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EXACT SOLUTIONS FOR A FAMILY OF VARIABLE-COEFFICIENT "REACTION–DUFFING" EQUATIONS VIA THE BÄCKLUND TRANSFORMATION

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The homogeneous balance method is extended and applied to a class of variable-coefficient "reaction–duffing" equations, and a Bäcklund transformation (BT) is obtained. Based on the BT, a nonlocal symmetry and several families of exact solutions of this equation are obtained, including soliton solutions that have important physical significance. The Fitzhugh–Nagumo and Chaffee–Infante equations are also considered as special cases.

Keywords: "reaction–duffing" equation, Bäcklund transformation, symmetry, exact solution, soliton solution

1. Introduction

Searching for exact solutions of nonlinear evolution equations plays an important role in soliton theory. Many powerful methods have been presented including the sine-cosine method, the homogeneous balance method, the Bäcklund transformation (BT), and Hirota's method [1]–[6]. It is important to seek BTs of nonlinear partial differential equations in mathematical physics [1], [6]. By virtue of a BT, an exact solution can be readily obtained.

We consider the variable-coefficient "reaction–duffing" equation

$$u_t + h_2(t)u_x + h_1(t)(Au_{xx} + Bu^3 + Eu^2 + Du) = 0, \tag{1}$$

where $h_1(t)$ and $h_2(t)$ are arbitrary functions of t and A, B, E, and D are constants. With different functions $h_1(t)$ and $h_2(t)$ and different values of the constants A, B, C, and D, Eq. (1) becomes many famous wave equations, such as the Newell–Whitehead equation, the Fitzhugh–Nagumo equation, the Chaffee–Infante equation, the KPP equation, and the Huxley equation [7]–[10].

In this paper, we extend the homogeneous balance method [7], [8], [11], [12] to Eq. (1) to find the BT of Eq. (1). Based on the BT, we derive symmetry and explicit exact solutions for Eq. (1), which contain soliton solutions of important physical significance and other types of new exact solutions. Exact solutions of other equations are also obtained.

2. Bäcklund transformation and symmetry

To make the two terms u_{xx} and u^3 balance in Eq. (1), we suppose that this equation has the formal solution

$$u(x,t) = f'(\phi)\phi_x + u_0 \equiv f'\phi_x + u_0, \tag{2}$$

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where the prime denotes $d/d\phi$ and $f(\phi)$, $\phi(x,t)$, and $u_0 = u_0(x,t)$ are unknown functions to be determined in what follows. Substituting Eq. (2) in Eq. (1), we obtain

$$
\begin{aligned}
u_t + h_2(t)u_x + h_1(t)(Au_{xx} + Bu^3 + Eu^2 + Du) = \\
= (Af''' + Bf'^3)h_1\phi_x^3 + 3Ah_1 f''\phi_x\phi_{xx} + f''\phi_x\phi_t + 3Bh_1 f'^2 u_0\phi_x^2 + Eh_1 f'^2\phi_x^2 + \\
+ h_2 f''\phi_x^2 + (\phi_{xt} + Ah_1\phi_{xxx} + 3Bh_1u_0^2\phi_x + 2Eh_1u_0\phi_x + Dh_1\phi_x + h_2\phi_{xx})f' + \\
+ u_{0t} + h_2u_{0x} + h_1(Au_{0xx} + Bu_0^3 + Eu_0^2 + Du_0) = 0.
\end{aligned} \tag{3}
$$

To determine the function $f(\phi)$, we set the coefficient of ϕ_x^3 equal to zero. Taking $h_1(t) \not\equiv 0$ into account, we then obtain

$$
Af''' + Bf'^3 = 0, \tag{4}
$$

which has the special solution

$$
f = \mu\sqrt{-\frac{2A}{B}}\log\phi, \quad \mu = \pm 1, \quad A < 0, \quad B > 0, \tag{5}
$$

which is useful in what follows. The relation

$$
f'^2 = -\mu\sqrt{-\frac{2A}{B}}f'' \tag{6}
$$

can be readily obtained from Eq. (5). Substituting Eqs. (5) and (6) in Eq. (3) yields

$$
\begin{aligned}
u_t + h_2(t)u_x + h_1(t)(Au_{xx} + Bu^3 + Eu^2 + Du) = \\
= \phi_x\left[3Ah_1\phi_{xx} + \phi_t - \mu\sqrt{-\frac{2A}{B}}(3Bh_1u_0\phi_x + Eh_1\phi_x) + h_2\phi_x\right]f'' + \\
+ (\phi_{xt} + Ah_1\phi_{xxx} + 3Bh_1u_0^2\phi_x + 2Eh_1u_0\phi_x + Dh_1\phi_x + h_2\phi_{xx})f' + \\
+ u_{0t} + h_2u_{0x} + h_1(Au_{0xx} + Bu_0^3 + Eu_0^2 + Du_0) = 0.
\end{aligned} \tag{7}
$$

Setting the coefficients of f'', f', and the remaining term to zero yields the set of equations

$$
3Ah_1\phi_{xx} + \phi_t - \mu\sqrt{-\frac{2A}{B}}(3Bh_1u_0\phi_x + Eh_1\phi_x) + h_2\phi_x = 0, \tag{8}
$$

$$
\phi_{xt} + Ah_1\phi_{xxx} + 3Bh_1u_0^2\phi_x + 2Eh_1u_0\phi_x + Dh_1\phi_x + h_2\phi_{xx} = 0, \tag{9}
$$

$$
u_{0t} + h_2u_{0x} + h_1(Au_{0xx} + Bu_0^3 + Eu_0^2 + Du_0) = 0. \tag{10}
$$

From (10), we know that u_0 is a solution of (1). Therefore, substituting (5) in (2), we obtain

$$
u(x,t) = \mu\sqrt{-\frac{2A}{B}}\frac{\partial}{\partial x}\log\phi + u_0 = \mu\sqrt{-\frac{2A}{B}}\frac{\phi_x}{\phi} + u_0. \tag{11}
$$

It is easily seen that if (ϕ, u_0) satisfies (8)–(10), then u defined by (11) satisfies (1) by using (7)–(10). We therefore call the transformation obtained in (11) the BT of (1).

Setting $\phi_x = \sigma$ in Eq. (9), we have

$$
\sigma_t + [h_2\partial + h_1(A\partial^2 + 3Bu_0^2 + 2Eu_0 + D)]\sigma = 0, \tag{12}
$$

where $\partial = \partial/\partial x$ and $\partial^2 = \partial^2/\partial x^2$. Equation (12) shows that σ is a nonlocal symmetry of Eq. (1).

3. Explicit and exact solutions

To find exact solutions for Eq. (1), we now consider BT (11). If we can obtain the solutions (ϕ, u_0) of (8)–(10), then we can obtain the corresponding solution of (1) by using (11). Therefore, we first reduce Eqs. (8)–(10) to an equation that can be solved using the known method. Differentiating Eq. (8) with respect to t and using Eq. (9), we obtain the linear differential equation

$$2A\phi_{xxx} - \mu\sqrt{-\frac{2A}{B}}(3Bu_0 + E)\phi_{xx} - \left(3Bu_0^2 + 2Eu_0 + D + 3B\mu\sqrt{-\frac{2A}{B}}u_{0x}\right)\phi_x = 0.$$

Setting $u_0 = \text{const}$, we can write this equation as the third-order linear ordinary differential equation with constant coefficients

$$2A\phi_{xxx} - \mu\sqrt{-\frac{2A}{B}}(3Bu_0 + E)\phi_{xx} - (3Bu_0^2 + 2Eu_0 + D)\phi_x = 0. \tag{13}$$

The characteristic equation for (13) is

$$2A\lambda^3 - \mu\sqrt{-\frac{2A}{B}}(3Bu_0 + E)\lambda^2 - (3Bu_0^2 + 2Eu_0 + D)\lambda = 0. \tag{14}$$

We now discuss the exact solutions of Eq. (1) in the following several cases.

Case 1. When $u_0 = 0$, which is a trivial solution of (1), and $D \neq 0$, Eq. (14) has the three roots

$$\lambda_1 = 0, \qquad \lambda_2 = -\sqrt{-\frac{1}{8AB}}\left(\mu E + \sqrt{E^2 - 4BD}\right),$$

$$\lambda_3 = -\sqrt{-\frac{1}{8AB}}\left(\mu E - \sqrt{E^2 - 4BD}\right).$$

The solutions of Eq. (13) can therefore be written as

$$\phi(x,t) = \alpha_1(t) + \alpha_2(t)e^{\lambda_2 x} + \alpha_3(t)e^{\lambda_3 x}. \tag{15}$$

Substituting (15) in Eqs. (8) and (9) yields

$$\alpha_1'(t) + [\alpha_2'(t) - k_2(t)\alpha_2(t)]e^{\lambda_2 x} + [\alpha_3'(t) - k_3(t)\alpha_3(t)]e^{\lambda_3 x} = 0,$$

where

$$k_2(t) = \frac{1}{4B}\left(E^2 - 6BD + \mu E\sqrt{E^2 - 4BD}\right)h_1(t) + \sqrt{-\frac{1}{8AB}}\left(\mu E + \sqrt{E^2 - 4BD}\right)h_2(t),$$

$$k_3(t) = \frac{1}{4B}\left(E^2 - 6BD - \mu E\sqrt{E^2 - 4BD}\right)h_1(t) + \sqrt{-\frac{1}{8AB}}\left(\mu E - \sqrt{E^2 - 4BD}\right)h_2(t).$$

Because 1, $e^{\lambda_2 x}$, and $e^{\lambda_3 x}$ are linearly independent, we have the system of equations

$$\alpha_1'(t) = 0, \qquad \alpha_2'(t) - k_2(t)\alpha_2(t) = 0, \qquad \alpha_3'(t) - k_3(t)\alpha_3(t) = 0,$$

which has the general solution

$$\alpha_1(t) = c_1, \qquad \alpha_2(t) = c_2 \exp\left[\int k_2(t)\,dt\right], \qquad \alpha_3(t) = c_3 \exp\left[\int k_3(t)\,dt\right], \tag{16}$$

where c_1, c_2, and c_3 are arbitrary constants. From Eqs. (11), (15), and (16), a family of solutions of Eq. (1) is obtained as

$$u(x,t) = \mu\sqrt{-\frac{2A}{B}}\frac{c_2\lambda_2 \exp\left[\lambda_2 x + \int k_2(t)\,dt\right] + c_3\lambda_3 \exp\left[\lambda_3 x + \int k_3(t)\,dt\right]}{c_1 + c_2 \exp\left[\lambda_2 x + \int k_2(t)\,dt\right] + c_3 \exp\left[\lambda_3 x + \int k_3(t)\,dt\right]}. \tag{17}$$

Case 2. When $u_0 = 0$ and $D = 0$, three roots of Eq. (14) are given by

$$\lambda_1 = \lambda_2 = 0, \qquad \lambda_3 = -\mu E\sqrt{-\frac{1}{2AB}}.$$

The general solution of Eq. (13) is therefore written as

$$\phi(x,t) = \alpha_1(t) + \alpha_2(t)x + \alpha_3(t)\exp\left(-\mu E\sqrt{-\frac{1}{2AB}}x\right). \tag{18}$$

As in the previous case, we easily find

$$\begin{aligned} \alpha_1(t) &= c_1\mu E\sqrt{-\frac{1}{2AB}}\int h_1\,dt - \int h_2\,dt, \qquad \alpha_2(t) = c_1, \\ \alpha_3(t) &= c_3\exp\left[\frac{E^2}{2B}\int h_1(t)\,dt + \mu E\sqrt{-\frac{1}{2AB}}\int h_2(t)\,dt\right]. \end{aligned} \tag{19}$$

Substituting (18) and (19) in (11), we obtain

$$u(x,t) = \mu\sqrt{-\frac{2A}{B}}\frac{c_1 - c_3\mu E\sqrt{-1/(2AB)}\,\alpha_3(t)\exp(-\mu E\sqrt{-1/(2AB)}\,x)}{c_1x + c_1\mu E\sqrt{-1/(2AB)}\int h_1\,dt - \int h_2\,dt + c_3\alpha_3(t)\exp(-\mu E\sqrt{-1/(2AB)}\,x)}. \tag{20}$$

Case 3. When $u_0 = -E/B$ and $D = 0$, Eq. (14) has the three roots

$$\lambda_1 = 0, \qquad \lambda_2 = \lambda_3 = \mu E\sqrt{-\frac{1}{2AB}}.$$

The solution of Eq. (13) is therefore written as

$$\phi(x,t) = \alpha_1(t) + [\alpha_2(t) + \alpha_3(t)x]\exp\left(\mu E\sqrt{-\frac{1}{2AB}}x\right). \tag{21}$$

As in the previous cases, we have

$$\alpha_1(t) = c_1, \qquad \alpha_2(t) = \int k_2(t)\,dt\,\exp\left\{\int k_3(t)\,dt\right\}, \qquad \alpha_3(t) = c_3\exp\left\{\int k_3(t)\,dt\right\}, \tag{22}$$

where

$$k_2(t) = \mu E\sqrt{-\frac{2A}{B}}h_1(t) - h_2(t)$$

and

$$k_3(t) = -\frac{E^2}{2B}h_1(t) - \mu E\sqrt{-\frac{1}{2AB}}h_2(t).$$

We set $\mu E\sqrt{-1/(2AB)} = M$. From (11), (21), and (22), the solution of Eq. (1) is therefore written as

$$u(x,t) = \mu\sqrt{-\frac{2A}{B}}\frac{\left[c_3 + M\left(\int k_2(t)\,dt + c_3x\right)\right]\exp\left[Mx + \int k_3\,dt\right]}{c_1 + \left(\int k_2(t)\,dt + c_3x\right)\exp\left[Mx + \int k_3\,dt\right]} - \frac{E}{B}. \tag{23}$$

Case 4. When

$$u_0 = \frac{-E \pm \sqrt{E^2 - 4BD}}{2B}, \qquad E^2 - 4BD > 0,$$

we can also derive an exact solution for Eq. (1) by the same procedure as above,

$$u = \mu\sqrt{-\frac{2A}{B}}\frac{c_2\lambda_2 \exp\left[\lambda_2 x + \int k_2(t)\,dt\right] + c_3\lambda_3 \exp\left[\lambda_3 x + \int k_3(t)\,dt\right]}{c_1 + c_2 \exp\left[\lambda_2 x + \int k_2(t)\,dt\right] + c_3 \exp\left[\lambda_3 x + \int k_3(t)\,dt\right]} + \frac{-E \pm \sqrt{E^2 - 4BD}}{2B}, \tag{24}$$

where

$$\lambda_2 = \mp\mu\sqrt{-\frac{1}{2AB}(E^2 - 4BD)}, \qquad \lambda_3 = \frac{\mu}{2}\sqrt{-\frac{1}{2AB}}\left(E \mp \sqrt{E^2 - 4BD}\right),$$

$$k_2(t) = \pm\left[\frac{E}{2B}\sqrt{E^2 - 4BD}h_1(t) + \mu\sqrt{-\frac{1}{2AB}(E^2 - 4BD)}h_2(t)\right],$$

$$k_3(t) = \frac{-E^2 + 6BD \pm E\sqrt{E^2 - 4BD}}{4B}h_1(t) + \frac{\mu}{2}\sqrt{-\frac{1}{2AB}}\left(-E \pm \sqrt{E^2 - 4BD}\right)h_2(t).$$

4. Conclusion

We found several types of explicit exact solutions for Eq. (1). These results may be useful for explaining some physical problems. We know that many famous nonlinear wave equations, such as the Newell–Whitehead equation, the Fitzhugh–Nagumo equation, the Chaffee–Infante equation, the KPP equation, and the Huxley equation are special cases of Eq. (1).

When $h_1(t) = 1$, $h_2(t) = 0$, $A = -1$, $B = \beta$, $E = 0$, and $D = -\beta$, Eq. (1) reduces to the Chaffee–Infante equation [8]–[10]:

$$u_t - u_{xx} + \beta u^3 - \beta u = 0.$$

Two double-soliton solutions can be derived from Eqs. (17) and (24):

$$\begin{cases} u_1(x,t) = \mu\dfrac{-c_2\exp(-\sqrt{\beta/2}\,x + 3\beta t/2) + c_3\exp(\sqrt{\beta/2}\,x + 3\beta t/2)}{c_1 + c_2\exp(-\sqrt{\beta/2}\,x + 3\beta t/2) + c_3\exp(\sqrt{\beta/2}\,x + 3\beta t/2)}, \\[2ex] u_2(x,t) = \mu\dfrac{\pm 2c_2\exp\left(\pm\sqrt{2\beta}x\right) \pm c_3\exp(\pm\sqrt{\beta/2}\,x - 3\beta t/2)}{c_1 + c_2\exp\left(\pm\sqrt{2\beta}x\right) + c_3\exp(\pm\sqrt{\beta/2}\,x - 3\beta t/2)} \pm 1. \end{cases}$$

We emphasize that $u_1(x,t)$ did not appear in [8].

In addition, we can also obtain four single-soliton solutions of Eq. (1),

$$\begin{cases} u_1(x,t) = \dfrac{1}{2}\mu\tanh\left(\dfrac{1}{2}\sqrt{\dfrac{\beta}{2}}x \mp \dfrac{3\beta}{4}t + c_0\right) + \dfrac{1}{2}\mu \pm 1, \\[2ex] u_2(x,t) = \dfrac{1}{2}\mu\coth\left(\dfrac{1}{2}\sqrt{\dfrac{\beta}{2}}x \mp \dfrac{3\beta}{4}t + c_0\right) + \dfrac{1}{2}\mu \pm 1, \\[2ex] u_3(x,t) = \mu\tanh\left(\sqrt{\dfrac{\beta}{2}}x + c_0\right) + \mu \pm 1, \\[2ex] u_4(x,t) = \mu\coth\left(\sqrt{\dfrac{\beta}{2}}x + c_0\right) + \mu \pm 1. \end{cases}$$

We now consider the Fitzhugh–Nagumo equation [5]–[7], [11], [13],

$$H_t - \frac{1}{2} H_{xx} + (\alpha - H)(1 - H^2) = 0, \quad -1 < \alpha < 0. \tag{25}$$

If $H(x,t) = \alpha - u$, then Eq. (25) is a special case of Eq. (1). Therefore, we can also find additional exact solutions that contain soliton solutions. We omit these conclusions here.

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