_3,a_\mu)\mathrm{pf}(d_1,a_\nu)+\mathrm{pf}(d_2,a_\mu)\mathrm{pf}(d_2,a_\nu)\right]\\
&\quad \times\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)\\
&= 3\sum_{\mu,\nu=1}^{L}(-1)^{\mu+\nu}\left[(p_\mu^4-p_\mu^2p_\nu^2)\mathrm{pf}(a_\mu,a_\nu)\right]\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots).
\end{aligned}
\tag{39}
$$

The second term in above bracket vanishes due to the fact of $\mathrm{pf}(a_\mu, a_\nu) = -\mathrm{pf}(a_\nu, a_\mu)$. Thus

$$
\begin{aligned}
-\left(4\frac{\partial^3 f}{\partial x^3}+2\frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial x}+3\left(\frac{\partial^2 f}{\partial x^2}\right)^2 &= 3\sum_{\mu,\nu=1}^{L}(-1)^{\mu+\nu}p_\mu^4\mathrm{pf}(a_\mu,a_\nu)\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)\\
&= 3\sum_{\mu=1}^{L}(-1)^{\mu}p_\mu^4\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\\
&\quad\times\left[\sum_{\nu=1}^{L}(-1)^{\nu}\mathrm{pf}(a_\mu,a_\nu)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)\right].
\end{aligned}
\tag{40}
$$

Moreover, note that the following identity can be substituted into the term within bracket

$$
\sum_{\nu=1}^{L}(-1)^{\nu}\mathrm{pf}(a_\mu,a_\nu)\mathrm{pf}(d_0,\ldots,\hat{a}_\nu,\ldots)=\mathrm{pf}(d_0,a_\mu)\mathrm{pf}(\bullet)+(-1)^{L+\mu+1}\mathrm{pf}(d_0,\ldots,\hat{b}_\mu,\ldots),
$$

which is obtained from the expansion of a vanishing pfaffian $(a_\mu, d_0, \bullet)$ on a_μ. Consequently, we have

$$
\begin{aligned}
-\left(4\frac{\partial^3 f}{\partial x^3}+2\frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial x}+3\left(\frac{\partial^2 f}{\partial x^2}\right)^2 &= 3\sum_{\mu=1}^{L}(-1)^{\mu}p_\mu^4\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\\
&\quad\times\left[\mathrm{pf}(d_0,a_\mu)\mathrm{pf}(\bullet)+(-1)^{L+\mu+1}\mathrm{pf}(d_0,\ldots,\hat{b}_\mu,\ldots)\right]\\
&= -3\mathrm{pf}(d_0,d_4,\bullet)\mathrm{pf}(\bullet)+3(-1)^{L+1}\\
&\quad\times\sum_{\mu=1}^{L}p_\mu^4\mathrm{pf}(d_0,\ldots,\hat{a}_\mu,\ldots)\mathrm{pf}(d_0,\ldots,\hat{b}_\mu,\ldots)\\
&= -\left(\frac{\partial^4 f}{\partial x^4}+2\frac{\partial^2 f}{\partial x\partial t}\right)f+3\sum_{1\le j<k\le N}c_{jk}g_jg_k.
\end{aligned}
\tag{41}
$$

Thus we have proved the first bilinear equation (15).

To prove Eq. (16), to which (31)–(35) are substituted, we have

$$
\begin{aligned}
(D_t-D_x^3)g_i\cdot f &= \left(\frac{\partial g_i}{\partial t}-\frac{\partial^3 g_i}{\partial x^3}\right)f+3\frac{\partial^2 g_i}{\partial x^2}\frac{\partial f}{\partial x}-3\frac{\partial g_i}{\partial x}\frac{\partial^2 f}{\partial x^2}-g_i\left(\frac{\partial f}{\partial t}-\frac{\partial^3 f}{\partial x^3}\right)\\
&= -3\mathrm{pf}(d_0,d_1,d_2,\bullet,\beta_i)\mathrm{pf}(\bullet)+3\mathrm{pf}(d_2,\bullet,\beta_i)\mathrm{pf}(d_0,d_1,\bullet)\\
&\quad -3\mathrm{pf}(d_1,\bullet,\beta_i)\mathrm{pf}(d_0,d_2,\bullet)+3\mathrm{pf}(d_0,\bullet,\beta_i)\mathrm{pf}(d_1,d_2,\bullet)
\end{aligned}
\tag{42}
$$

which vanishes by the pfaffian identity. Thus the proof is finished.

In the last, we point out that the pfaffian solution for $N = 2$ contains the multi-soliton solution for the generalized Hirota–Satsuma coupled KdV equations (6)–(8). By setting $u \to -u$, $\phi_1 \to \sqrt{3}v$, $\phi_2 \to \sqrt{3}w$, $t \to -t$ and $c_{12} = \frac{1}{3}$, the results we found here are equivalent to (24) and (30) in Ref. [24]. Therefore, it is clear that the general multi-soliton solution given by Wu et al. [24] can be expressed by pfaffians.

Regarding the collision properties, the component ϕ_i increases/decreases exponentially as t approaches ∞ as shown in [24] for $N = 2$. Therefore, it does not make sense to discuss the elastic/inelastic collision property for the component ϕ_i. However, the component u always undergoes elastic collision. The details for the asymptotic analysis are omitted here.

5. Conclusion and summary

In the present paper, we study multi-component generalizations of the Hirota–Satsuma coupled KdV equation. We first propose a matrix generalization of the Hirota–Satsuma coupled KdV equation which is shown to be integrable in the Lax sense. Then by defining the submatrices in the Lax pair recursively, the matrix equation is reduced to a vector Hirota–Satsuma coupled KdV equation. It is worthy to point out that this vector equation for $N = 2$ comprises the Hirota–Satsuma coupled KdV equation and its generalized versions studied so far [1–3]. In general, it is not easy to derive the multi-soliton solution for the vector Hirota–Satsuma coupled KdV equations (13)–(14). However, if we restrict ourselves to a symmetric case (15)–(16), its multi-soliton solution in a pfaffian form can be deduced by Hirota's bilinear method. Furthermore, this pfaffian solution is proved by using pfaffian techniques.

In the last, we would like to mention that both the coupled modified KdV equation [26] and the vector Hirota–Satsuma coupled KdV equation studied in this paper possess multi-soliton solutions in a pfaffian form of the same structure. It is conjectured that these coupled soliton equations together with their pfaffian solutions can be reduced from multi-component BKP/DKP hierarchy. The details deserve further exploration in the future.

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