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A New Generalization of Extended Tanh-Function Method for Solving Nonlinear Evolution Equations*

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Abstract *Making use of a new generalized ansätze and a proper transformation, we generalized the extended tanh-function method. Applying the generalized method with the aid of Maple, we consider some nonlinear evolution equations. As a result, we can successfully recover the previously known solitary wave solutions that had been found by the extended tanh-function method and other more sophisticated methods. More importantly, for some equations, we also obtain other new and more general solutions at the same time. The results include kink-profile solitary-wave solutions, bell-profile solitary-wave solutions, periodic wave solutions, rational solutions, singular solutions and new formal solutions.*

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1 Introduction

Nonlinear partial differential equations (PDEs) are widely used to describe complex phenomena in various fields of science, such as in physics, mechanics, chemistry, biology, etc. Seeking exact solutions of the nonlinear evolution equations has been an important topic in nonlinear science physics. In recent years, important progress has been made in understanding the nonlinear evolution equations (NEEs). Various methods have been proposed to explore different kinds of solutions of various physical models described by NEEs, such as Bäcklund transformation, Darboux transformation, Painlevé method, Hirota method,[1] rank analysis method,[3] homogenous balance method,[4−6] variable-coefficient balancing-act method,[7,8] and so on. Particularly, various ansätze have been proposed in order to obtain new forms of solutions. Directly searching for exact solutions of NEEs has become more and more attractive. The availability of computer systems like *Maple* and *Mathematica*, which allows us to perform some complicated and tedious algebraic calculation on a computer, helps us to find new exact solutions of NEEs.

Recently, based on the well-known Riccati equation, Fan presented a useful extended tanh method to find exact solutions of given NEEs.[6] More recently, Fan[9] and Yan[10] further developed this idea and made it much more lucid and straightforward for a class of NEEs. Not long since, S.A. Elwakil[11] *et al.* modified extended tanh-function method, and obtained some new exact solutions. The present work is motivated by the desire to extend the work made in Refs. [9] and [11]. By introducing a new ansätze and a more general proper transformation, we generalize the tanh-method and its modification. Applying the generalized method with the aid of *Maple*, we consider some NEEs, some of which have nonlinear terms of any order, especially as $p = 1$ and $p = 2$.

The plan of this paper is as follows. In Sec. 2 we describe briefly the method. In Sec. 3 we apply the generalized extended tanh-function method to some NEEs and bring out some interesting solutions. Conclusions will be presented in Sec. 4.

2 Method

Now, we simply describe the new generalization of extended tanh-method. Consider a given NEEs, say, in two variables x, t,

$$H(u, u_t, u_x, u_{xt}, u_{tt}, \ldots) = 0\,, \tag{1}$$

we first consider the following formal travelling wave solutions $u(x,t) = \phi(\xi)$, $\xi = x-\lambda t$, where λ is a constant to be determined later. Then equation (1) becomes a nonlinear ordinary differential equation

$$F(\phi, \phi', \phi'', \phi''' \cdots) = 0\,, \tag{2}$$

where " $'$ " denotes $\mathrm{d}/\mathrm{d}\xi$. In order to seek the travelling wave solutions of Eq. (2), we introduce the following ansätze

$$\phi(\xi) = A_0 + \sum_{i=1}^{m} \{\omega^{i-1}[A_i\omega + C_i\sqrt{R+\omega^2}] + B_i\omega^{-i}\} \tag{3}$$

with the new variable $\omega = \omega(\xi)$ satisfying

$$\omega' = \mathrm{d}\omega/\mathrm{d}\xi = R + \omega^2\,, \tag{4}$$

where A_0, A_i, B_i, C_i $(i = 0, 1, 2, \ldots, m)$ and R are constants to be determined later. The value of m in Eq. (3) can be determined by balancing the highest-order derivative term with the nonlinear term[4−6] in Eq. (1) or Eq. (2). If m is not a positive number, we can take the following transformation.

i) if $m = q/p$ (where $m = q/p$ is a fraction in the lowest terms), we let

$$\phi(\xi) = \varphi^{q/p}(\xi)\,, \tag{5}$$

then substitute Eq. (5) into Eq. (2) and then determine the value of m in new Eq. (2).

ii) if m is a negative integer, we let

$$\phi(\xi) = \varphi^{m}(\xi)\,, \tag{6}$$

then substitute Eq. (6) into Eq. (2) and then determine the value of m again. After that, we substitute Eq. (3) into Eq. (2), the corresponding ODEs (ordinary differential

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equations), and then let all coefficients of $\omega^i\sqrt{(R+\omega^2)}^j$ and ω^r $(j=0,1; i,r=0,1,2,\ldots)$ to be zero to get an over-determined system of nonlinear algebraic equations with respect to λ, R, A_0, A_i, B_i, C_i $(i=1,2,\ldots,m)$. With the aid of *Maple*, we apply Wu-elimination method[12] to solve the above-mentioned over-determined system of nonlinear algebraic equations to yields the values of λ, R, A_0, A_i, B_i, C_i $(i=1,2,\ldots,m)$. Because the Raccati equation (4) has the following general solutions:

i) if $R<0$,

$$\omega(\xi) = -\sqrt{-R}\tanh(\sqrt{-R}\,\xi)\,, \tag{7}$$

$$\omega(\xi) = -\sqrt{-R}\coth(\sqrt{-R}\,\xi)\,; \tag{8}$$

ii) if $R=0$,

$$\omega(\xi) = -1/\xi\,; \tag{9}$$

iii) if $R>0$,

$$\omega(\xi) = \sqrt{R}\tan(\sqrt{R}\,\xi)\,, \tag{10}$$

$$\omega(\xi) = -\sqrt{R}\cot(\sqrt{R}\,\xi)\,; \tag{11}$$

So combining the values of λ, R, A_0, A_i, B_i, C_i $(i=1,2,\ldots,m)$ with Eq. (3), Eq. (5) or Eq. (6) and Eqs. (7) $\sim$ (11), more travelling wave solutions of Eq. (1) are obtained.

Note Since coth- and cot-type solutions appear in pairs with tanh- and tan-type solutions, respectively, we omit them in this paper.

3 Applications

Example 1 Consider a (2+1)-dimensional generalization of Boussinesq equation[13]

$$u_{tt} - u_{xx} - u_{yy} - (u^2)_{xx} - u_{xxxx} = 0\,. \tag{12}$$

As described in Sec. 2, considering the travelling wave solution of the above equations, $u(x,t)=\phi(\xi)$, $\xi = x+\beta y-\lambda t$, we have

$$\phi'' + \phi^2 + (1+\beta^2-\lambda^2)\phi = 0\,. \tag{13}$$

By balancing ϕ'' and ϕ^2, we can get $m=2$. As described in Sec. 2, we take the following ansätze

$$\phi(\xi) = A_0 + A_1\omega + C_1\sqrt{R+\omega^2} + B_1/\omega + A_2\omega^2 + C_2\omega\sqrt{R+\omega^2} + B_2/\omega^2\,, \tag{14}$$

and $\omega=\omega(\xi)$ satisfies equation (4), where A_0, A_1, B_1, A_2, B_2, C_1, C_2, R are constants to be determined later. With the aid of *Maple*, substituting Eq. (14) into Eq.(13) and making use of Eq.(4), we get a system of nonlinear algebraic equations with respect to the unknowns A_0, A_1, B_1, A_2, B_2, C_1, C_2, R, λ, β,

$$6\,B_2R^2 + {B_2}^2 = 0\,, \tag{15}$$

$$2\,B_1B_2 + 2\,B_1R^2 = 0\,, \tag{16}$$

$$C_2 + 5\,C_2R - \lambda^2C_2 + \beta^2C_2 + 2\,A_0C_2 + 2\,A_1C_1 = 0\,, \tag{17}$$

$$C_1R + 2\,A_0C_1 + C_1 + \beta^2C_1 + 2\,B_1C_2 - \lambda^2C_1 = 0\,, \tag{18}$$

$$6\,C_2 + 2\,A_2C_2 = 0\,, \tag{19}$$

$$2\,A_1C_2 + 2\,C_1 + 2\,A_2C_1 = 0\,, \tag{20}$$

$$2\,B_1C_1 + 2\,B_2C_2 = 0\,, \tag{21}$$

$$2\,B_2C_1 = 0\,, \tag{22}$$

$${C_2}^2 + 6\,A_2 + {A_2}^2 = 0\,, \tag{23}$$

$$-\lambda^2A_2 + {C_2}^2R + {A_1}^2 + \beta^2A_2 + {C_1}^2 + A_2 + 8\,A_2R + 2\,A_0A_2 = 0\,, \tag{24}$$

$$B_1 + \beta^2B_1 + 2\,B_1R + 2\,A_1B_2 + 2\,A_0B_1 - \lambda^2B_1 = 0\,, \tag{25}$$

$$2\,A_2B_2 + 2\,B_2 - \lambda^2A_0 + 2\,A_1B_1 + 2\,A_2R^2 + \beta^2A_0 + A_0 + {A_0}^2 + {C_1}^2R = 0\,, \tag{26}$$

$$-\lambda^2B_2 + \beta^2B_2 + B_2 + 2\,A_0B_2 + {B_1}^2 + 8\,B_2R = 0\,, \tag{27}$$

$$2\,C_1C_2 + 2\,A_1A_2 + 2\,A_1 = 0\,, \tag{28}$$

$$-\lambda^2A_1 + 2\,B_1A_2 + 2\,A_0A_1 + 2\,C_1C_2R + \beta^2A_1 + 2\,A_1R + A_1 = 0\,. \tag{29}$$

From Eqs. (15) $\sim$ (29), by Wu's elimination method,[12] we find

Case 1 $A_1=B_1=B_2=C_1=C_2=0$, $A_0=-3\beta^2/2+3\lambda^2/2-3/2$, $A_2=-6$, $R=\beta^2/4-\lambda^2/4+1/4$.

Case 2 $A_1=B_1=B_2=C_1=0$, $A_0=-3\beta^2+3\lambda^2-3$, $A_2=-3$, $C_2=\pm3$, $R=\beta^2-\lambda^2+1$.

Case 3 $A_1=B_1=B_2=C_1=0$, $A_0=2+2\beta^2-2\lambda^2$, $A_2=-3$, $C_2=\pm3$, $R=-\beta^2+\lambda^2-1$.

Case 4 $A_1=A_2=B_1=C_1=C_2=0$, $A_0=\beta^2/2-\lambda^2/2+1/2$, $B_2=-3\beta^4/8-3\beta^2/4+3\beta^2\lambda^2/4-3\lambda^4/8+3\lambda^2/4-3/8$, $R=-\beta^2/4+\lambda^2/4-1/4$.

Case 5 $A_1=B_1=C_1=C_2=0$, $A_0=-\beta^2/4+\lambda^2/4-1/4$, $A_2=-6$, $B_2=-3\beta^4/128-3\beta^2/64+3\beta^2\lambda^2/64-3\lambda^4/128+3\lambda^2/64-3/128$, $R=-\beta^2/16+\lambda^2/16-1/16$.

Case 6 $A_1=A_2=B_1=C_1=C_2=0$, $A_0=-3\beta^2/2+3\lambda^2/2-3/2$, $B_2=-3\beta^4/8-3\beta^2/4+3\beta^2\lambda^2/4-3\lambda^4/8+3\lambda^2/4-3/8$, $R=\beta^2/4-\lambda^2/4+1/4$.

Case 7 $A_1=B_1=C_1=C_2=0$, $A_0=-3\beta^2/4+3\lambda^2/4-3/4$, $A_2=-6$, $B_2=-3\beta^4/128-3\beta^2/64+3\beta^2\lambda^2/64-3\lambda^4/128+3\lambda^2/64-3/128$, $R=\beta^2/16-\lambda^2/16+1/16$.

Case 8 $A_1=B_1=B_2=C_1=C_2=0$, $A_0=\beta^2/2-\lambda^2/2+1/2$, $R=-\beta^2/4+\lambda^2/4-1/4$, $A_2=-6$.

Case 9 $A_0=A_1=B_1=B_2=C_1=C_2=0$, $A_2=-6$, $R=0$, $\lambda=\pm\sqrt{\beta^2+1}$.

Therefore, combining Eqs. (7) $\sim$ (11) along with Cases 1 $\sim$ 9, we obtain the travelling wave solutions of (2+1)-dimensional generalization of Boussinesq equation as follows:

Case 1

$$u_{11} = \left(-\tfrac{3}{2}\beta^2 + \tfrac{3}{2}\lambda^2 - \tfrac{3}{2}\right)\sec^2\left(\tfrac{1}{2}\sqrt{\beta^2-\lambda^2+1}\,\xi\right)\,, \tag{30}$$

$$u_{12} = \left(-\tfrac{3}{2}\beta^2 + \tfrac{3}{2}\lambda^2 - \tfrac{3}{2}\right)\mathrm{sech}^2\left(\sqrt{-\tfrac{1}{4}(\beta^2-\lambda^2+1)}\,\xi\right)\,. \tag{31}$$

Case 2

$$u_{21} = -3\beta^2 + 3\lambda^2 - 3 - 3(\beta^2 - \lambda^2 + 1)\tan^2\big(\sqrt{\beta^2 - \lambda^2 + 1}\,\xi\big)$$
$$\pm 3(\beta^2 - \lambda^2 + 1)\tan\big(\sqrt{\beta^2 - \lambda^2 + 1}\,\xi\big)\sec\big(\sqrt{\beta^2 - \lambda^2 + 1}\,\xi\big)\,, \tag{32}$$
$$u_{22} = -3\beta^2 + 3\lambda^2 - 3 + 3(\beta^2 - \lambda^2 + 1)\tanh^2\big(\sqrt{-(\beta^2 - \lambda^2 + 1)}\,\xi\big)$$
$$\pm 3\mathrm{i}(\beta^2 - \lambda^2 + 1)\tanh\big(\sqrt{-(\beta^2 - \lambda^2 + 1)}\,\xi\big)\,\mathrm{sech}\big(\sqrt{-(\beta^2 - \lambda^2 + 1)}\,\xi\big)\,. \tag{33}$$

Case 3

$$u_{31} = 2 + 2\beta^2 - 2\lambda^2 - 3(-\beta^2 + \lambda^2 - 1)\tan^2\big(\sqrt{-\beta^2 + \lambda^2 - 1}\,\xi\big)$$
$$\pm 3(-\beta^2 + \lambda^2 - 1)\tan\big(\sqrt{-\beta^2 + \lambda^2 - 1}\,\xi\big)\sec\big(\sqrt{-\beta^2 + \lambda^2 - 1}\,\xi\big)\,, \tag{34}$$
$$u_{32} = 2 + 2\beta^2 - 2\lambda^2 + 3(-\beta^2 + \lambda^2 - 1)\tanh^2\big(\sqrt{\beta^2 - \lambda^2 + 1}\,\xi\big)$$
$$\pm 3\mathrm{i}(-\beta^2 + \lambda^2 - 1)\tanh\big(\sqrt{\beta^2 - \lambda^2 + 1}\,\xi\big)\mathrm{sech}\big(\sqrt{\beta^2 - \lambda^2 + 1}\,\xi\big)\,. \tag{35}$$

Case 4

$$u_{41} = -(\beta^2 - \lambda^2 + 1) + \tfrac{3}{2}(\beta^2 - \lambda^2 + 1)\csc^2\big(\tfrac{1}{2}\sqrt{-\beta^2 + \lambda^2 - 1}\,\xi\big)\,, \tag{36}$$
$$u_{42} = -(\beta^2 - \lambda^2 + 1) - \tfrac{3}{2}(\beta^2 - \lambda^2 + 1)\,\mathrm{csch}^2\big(\tfrac{1}{2}\sqrt{-\beta^2 + \lambda^2 - 1}\,\xi\big)\,. \tag{37}$$

Case 5

$$u_{51} = (\beta^2 - \lambda^2 + 1)\big[-1 + \tfrac{3}{8}\sec^2\big(\tfrac{1}{4}\sqrt{-\beta^2 + \lambda^2 - 1}\,\xi\big)\csc^2\big(\tfrac{1}{4}\sqrt{-\beta^2 + \lambda^2 - 1}\,\xi\big)\,, \tag{38}$$
$$u_{52} = (\beta^2 - \lambda^2 + 1)\big[-1 + \tfrac{3}{8}\,\mathrm{sech}^2\big(\tfrac{1}{4}\sqrt{-\beta^2 + \lambda^2 - 1}\,\xi\big)\,\mathrm{csch}^2\big(\tfrac{1}{4}\sqrt{-\beta^2 + \lambda^2 - 1}\,\xi\big)\,. \tag{39}$$

Case 6

$$u_{61} = -\tfrac{3}{2}(\beta^2 - \lambda^2 + 1)\csc^2\big(\tfrac{1}{2}\sqrt{\beta^2 - \lambda^2 + 1}\,\xi\big)\,, \tag{40}$$
$$u_{62} = \tfrac{3}{2}(\beta^2 - \lambda^2 + 1)\,\mathrm{csch}^2\big(\tfrac{1}{2}\sqrt{\beta^2 - \lambda^2 + 1}\,\xi\big)\,. \tag{41}$$

Case 7

$$u_{71} = -\tfrac{3}{8}(\beta^2 - \lambda^2 + 1)\sec^2\big(\tfrac{1}{4}\sqrt{\beta^2 - \lambda^2 + 1}\,\xi\big)\csc^2\big(\tfrac{1}{4}\sqrt{\beta^2 - \lambda^2 + 1}\,\xi\big)\,, \tag{42}$$
$$u_{72} = \tfrac{3}{8}(\beta^2 - \lambda^2 + 1)\,\mathrm{sech}^2\big(\tfrac{1}{4}\sqrt{\beta^2 - \lambda^2 + 1}\,\xi\big)\,\mathrm{csch}^2\big(\tfrac{1}{4}\sqrt{\beta^2 - \lambda^2 + 1}\,\xi\big)\,. \tag{43}$$

Case 8

$$u_{81} = -(\beta^2 - \lambda^2 + 1) + \tfrac{3}{2}(\beta^2 - \lambda^2 + 1)\sec^2\big(\tfrac{1}{2}\sqrt{-\beta^2 + \lambda^2 - 1}\,\xi\big)\,, \tag{44}$$
$$u_{82} = -(\beta^2 - \lambda^2 + 1) + \tfrac{3}{2}(\beta^2 - \lambda^2 + 1)\,\mathrm{sech}^2\big(\tfrac{1}{2}\sqrt{-\beta^2 + \lambda^2 - 1}\,\xi\big)\,. \tag{45}$$

Case 9

$$u_{91} = -6\frac{1}{(x + \beta y \pm \sqrt{\beta^2 + 1}\,t)^2}\,, \tag{46}$$

where $\xi = x + \beta y - \lambda t$.

Remark 1 It is easily seen that the obtained solutions in cases 1, 4, 6, and 8 are just the solutions (22) and (23) obtained by Senthilvelanin in Ref. [14]. But to our knowledge, the obtained solutions of Eq. (12) in cases 5, 7 were not found before.

Example 2 Consider the system of variant Boussinesq equations[15,16]

$$H_t + (Hu)_x + u_{xxx} = 0\,, \tag{47}$$
$$u_t + H_x + uu_x = 0\,. \tag{48}$$

Using $H(x,t) = \phi(\xi)$, $u(x,t) = \theta(\xi)$, $\xi = x - \lambda t$, we have

$$-\lambda\phi' + \phi'\theta + \phi\theta' + \theta''' = 0\,, \tag{49}$$
$$-\lambda\theta' + \phi' + \theta\theta' = 0\,. \tag{50}$$

By balancing the highest-order derivative term with the nonlinear term in Eqs. (47) and (48) or Eqs. (49) and (50), we get $m = 2$, $n = 1$, and so we take the following ansätze

$$\phi = A_0 + A_1\omega + \frac{B_1}{\omega} + C_1\sqrt{R + \omega^2} + A_2\omega^2 + \frac{B_2}{\omega^2} + C_2\omega\sqrt{R + \omega^2}\,, \tag{51}$$
$$\theta = a_0 + a_1\omega + \frac{b_1}{\omega} + c_1\sqrt{R + \omega^2}\,, \tag{52}$$

where $\omega = \omega(\xi)$ satisfies Eq. (4), and $R, A_0, A_1, A_2, B_1, B_2, C_1, C_2, a_0, a_1, b_1, c_1$ are constants to be determined later. Substituting Eqs. (51) and (52) into Eqs. (49) and (50) and considering Eq. (4) we have the following system:

$$-3B_2Rb_1 - 6b_1R^3 = 0\,, \tag{53}$$
$$4C_2c_1R + A_2b_1 + A_1a_0 + A_0a_1 + 3A_2Ra_1 - \lambda A_1 + 8a_1R = 0\,, \tag{54}$$
$$-2B_1Rb_1 + 2\lambda B_2R - 2B_2Ra_0 = 0\,, \tag{55}$$
$$2A_2a_0 - 2\lambda A_2 + 2A_1a_1 + 2C_1c_1 = 0\,, \tag{56}$$
$$-B_2c_1 = 0\,, \tag{57}$$
$$-2\lambda C_2 + 2C_2a_0 + 2C_1a_1 + 2A_1c_1 = 0\,, \tag{58}$$
$$-2B_2Rc_1 = 0\,, \tag{59}$$
$$2A_2Ra_0 + 2C_1c_1R - 2\lambda A_2R + 2A_1Ra_1 = 0\,, \tag{60}$$
$$A_1Rc_1 + C_1a_1R + C_2a_0R - \lambda C_2R = 0\,, \tag{61}$$
$$-C_1b_1R - B_1Rc_1 = 0\,, \tag{62}$$
$$C_2b_1 + 2C_2a_1R + A_0c_1 + 2A_2Rc_1 + 5c_1R - \lambda C_1 + C_1a_0 = 0\,, \tag{63}$$
$$-B_2Ra_1 - 8b_1R^2 + \lambda B_1R - 3B_2b_1 - A_0b_1R - B_1Ra_0 = 0\,, \tag{64}$$
$$6c_1 + 3C_2a_1 + 3A_2c_1 - 2B_1b_1 + 2\lambda B_2 - 2B_2a_0 = 0\,, \tag{65}$$

$$-\lambda A_1R + A_1Ra_0 + \lambda B_1 + 2a_1R^2 - B_1a_0 + A_0a_1R + A_2Rb_1 - B_2a_1 - 2b_1R - A_0b_1 + C_2c_1R^2 = 0\,, \tag{66}$$

$$3A_2a_1 + 3C_2c_1 + 6a_1 = 0\,, \tag{67}$$

$$-2B_2R - {b_1}^2R = 0\,, \tag{68}$$

$${a_1}^2 + {c_1}^2 + 2A_2 = 0\,, \tag{69}$$

$$\lambda b_1R - B_1R - a_0b_1R = 0\,, \tag{70}$$

$$c_1a_1R + C_2R = 0\,, \tag{71}$$

$$-c_1b_1R = 0\,, \tag{72}$$

$$a_0a_1 - \lambda a_1 + A_1 = 0\,, \tag{73}$$

$$C_1 + a_0c_1 - \lambda c_1 = 0\,, \tag{74}$$

$$2C_2 + 2a_1c_1 = 0\,, \tag{75}$$

$$-2B_2 - {b_1}^2 = 0\,, \tag{76}$$

$$\lambda b_1 - B_1 + A_1R + a_0a_1R - \lambda a_1R - a_0b_1 = 0\,, \tag{77}$$

$${c_1}^2R + {a_1}^2R + 2A_2R = 0\,. \tag{78}$$

From Eqs. (53) $\sim$ (78), we find

Case 1 $A_1 = B_1 = C_1 = B_2 = C_2 = b_1 = c_1 = 0$, $a_0 = \lambda$, $a_1 = \pm 2$, $A_0 = -2R$, $A_2 = -2$.

Case 2 $A_1 = B_1 = C_1 = C_2 = c_1 = 0$, $a_0 = \lambda$, $a_1 = \pm 2$, $b_1 = \mp 2R$, $A_0 = -4R$, $A_2 = -2$, $B_2 = -2R^2$.

Case 3 $A_1 = B_1 = C_1 = B_2 = b_1 = 0$, $a_0 = \lambda$, $a_1 = \pm 1$, $c_1 = \pm 1$, $A_0 = -R$, $A_2 = -1$, $C_2 = -1$.

Case 4 $A_1 = B_1 = C_1 = B_2 = b_1 = 0$, $a_0 = \lambda$, $a_1 = \pm 1$, $c_1 = \mp 1$, $A_0 = -R$, $A_2 = -1$, $C_2 = 1$.

Case 5 $A_1 = B_1 = C_1 = A_2 = C_2 = a_1 = c_1 = 0$, $a_0 = \lambda$, $b_1 = \pm 2R$, $A_0 = -2R$, $B_2 = -2R^2$.

Case 6 $A_1 = B_1 = C_1 = B_2 = C_2 = a_1 = b_1 = 0$, $a_0 = \lambda$, $c_1 = \pm 2$, $A_0 = -R$, $A_2 = -2$.

Case 7 $A_1 = B_1 = C_1 = A_0 = C_2 = c_1 = 0$, $a_0 = \lambda$, $a_1 = \pm 2$, $b_1 = \pm 2R$, $A_2 = -2$, $B_2 = -2R^2$.

Case 8 $A_0 = A_1 = B_1 = C_1 = B_2 = C_2 = b_1 = c_1 = R = 0$, $a_0 = \lambda$, $a_1 = \pm 2$, $A_2 = -2$.

Therefore, combining Eqs. (7), (9) and (10) along with cases $1 \sim 8$, we obtain the travelling wave solutions of the system of variant Boussinesq equations as follows:

Case 1

$$H_{11} = -2R - 2R\tan^2(\sqrt{R}\,\xi)\,, \tag{79}$$

$$u_{11} = \lambda \pm 2\sqrt{R}\tan(\sqrt{R}\,\xi)\,, \tag{80}$$

$$H_{12} = -2R + 2R\tanh^2(\sqrt{-R}\,\xi)\,, \tag{81}$$

$$u_{12} = \lambda \pm 2\sqrt{-R}\tanh(\sqrt{-R}\,\xi)\,. \tag{82}$$

Case 2

$$H_{21} = -2R\sec^2(\sqrt{R}\,\xi)\csc^2(\sqrt{R}\,\xi)\,, \tag{83}$$

$$u_{21} = \lambda \mp 2\sqrt{R}\tan(\sqrt{R}\,\xi) \pm 2\sqrt{R}\cot(\sqrt{R}\,\xi)\,, \tag{84}$$

$$H_{22} = 2R\,\mathrm{sech}^2(\sqrt{R}\,\xi)\,\mathrm{csch}^2(\sqrt{R}\,\xi)\,, \tag{85}$$

$$u_{22} = \lambda \mp 2\sqrt{-R}\tanh(\sqrt{-R}\,\xi) \pm 2\,\frac{R\coth(\sqrt{-R}\,\xi)}{\sqrt{-R}}\,. \tag{86}$$

Case 3

$$H_{31} = -R - R\tan^2(\sqrt{R}\,\xi) - R\tan(\sqrt{R}\,\xi)\sec(\sqrt{R}\,\xi)\,, \tag{87}$$

$$u_{31} = \lambda \pm \sqrt{R}\tan(\sqrt{R}\,\xi) \pm \sqrt{R}\sec(\sqrt{R}\,\xi)\,, \tag{88}$$

$$H_{32} = -R + R\tanh^2(\sqrt{-R}\,\xi) - \mathrm{i}R\tanh(\sqrt{-R}\,\xi)\,\mathrm{sech}(\sqrt{-R}\,\xi)\,, \tag{89}$$

$$u_{32} = \lambda \mp \sqrt{-R}\tanh(\sqrt{-R}\,\xi) \pm \mathrm{i}\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\,\xi)\,. \tag{90}$$

Case 4

$$H_{41} = -R - R\tan^2(\sqrt{R}\,\xi) + R\tan(\sqrt{R}\,\xi)\sec(\sqrt{R}\,\xi)\,, \tag{91}$$

$$u_{41} = \lambda \mp \sqrt{R}\tan(\sqrt{R}\,\xi) \pm \sqrt{R}\sec(\sqrt{R}\,\xi)\,, \tag{92}$$

$$H_{42} = -R + R\tanh^2(\sqrt{-R}\,\xi) + \mathrm{i}R\tanh(\sqrt{-R}\,\xi)\,\mathrm{sech}(\sqrt{-R}\,\xi)\,, \tag{93}$$

$$u_{42} = \lambda \pm \sqrt{-R}\tanh(\sqrt{-R}\,\xi) \pm \mathrm{i}\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\,\xi)\,. \tag{94}$$

Case 5

$$H_{51} = -2R\csc^2(\sqrt{R}\,\xi)\,, \tag{95}$$

$$u_{51} = \lambda \pm 2\sqrt{R}\cot(\sqrt{R}\,\xi)\,, \tag{96}$$

$$H_{52} = 2R\,\mathrm{csch}^2(\sqrt{R}\,\xi)\,, \tag{97}$$

$$u_{52} = \lambda \pm 2\,\frac{R\coth(\sqrt{-R}\,\xi)}{\sqrt{-R}}\,. \tag{98}$$

Case 6

$$H_{61} = R - 2R\sec^2(\sqrt{R}\,\xi)\,, \tag{99}$$

$$u_{61} = \lambda \pm 2\sqrt{R}\sec(\sqrt{R}\,\xi)\,, \tag{100}$$

$$H_{62} = R - 2R\sec^2(\sqrt{R}\,\xi)\,, \tag{101}$$

$$u_{62} = \lambda \pm 2\,\mathrm{i}\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\,\xi)\,. \tag{102}$$

Case 7

$$H_{71} = 4R - 2R\sec^2(\sqrt{R}\,\xi)\csc^2(\sqrt{R}\,\xi)\,, \tag{103}$$

$$u_{71} = \lambda \pm 2\sqrt{R}\sec(\sqrt{R}\,\xi)\csc(\sqrt{R}\,\xi)\,, \tag{104}$$

$$H_{72} = 4R + 2R\,\mathrm{sech}^2(\sqrt{-R}\,\xi)\,\mathrm{csch}^2(\sqrt{-R}\,\xi)\,, \tag{105}$$

$$u_{72} = \lambda \pm 2\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\,\xi)\,\mathrm{csch}(\sqrt{-R}\,\xi)\,. \tag{106}$$

Case 8

$$H_{81} = -2/\xi^2, \qquad u_{81} = \lambda \pm 2/\xi\,. \tag{107}$$

Remark 2 Our obtained solitary wave solutions in cases 1, $3 \sim 6$ for Eqs. (47) and (48) include the solutions in Ref. [14] and [15]. And to our knowledge, the solutions in cases 2 and 7 were not found before.

Example 3 Consider the generalized BBM equation

$$u_t + au_x + bu^pu_x + cu^{2p}u_x - \delta u_{xxt} = 0\,, \tag{108}$$

where a, b, c, and δ are constants with nonlinear term and dispersion coefficients, $c \neq 0$ and $\delta \neq 0$. In the study of many problems of mechanical and physical sciences, various BBM equations or the regularized long-wave equation have been proposed and well documented.[17–25] In Ref. [25], Zhang *et al.* considered the general form of the BBM equation

$$u_t + au_x + buu_x - \delta u_{xxt} = 0 \tag{109}$$

and obtained the solitary-wave solutions of Eq. (109) by variable-coefficient balancing-act method. It is not difficult to see that equation (108) with $p = 1/2$, $b = 0$ and $c = b$ is just Eq. (109). For convenience, we just give the solutions of Eq. (108) for $p = 1$ and $p = 2$ as follows:

Case 1

$$p=1,\quad A_0=-\frac{b}{2c},\quad A_1=\pm\frac{\sqrt{-6\,c\delta\,a+b^2\delta}}{2c},$$

$$B_1=0,\qquad C_1=\pm\frac{\sqrt{-6\,c\delta\,a+b^2\delta}}{2c},$$

$$R=\frac{b^2}{\delta(6ac-b^2)},\qquad \lambda=\frac{6ac-b^2}{6c},$$

$$u_{11}=-\frac{b}{2c}\pm\frac{\mathrm{i}b}{2c}\tan\left(\sqrt{\frac{b^2}{\delta\,(6\,ac-b^2)}}\,\breve{\xi}\right)\pm\frac{\mathrm{i}b}{2c}\sec\left(\sqrt{\frac{b^2}{\delta\,(6\,ac-b^2)}}\,\breve{\xi}\right),\tag{110}$$

$$u_{12}=-\frac{b}{2c}\mp\frac{\mathrm{i}b}{2c}\tanh\left(\sqrt{\frac{b^2}{\delta\,(6\,ac-b^2)}}\,\breve{\xi}\right)\mp\frac{\mathrm{i}b}{2c}\mathrm{sech}\left(\sqrt{\frac{b^2}{\delta\,(6\,ac-b^2)}}\breve{\xi}\right).\tag{111}$$

Case 2

$$p=1,\qquad A_0=-\frac{b}{2c},\qquad A_1=0,$$

$$B_1=\pm\frac{b^2}{4\sqrt{-6\,c\delta\,a+b^2\delta c}},\quad C_1=0,$$

$$R=\frac{b^2}{4\delta\,(6\,ac-b^2)},\qquad \lambda=\frac{6\,ac-b^2}{6c},$$

$$u_{21}=-\frac{b}{2c}\pm\frac{\mathrm{i}b}{2c}\cot\left(\frac{1}{2}\sqrt{\frac{b^2}{\delta\,(6\,ac-b^2)}}\,\breve{\xi}\right),\tag{112}$$

$$u_{22}=-\frac{b}{2c}\pm\frac{b}{2c}\coth\left(\frac{1}{2}\sqrt{-\frac{b^2}{\delta\,(6\,ac-b^2)}}\,\breve{\xi}g\right).\tag{113}$$

Case 3

$$p=1,\qquad A_0=-\frac{b}{2c},$$

$$A_1=\pm\frac{\sqrt{-6\,c\delta\,a+b^2\delta}}{2c},\quad B_1=0,$$

$$C_1=\mp\frac{\sqrt{-6\,c\delta\,a+b^2\delta}}{2c},$$

$$R=\frac{b^2}{\delta\,(6\,ac-b^2)},\quad \lambda=\frac{6\,ac-b^2}{6c},$$

$$u_{31}=-\frac{b}{2c}\pm\frac{\mathrm{i}b}{2c}\tan\left(\sqrt{\frac{b^2}{\delta\,(6\,ac-b^2)}}\,\breve{\xi}\right)\mp\frac{\mathrm{i}b}{2c}\sec\left(\sqrt{\frac{b^2}{\delta\,(6\,ac-b^2)}}\,\breve{\xi}\right),\tag{114}$$

$$u_{32}=-\frac{b}{2c}\mp\frac{b}{2c}\tanh\left(\sqrt{-\frac{b^2}{\delta\,(6\,ac-b^2)}}\,\breve{\xi}\right)\mp\frac{\mathrm{i}b}{2c}\mathit{sech}\left(\sqrt{-\frac{b^2}{\delta\,(6\,ac-b^2)}}\,\breve{\xi}\right).\tag{115}$$

Case 4

$$p=1,\qquad A_0=-\frac{b}{2c},$$

$$A_1=\pm\frac{\sqrt{-6\,c\delta\,a+b^2\delta}}{c},$$

$$B_1=0,\qquad C_1=0,$$

$$R=\frac{b^2}{4\delta\,(6\,ac-b^2)},\quad \lambda=\frac{6\,ac-b^2}{6c},$$

$$u_{41}=-\frac{b}{2c}\mp\frac{\mathrm{i}b}{2c}\tan\left(\frac{1}{2}\sqrt{\frac{b^2}{\delta\,(6\,ac-b^2)}}\,\breve{\xi}\right),\tag{116}$$

$$u_{42}=-\frac{b}{2c}\pm\frac{b}{2c}\tanh\left(\sqrt{-\frac{b^2}{2\delta\,(6\,ac-b^2)}}\,\breve{\xi}\right),\tag{117}$$

where $\breve{\xi}=x-(6\,ac-b^2)t/6c$.

Case 5

$$p=2,\qquad A_0=-\frac{5b}{8c},\qquad A_1=\pm\frac{\sqrt{25\,b^2\delta-240\,c\delta\,a}}{8c},\qquad B_1=0,\qquad C_1=0,$$

$$R=\frac{5b^2}{\delta\,(48\,ca-5\,b^2)},\qquad \lambda=\frac{48\,ca-5\,b^2}{48c},$$

$$u_{51}=\left\{-\frac{5b}{8c}\pm\frac{5\mathrm{i}b}{8c}\tan\left(\sqrt{\frac{5b^2}{\delta\,(48\,ca-5\,b^2)}}(x-\lambda t)\right)\right\}^{1/2},\tag{118}$$

$$u_{52}=\left\{-\frac{5b}{8c}\mp\frac{5b}{8c}\tanh\left(\sqrt{-\frac{5b^2}{\delta\,(48\,ca-5\,b^2)}}(x-\lambda t)\right)\right\}^{1/2}.\tag{119}$$

Case 6

$$p=2,\qquad A_0=-\frac{5b}{8c},\qquad A_1=\pm\frac{\sqrt{-960\,c\delta\,a+125\,b^2\delta}}{16c},\qquad \lambda=\frac{192\,ca-25\,b^2}{192c},$$

$$C_1=0,\qquad B_1=\pm\frac{25b^2}{16c\sqrt{-960\,c\delta\,a+125\,b^2\delta}},\qquad R=-\frac{5b^2}{\delta\,(192\,ca-25\,b^2)},$$

$$u_{61}=\left\{-\frac{5b}{8c}\pm\frac{5b}{16c}\sec\big(\sqrt{R}(x-\lambda t)\big)\csc\big(\sqrt{R}(x-\lambda t)\big)\right\}^{1/2},\tag{120}$$

$$u_{62}=\left\{-\frac{5b}{8c}\pm\frac{5\mathrm{i}b}{16c}\mathrm{sech}\big(\sqrt{-R}(x-\lambda t)\big)\mathrm{csch}\big(\sqrt{-R}(x-\lambda t)\big)\right\}^{1/2}.\tag{121}$$

Case 7

$$p=2,\qquad A_0=-\frac{5b}{8c},\qquad A_1=\pm\frac{\sqrt{25\,b^2\delta-240\,c\delta\,a}}{8c},\qquad B_1=\pm\frac{25b^2}{32c\sqrt{25\,b^2\delta-240\,c\delta\,a}}\,,$$

$$C_1=0,\qquad R=\frac{5b^2}{4\delta\,(48\,ca-5\,b^2)},\qquad \lambda=\frac{48\,ca-5\,b^2}{48c}\,,$$

$$u_{71}=\Big\{-\frac{5b}{8c}\pm\frac{5\mathrm{i}b}{16c}\sec\big(\sqrt{R}(x-\lambda t)\big)\csc\big(\sqrt{R}(x-\lambda t)\big)\Big\}^{1/2}, \tag{122}$$

$$u_{72}=\Big\{-\frac{5b}{8c}\pm\frac{5b}{16c}\mathrm{sech}\big(\sqrt{-R}(x-\lambda t)\big)\mathrm{csch}\big(\sqrt{-R}(x-\lambda t)\big)\Big\}^{1/2}. \tag{123}$$

Case 8

$$p=2,\qquad A_0=-5b/8c,\qquad A_1=0,\qquad B_1=\pm 26b^2/8c\sqrt{25\,b^2\delta-240\,c\delta\,a}\,,$$

$$C_1=0,\qquad R=5b^2/\delta\,(48\,ca-5\,b^2),\qquad \lambda=(48\,ca-5\,b^2)/48c\,,$$

$$u_{81}=\big\{-5b/8c\pm(5\mathrm{i}b/8c)\cot\big(\sqrt{5b^2/\delta\,(48\,ca-5\,b^2)}(x-\lambda t)\big)\big\}^{1/2}, \tag{124}$$

$$u_{82}=\big\{-5b/8c\pm(5b/8c)\coth\big(\sqrt{-5b^2/\delta\,(48\,ca-5\,b^2)}(x-\lambda t)\big)\big\}^{12}. \tag{125}$$

Remark 3 So the generalized extended tanh-fun-ction method is more powerful than the method by Fan and Yan[9,10] and the method by S.A. Elwakil.[11] It is easily seen that the above solutions are not included in Ref. [25], and to our knowledge, the solutions of Eq. (108) in cases $5\sim 8$ were not found before.

4 Conclusions

In this paper, by introducing a proper transformation, the extended tanh-method is further extended into the nonlinear evolution equations whose balancing numbers may be any nonzero real numbers. On the other hand, by making use of a new general ansätze, we explore some new travelling wave solutions for certain physically interesting models described by nonlinear evolution equations. Particularly, for our analysis we have considered the (2+1)-dimensional generalization of Boussinesq equation[13] and the system of variant Boussinesq equations[14,15] and the generalized BBM[17−25] equation. It should be noted that this method is a combination of extended tanh-method and its modification, and the results by this method not only cover the solutions obtained by the above method, but also some new formal solutions which cannot be obtained by extended tanh-method and its modification. Moreover, because it is computerizable, the method allows us to perform the complicated and tedious algebraic calculations on a computer.

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