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New Explicit Solitary Wave Solutions and Periodic Wave Solutions for the Generalized Coupled Hirota–Satsuma KdV System*

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Abstract *In this paper, we study the generalized coupled Hirota–Satsuma KdV system by using the new generalized transformation in homogeneous balance method. As a result, many explicit exact solutions, which contain new solitary wave solutions, periodic wave solutions, and the combined formal solitary wave solutions, and periodic wave solutions, are obtained*

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1 Introduction

In recent years, searching for explicit exact solutions, in particular, solitary wave solutions of nonlinear evolution equations (NEEs) in mathematical physics plays an important role in the soliton theory.[1−11,14] Various powerful methods have been presented, such as Backlund transformation, Darboux transformation, Cole–Hopf transformation, tanh method, sine-cosine method, Painleve method, homogeneous balance method (HBM), Hirota method,[12] Lie group analysis, similarity reduced method and so on. Based upon the well-known Riccati equation, homogeneous balance method[6,7] is used to find exact solutions of certain nonlinear PDEs. Fan and Zhang[13,15] improved considerably the steps of the key step of the HBM. Particularly, more general ansatz has been proposed in order to obtain new form of solutions. In 1981, Hirota–Satsuma first proposed the well-known coupled Hirota–Satsuma KdV equation[16]

$$u_t = \frac{1}{4}u_{xxx} + 3uu_x - 6vv_x\,, \tag{1a}$$

$$v_t = -\frac{1}{2}v_{xxx} - 3uv_x\,, \tag{1b}$$

and further showed that system (1) is a special case of the generalized coupled Hirota–Satsuma KdV system[17,18]

$$u_t = \frac{1}{4}u_{xxx} + 3uu_x + 3(-v^2 + w)_x\,, \tag{2a}$$

$$v_t = -\frac{1}{2}v_{xxx} - 3uv_x\,, \tag{2b}$$

$$w_t = -\frac{1}{2}w_{xxx} - 3uw_x\,, \tag{2c}$$

which can be obtained from the four-reduction of KP hierarchy by the dependent variable transform

$$u = (\ln f)_{xx}\,, \quad v = \frac{1}{2}\frac{f_y}{f}\,, \quad w = \frac{1}{2}\frac{f_{yy}}{f} \tag{3}$$

with y being an auxiliary variable.

In this paper, we would like to discuss Eqs. (2) further by our improved method, in which we presented a new generalized transformation.[19] As a result, many exact solutions are obtained.

2 Summary of Our Method

Our method is summed up as follows:

For a given system of nonlinear evolution equations, say, in two variables

$$F(u, H, u_t, H_t, u_x, H_x, u_{xt}, H_{xt}, u_{tt}, H_{tt}, u_{xx}, H_{xx}, \ldots) = 0\,, \tag{4a}$$

$$G(u, H, u_t, H_t, u_x, H_x, u_{xt}, H_{xt}, u_{tt}, H_{tt}u_{xx}, H_{xx}, \ldots) = 0\,, \tag{4b}$$

we seek for the following formal travelling wave solutions

$$u(x,t) = u(\xi)\,, \quad H(x,t) = v(\xi)\,, \quad \xi = x + \lambda t + c\,, \tag{5}$$

where λ is a constant to be determined later, c is an arbitrary constant. Then the system (4) reduces to a system of nonlinear ordinary differential equation

$$F_0(u, H, u', H', u'', H'', \ldots) = 0\,, \qquad G_0(u, H, u', H', u'', H'', \ldots) = 0\,, \tag{6}$$

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where the prime denotes $\mathrm{d}/\mathrm{d}\xi$. In order to seek for the travelling wave solutions of system (6), we take the following transformations

$$u(\xi)=\sum_{i=1}^{m}\omega^{i-1}(\xi)\big[A_i\omega(\xi)+B_i\sqrt{\mu_1(1+\mu_2\omega^2(\xi))}\,\big]+A_0\,, \tag{7a}$$

$$H(\xi)=\sum_{i=1}^{n}\omega^{i-1}(\xi)\big[a_i\omega(\xi)+b_i\sqrt{\mu_1(1+\mu_2\omega^2(\xi))}\,\big]+a_0 \tag{7b}$$

and the new variable $\omega=\omega(\xi)$ satisfies

$$\omega'-R(1+\mu_2\omega^2)=\frac{\mathrm{d}\omega}{\mathrm{d}\xi}-R(1+\mu_2\omega^2)=0\,, \tag{8}$$

where $\mu_j=\pm1$ $(j=1,2)$, m and n are integers to be determined and A_i, B_i, a_j, b_j $(i=0,1,2,\ldots,m;\ j=0,1,2,\ldots,n)$, R are constants to be determined later.

When $B_i=0$, $b_j=0$ $(i=1,2,\ldots,m;\ j=1,2,\ldots,n)$ in Eqs. (7a) and (7b), equations (7) become the transformation proposed by Ma *et al.*[10] But as $B_i\neq0$, $b_j\neq0$ $(i=1,2,\ldots,m;\ j=1,2,\ldots,n)$, we can find many new exact solutions of system (4).

There exist the following steps to be considered further.

Step 1 Determining the values of m and n of system (7) by respectively balancing the highest order partial derivative term and the nonlinear term in system (4) (or (6)), it is easy to get the value of m and n.

Step 2 With the aid of Mathematica, substituting system (7) along with the condition (8) into system (6), yields a system of algebraic equations with respect to $\omega^i(\mu_1+\mu_1\mu_2\omega^2)^{j/2}$ $(j=0,1;\ i=0,1,2,\ldots)$.

Step 3 Collect all terms with the same power in $\omega^i(\mu_1+\mu_1\mu_2\omega^2)^{j/2}$ $(j=0,1;i=0,1,2,\ldots)$ and set the coefficients of the terms $\omega^i(\mu_1+\mu_1\mu_2\omega^2)^{j/2}$ $(j=0,1;i=0,1,2,\ldots)$ to zero to get an over-determined system of nonlinear algebraic equations with respect to the unknown variables $\lambda, R, A_0, a_0, A_i, B_i, a_j, b_j$ $(i=1,2,\ldots,m;\ j=1,2,\ldots,n)$.

Step 4 We apply Wu-elimination method[20,21] to solve the above over-determined system of nonlinear algebraic equations obtained in step 4, and yields the values of λ, R, A_0, a_0, A_i, B_i, a_j, b_j $(i=1,2,\ldots,m;\ j=1,2,\ldots,n)$.

Step 5 It is well known that the general solutions of Eq. (8) are

i) When taking $\mu_2=-1$,

$$\omega=\omega(\xi)=\frac{A-B\exp(-2R\xi)}{A+B\exp(-2R\xi)}=\begin{cases}1\,, & \text{for } B=0\,,\\ -1\,, & \text{for } A=0\,,\\ \tanh\Big[\Big(R\xi-\dfrac{1}{2}\ln\Big(\dfrac{A}{B}\Big)\Big],& \text{for } AB>0\,,\\ \coth\Big[\Big(R\xi-\dfrac{1}{2}\ln\Big(-\dfrac{A}{B}\Big)\Big],& \text{for } AB<0\,.\end{cases} \tag{9}$$

When A, B are arbitrary constants satisfying $A^2+B^2\neq0$. This solution may be obtained by three tricks: a Möbius transformation, a Cole–Hopf transformation and a relation

$$\frac{(\omega_1-\omega_2)(\omega_3-\omega_4)}{(\omega_1-\omega_3)(\omega_2-\omega_4)}=C=\text{const}\,. \tag{10}$$

of the solutions $\omega_i, 1\leq i\leq4$, beginning with three known solutions $1, -1, \tanh(R\xi)$.

ii) When $\mu_2=1$,

$$\omega=\omega(\xi)=\begin{cases}\tan(R\,\xi+\xi_0)\,,\\ -\cot(R\,\xi+\xi_0)\,.\end{cases} \tag{11}$$

Thus according to Eqs. (5), (7), (9), and (11) and the conclusions in step 4, we can obtain several travelling wave solutions of system (4).

3 Solitary Wave Solution and Periodic Wave Solution for the Generalized Coupled Hirota–Satsuma KdV System

According to the above steps, we firstly make the following wave transformation

$$u(x,t)=u(\xi)\,,\qquad v(x,t)=v(\xi)\,,$$
$$w(x,t)=w(\xi)\,,\qquad \xi=\alpha x-\lambda t\,, \tag{12}$$

where α,λ are constants to be determined.

Substituting Eqs. (12) into system (2) and integrating system (2a) reads

$$\frac{1}{4}\alpha^3u''+\frac{3}{2}\alpha u^2-3\alpha v^2+3\alpha w+\lambda u=0\,, \tag{13a}$$

$$-\frac{1}{2}\alpha^3v'''-3\alpha uv'+\lambda v'=0\,, \tag{13b}$$

$$-\frac{1}{2}\alpha^3w'''-3\alpha uw'+\lambda w'=0\,. \tag{13c}$$

According to step 1 in Sec. 2, we suppose that system (13) has the following formal solutions

$$u=a_0+a_1\omega+b_1\sqrt{\mu_1(1+\mu_2\omega^2)}+a_2\omega^2+b_2\omega\sqrt{\mu_1(1+\mu_2\omega^2)}\ , \tag{14a}$$

$$v=A_0+A_1\omega+B_1\sqrt{\mu_1(1+\mu_2\omega^2)}\ , \tag{14b}$$

$$w=C_0+C_1\omega+D_1\sqrt{\mu_1(1+\mu_2\omega^2)}\ , \tag{14c}$$

and $\omega = \omega(\xi)$ satisfies Eq. (8), where a_0, a_1, a_2, b_1, b_2, A_0, A_1, B_1, C_0, C_1, D_1 are constants to be determined later.

With the aid of Mathematica, substituting Eqs. (14) into system (13) along with Eq. (8) and collecting all terms with the same power in $\omega^i(\mu_1 + \mu_1\mu_2\omega^2)^{j/2}$ ($j = 0, 1$; $i = 0, 1, 2, 3, 4$), yield a system of equations with respect to $\omega^i(\mu_1 + \mu_1\mu_2\omega^2)^{j/2}$. Setting the coefficients of $\omega^i(\mu_1 + \mu_1\mu_2\omega^2)^{j/2}$ ($j = 0, 1$; $i = 0, 1, 2, 3, 4$) in the obtained system of equations to zero, we can deduce the following set of over-determined algebraic polynomials with respect to the unknowns $\alpha, \lambda, a_0, a_1, a_2, b_1, b_2, A_0, A_1, B_1, C_0, C_1, D_1$

$$6a_0^2\alpha - 12A_0^2\alpha + 12C_0\alpha + 2a_2R^2\alpha^3 + 4a_0\lambda + 6b_1^2\alpha\mu_1 - 12B^2\alpha\mu_1 = 0\,,$$

$$12a_0a_1\alpha - 24A_0A_1\alpha + 12C_1\alpha + 4a_1\lambda + 12b_1b_2\alpha\mu_1 + 2a_1R^2\alpha^3\mu_2 = 0\,,$$

$$6a_1^2\alpha - 12A_1^2\alpha + 12a_0a_2\alpha + 4a_2\lambda + 6b_2^2\alpha\mu_1 + 8a_2R^2\alpha^3\mu_2 + 6b_1^2\alpha\mu_1\mu_2 - 12B_1^2\alpha\mu_1\mu_2 = 0\,,$$

$$12a_1a_2\alpha + 12b_1b_2\alpha\mu_1\mu_2 + 2a_1R^2\alpha^3 = 0\,,$$

$$6a_2^2\alpha + 6b_2^2\alpha\mu_1\mu_2 + 6a_2R^2\alpha^3 = 0\,,$$

$$12a_0b_1\alpha - 24A_0B_1\alpha + 12D_1\alpha + 4b_1\lambda + b_1R^2\alpha^3\mu_2 = 0\,,$$

$$12a_1b_1\alpha - 24A_1B_1\alpha + 12a_0b_2\alpha + 4b_2\lambda + 5b_2R^2\alpha^3\mu_2 = 0\,,$$

$$12a_2b_1\alpha + 12a_1b_2\alpha + 2b_1R^2\alpha^3 = 0\,,$$

$$12a_2b_2\alpha + 6b_2R^2\alpha^3 = 0\,,$$

$$-6a_0A_1R\alpha + 2A_1R\lambda - 2A_1R^3\alpha^3\mu_2 = 0\,,$$

$$-6a_1A_1R\alpha - 6b_1B_1R\mu_1\mu_2 = 0\,,$$

$$-6A_1a_2R\alpha - 6a_0A_1R\alpha\mu_2 + 2A_1R\lambda\mu_2 - 6B_1b_2R\alpha\mu_1\mu_2 - 8A_1R^2\alpha^3 = 0\,,$$

$$-6a_1A_1R\alpha\mu_2 - 6b_1B_1R\mu_1 = 0\,,$$

$$-6A_1a_2R\alpha\mu_2 - 6B_1b_2R\alpha\mu_1 - 6A_1R_3\alpha^3\mu_2 = 0\,,$$

$$-6A_1b_1R\alpha = 0\,,$$

$$-6A_1b_2R\alpha - 6a_0B_1R\alpha\mu_2 + 2B_1R\lambda\mu_2 - 5B_1R^3\alpha^3 = 0\,,$$

$$-6A_1b_1R\alpha\mu_2 - 6a_1B_1R\alpha\mu_2 = 0\,,$$

$$-6a_2B_1R\alpha\mu_2 - 6A_1b_2R\alpha\mu_2 - 6B_1R^3\alpha^3\mu_2 = 0\,,$$

$$-6a_0C_1R\alpha + 2C_1R\lambda - 2C_1R^3\alpha^3\mu_2 = 0\,,$$

$$-6a_1C_1R\alpha - 6b_1D_1R\mu_1\mu_2 = 0\,,$$

$$-6C_1a_2R\alpha - 6a_0C_1R\alpha\mu_2 + 2C_1R\lambda\mu_2 - 6D_1b_2R\alpha\mu_1\mu_2 - 8C_1R^3\alpha^3 = 0\,,$$

$$-6C_1a_1R\alpha\mu_2 - 6b_1D_1R\mu_1 = 0\,,$$

$$-6C_1a_2R\alpha\mu_2 - 6D_1b_2R\alpha\mu_1 - 6C_1R^3\alpha^3\mu_2 = 0\,,$$

$$-6C_1b_1R\alpha = 0\,,$$

$$-6C_1b_2R\alpha - 6a_0D_1R\alpha\mu_2 + 2D_1R\lambda\mu_2 - 5D_1R^3\alpha^3 = 0\,,$$

$$-6C_1b_1R\alpha\mu_2 - 6a_1D_1R\alpha\mu_2 = 0\,,$$

$$-6a_2D_1R\alpha\mu_2 - 6C_1b_2R\alpha\mu_2 - 6D_1R^3\alpha^3\mu_2 = 0\,,$$

from which we have

Case 1

$$a_0 = \frac{\lambda - R^2\alpha^3\mu_2}{3\alpha}a_1 = 0\,, \quad a_2 = -\frac{R^2\alpha^2}{2}\,, \quad b_1 = 0\,, \quad b_2 = \frac{R^2\alpha^2\sqrt{\mu_1\mu_2}}{2\mu_1\mu_2}\,,$$

$$A_0 = \frac{\rho}{2\delta}\,, \quad A_1 = \mp\sqrt{-(R^4\alpha^4\mu_2 + 8R^2\alpha\lambda)/48}\,, \quad B_1 = \sqrt{-(R^4\alpha^4 + 8R^2\alpha\lambda\mu_2)\mu_1/48}\,,$$

$$C_0 = -\frac{R^4\alpha^6\delta_2 - 24\delta^2\lambda^2 + 8R^2\alpha^3\delta^2\lambda\mu_2 + 36\alpha^2\rho^2}{144\alpha^2\delta^2}\,, \quad C_1 = -\rho\sqrt{\mu_1\mu_2}\,, \quad D_1 = \rho\,;$$

Case 2

$$a_0 = \frac{\lambda - R^2\alpha^3\mu_2}{3\alpha}\,, \quad a_1 = 0\,, \quad a_2 = -\frac{R^2\alpha^2}{2}\,, \quad b_1 = 0\,,$$

$$b_2 = \frac{R^2\alpha^2\sqrt{\mu_1\mu_2}}{2\mu_1\mu_2}\,, \quad A_0 = 0\,, \quad A_1 = \pm\sqrt{-(R^4\alpha^4\mu_2 + 8R^2\alpha\lambda)/48}\,,$$

$$B_1 = \sqrt{-(R^4\alpha^4 + 8R^2\alpha\lambda\mu_2)\mu_1/48}\,, \quad C_0 = -\frac{R^4\alpha^6 - 24\lambda^2 + 8R^2\alpha^3\delta^2\lambda\mu_2}{144\alpha^2}\,, \quad C_1 = 0\,, \quad D_1 = 0\,;$$

Case 3

$$a_0 = \frac{\lambda - R^2\alpha^3\mu_2}{3\alpha}\,, \quad a_1 = 0\,, \quad a_2 = -R^2\alpha^2\,, \quad b_1 = 0\,, \quad b_2 = 0\,, A_0 = 0\,,$$

$$A_1 = \pm\sqrt{-(R^4\alpha^4\mu_2 + 2R^2\alpha\lambda)/3}\,, \quad B_1 = 0\,, \quad C_0 = \frac{2R^4\alpha^6 - 3\lambda^2 + 4R^2\alpha^3\lambda\mu_2}{18\alpha^2}\,, \quad C_1 = 0\,, \quad D_1 = 0\,;$$

Case 4

$$a_0 = 0\,, \quad a_1 = 0\,, \quad a_2 = -R^2\alpha^2\,, \quad b_1 = 0\,, \quad b_2 = 0\,, \quad A_0 = \rho\,, \quad A_1 = 0\,,$$

$$B_1 = \pm\sqrt{-9R^4\alpha^4\mu_1/6}\,, \quad C_0 = \frac{6\rho^2 - 8R^4\alpha^4}{6}\,, \quad C_1 = 0\,, \quad D_1 = \pm\rho\sqrt{-6R^4\alpha^4\mu_1}\,, \quad \lambda = \frac{5R^2\alpha^3\mu_2}{2}\,;$$

Case 5

$$a_0=\frac{2\lambda\mu_2-5R^2\alpha^3}{6\alpha}\,,\quad a_1=0\,,\quad a_2=-R^2\alpha^2\,,\quad b_1=0\,,\quad b_2=0\,,\quad A_0=0\,,\quad A_1=0\,,$$
$$B_1=\pm\sqrt{-3R^4\alpha^4\mu_1/2}\,,\quad C_0=\frac{4R^4\alpha^4}{3}\,,\quad C_1=0\,,\quad D_1=0\,,\quad \lambda=\frac{5R^2\alpha^3\mu_2}{2}\,;$$

Case 6

$$a_0=0\,,\quad a_1=0\,,\quad a_2=-R^2\alpha^2 2\,,\quad b_1=0\,,\quad b_2=\frac{-R^2\alpha^2\sqrt{\mu_1\mu_2}}{2\mu_1\mu_2}\,,\quad A_0=\frac{\rho}{2\delta}\,,\quad A_1=\delta\sqrt{\mu_1\mu_2}\,,$$
$$B_1=\delta\,,\quad C_0=\frac{1}{12}\Big(R^4\alpha^4+\frac{3(4\delta^4\mu_1+\rho^2)}{\delta^2}\Big)\,,\quad C_1=\rho\sqrt{\mu_1\mu_2}\,,\quad D_1=\rho\,,\quad \lambda=R^2\alpha^3\mu_2\,.$$

Therefore according to step 5, twenty-four families of explicit exact travelling wave solutions, which contain solitary wave solutions, periodic wave solutions and new travelling wave solutions, are found as follows for system (2).

Case 1

$$u_{11}=\frac{\lambda+R^2\alpha^3}{3\alpha}-\frac{R^2\alpha^2}{2}\tanh^2[R(\alpha x-\lambda t)]\mp \mathrm{i}\frac{R^2\alpha^2}{2}\tanh[R(\alpha x-\lambda t)]\,\mathrm{sech}\,[R(\alpha x-\lambda t)]\,,$$
$$v_{11}=\frac{\rho}{2\delta}\mp \mathrm{i}\sqrt{(-R^4\alpha^4+8R^2\alpha\lambda)/48}\,\tanh[R(\alpha x-\lambda t)]\pm\sqrt{(-R^4\alpha^4+8R^2\alpha\lambda)/48}\,\mathrm{sech}\,[R(\alpha x-\lambda t)]\,,$$
$$w_{11}=-\frac{R^4\alpha^6\delta^2-24\delta^2\lambda^2-8R^2\alpha^3\delta^2\lambda+36\alpha^2\rho^2}{144\alpha^2\delta^2}-\mathrm{i}\rho\tanh[R(\alpha x-\lambda t)]\mp\rho\,\mathrm{sech}\,[R(\alpha x-\lambda t)]\,,$$
$$u_{12}=\frac{\lambda+R^2\alpha^3}{3\alpha}-\frac{R^2\alpha^2}{2}\coth^2[R(\alpha x-\lambda t)]\mp\frac{R^2\alpha^2}{2}\coth[R(\alpha x-\lambda t)]\,\mathrm{csch}\,[R(\alpha x-\lambda t)]\,,$$
$$v_{12}=\frac{\rho}{2\delta}\mp\sqrt{(-R^4\alpha^4+8R^2\alpha\lambda)/48}\,\coth[R(\alpha x-\lambda t)]\pm\sqrt{(-R^4\alpha^4+8R^2\alpha\lambda)/48}\,\mathrm{csch}\,[R(\alpha x-\lambda t)]\,,$$
$$w_{12}=-\frac{R^4\alpha^6\delta^2-24\delta^2\lambda^2-8R^2\alpha^3\delta^2\lambda+36\alpha^2\rho^2}{144\alpha^2\delta^2}-\rho\coth[R(\alpha x-\lambda t)]\mp\rho\,\mathrm{csch}\,[R(\alpha x-\lambda t)]\,,$$
$$u_{13}=\frac{\lambda-R^2\alpha^3}{3\alpha}-\frac{R^2\alpha^2}{2}\cot^2[R(\alpha x-\lambda t)]\mp\frac{R^2\alpha^2}{2}\cot[R(\alpha x-\lambda t)]\csc[R(\alpha x-\lambda t)]\,,$$
$$v_{13}=\frac{\rho}{2\delta}\mp\sqrt{-(R^4\alpha^4+8R^2\alpha\lambda)/48}\,\cot[R(\alpha x-\lambda t)]\pm\sqrt{-(R^4\alpha^4+8R^2\alpha\lambda)/48}\,\csc[R(\alpha x-\lambda t)]\,,$$
$$w_{13}=-\frac{R^4\alpha^6\delta^2-24\delta^2\lambda^2-8R^2\alpha^3\delta^2\lambda+36\alpha^2\rho^2}{144\alpha^2\delta^2}-\rho\cot[R(\alpha x-\lambda t)]\mp\rho\csc[R(\alpha x-\lambda t)]\,,$$
$$u_{14}=\frac{\lambda+R^2\alpha^3}{3\alpha}-\frac{R^2\alpha^2}{2}\tan^2[R(\alpha x-\lambda t)]\mp\frac{R^2\alpha^2}{2}\tan[R(\alpha x-\lambda t)]\sec[R(\alpha x-\lambda t)]\,,$$
$$v_{14}=\frac{\rho}{2\delta}\mp\sqrt{-(R^4\alpha^4+8R^2\alpha\lambda)/48}\,\tan[R(\alpha x-\lambda t)]\pm\sqrt{-(R^4\alpha^4+8R^2\alpha\lambda)/48}\,\sec[R(\alpha x-\lambda t)]\,,$$
$$w_{14}=-\frac{R^4\alpha^6\delta^2-24\delta^2\lambda^2-8R^2\alpha^3\delta^2\lambda+36\alpha^2\rho^2 2}{144\alpha^2\delta^2}-\rho\tan[R(\alpha x-\lambda t)]\mp\rho\sec[R(\alpha x-\lambda t)]\,;$$

Case 2

$$u_{21}=\frac{\lambda+R^2\alpha^3}{3\alpha}-\frac{R^2\alpha^2}{2}\tanh^2[R(\alpha x-\lambda t)]\pm \mathrm{i}\frac{R^2\alpha^2}{2}\tanh[R(\alpha x-\lambda t)]\,\mathrm{sech}\,[R(\alpha x-\lambda t)]\,,$$
$$v_{21}=\pm\sqrt{-(R^4\alpha^4+8R^2\alpha\lambda)/48}\,\tanh[R(\alpha x-\lambda t)]\pm \mathrm{i}\sqrt{R^4\alpha^4+8R^2\alpha\lambda)/48}\,\mathrm{sech}\,[R(\alpha x-\lambda t)]\,,$$
$$w_{21}=-\frac{R^4\alpha^6-24\lambda^2+8R^2\alpha^3\delta^2\lambda}{144\alpha^2}\,,$$
$$u_{22}=\frac{\lambda+R^2\alpha^3}{3\alpha}-\frac{R^2\alpha^2}{2}\coth^2[R(\alpha x-\lambda t)]\pm\frac{R^2\alpha^2}{2}\coth[R(\alpha x-\lambda t)]\,\mathrm{csch}\,[R(\alpha x-\lambda t)]\,,$$
$$v_{22}=\pm\sqrt{(R^4\alpha^4-8R^2\alpha\lambda)/48}\,\coth[R(\alpha x-\lambda t)]\pm\sqrt{(R^4\alpha^4-8R^2\alpha\lambda)/48}\,\mathrm{csch}\,[R(\alpha x-\lambda t)]\,,$$
$$w_{22}=-\frac{R^4\alpha^6-24\lambda^2+8R^2\alpha^3\delta^2\lambda}{144\alpha^2}\,,$$
$$u_{23}=\frac{\lambda-R^2\alpha^3}{3\alpha}-\frac{R^2\alpha^2}{2}\cot^2[R(\alpha x-\lambda t)]\pm\frac{R^2\alpha^2}{2}\cot[R(\alpha x-\lambda t)]\csc[R(\alpha x-\lambda t)]\,,$$
$$v_{23}=-\sqrt{-(R^4\alpha^4+8R^2\alpha\lambda)/48}\,\cot[R(\alpha x-\lambda t)]\pm\sqrt{-(R^4\alpha^4+8R^2\alpha\lambda)/48}\,\csc[R(\alpha x-\lambda t)]\,,$$

$$w_{23} = -\frac{R^4\alpha^6 - 24\lambda^2 + 8R^2\alpha^3\delta^2\lambda}{144\alpha^2}\,,$$
$$u_{24} = \frac{\lambda - R^2\alpha^3}{3\alpha} - \frac{R^2\alpha^2}{2}\tan^2[R(\alpha x - \lambda t)] \pm \frac{R^2\alpha^2}{2}\tan[R(\alpha x - \lambda t)]\sec[R(\alpha x - \lambda t)]\,,$$
$$v_{24} = -\sqrt{-(R^4\alpha^4 + 8R^2\alpha\lambda)/48}\,\tan[R(\alpha x - \lambda t)] \pm \sqrt{-(R^4\alpha^4 + 8R^2\alpha\lambda)/48}\,\sec[R(\alpha x - \lambda t)]\,,$$
$$w_{24} = -\frac{R^4\alpha^6 - 24\lambda^2 + 8R^2\alpha^3\delta^2\lambda}{144\alpha^2}\,;$$

Case 3

$$u_{31} = \frac{\lambda + R^2\alpha^3}{3\alpha} - R^2\alpha^2\tanh^2[R(\alpha x - \lambda t)]\,,$$
$$v_{31} = \pm\sqrt{(R^4\alpha^4 - 2R^2\alpha\lambda)/3}\,\tanh[R(\alpha x - \lambda t)]\,,$$
$$w_{31} = \frac{2R^4\alpha^6 - 3\lambda^2 - 4R^2\alpha^3\lambda}{18\alpha^2}\,,$$
$$u_{32} = \frac{\lambda + R^2\alpha^3}{3\alpha} - R^2\alpha^2\coth^2[R(\alpha x - \lambda t)]\,,$$
$$v_{32} = \pm\sqrt{(R^4\alpha^4 - 2R^2\alpha\lambda)/3}\,\coth[R(\alpha x - \lambda t)]\,,$$
$$w_{32} = \frac{2R^4\alpha^6 - 3\lambda^2 - 4R^2\alpha^3\lambda}{18\alpha^2}\,,$$
$$u_{33} = \frac{\lambda - R^2\alpha^3}{3\alpha} - R^2\alpha^2\cot^2[R(\alpha x - \lambda t)]\,,$$
$$v_{33} = \pm\sqrt{-(R^4\alpha^4 + 2R^2\alpha\lambda)/3}\,\cot[R(\alpha x - \lambda t)]\,,$$
$$w_{33} = \frac{2R^4\alpha^6 - 3\lambda^2 + 4R^2\alpha^3\lambda}{18\alpha^2}\,,$$
$$u_{34} = \frac{\lambda - R^2\alpha^3}{3\alpha} - R^2\alpha^2\tan^2[R(\alpha x - \lambda t)]\,,$$
$$v_{34} = \pm\sqrt{-(R^4\alpha^4 + 2R^2\alpha\lambda)/3}\,\tan[R(\alpha x - \lambda t)]\,,$$
$$w_{34} = \frac{2R^4\alpha^6 - 3\lambda^2 + 4R^2\alpha^3\lambda}{18\alpha^2}\,;$$

Case 4

$$u_{41} = -R^2\alpha^2\tanh^2\Big(R\alpha - \frac{-5R^2\alpha^3}{2}t\Big)\,,$$
$$v_{41} = \rho \pm \mathrm{i}\sqrt{\frac{3}{2}}\,R^2\alpha^2\,\mathrm{sech}\,\Big[R\Big(\alpha x - \frac{-5R^2\alpha^3}{2}t\Big)\Big]\,,$$
$$w_{41} = \frac{6\rho^2 - 8R^4\alpha^4}{6} \pm \mathrm{i}\sqrt{6}\rho\,\alpha^2R^2\,\mathrm{sech}\Big[R\Big(\alpha x - \frac{-5R^2\alpha^3}{2}t\Big)\Big],$$
$$u_{42} = -R^2\alpha^2\coth^2\Big[R\Big(\alpha x - \frac{-5R^2\alpha^3}{2}t\Big)\Big]\,,$$
$$v_{42} = \rho \pm \sqrt{\frac{3}{2}}\,R^2\alpha^2\,\mathrm{csch}\,\Big[R\Big(\alpha x - \frac{-5R^2\alpha^3}{2}t\Big)\Big]\,,$$
$$w_{42} = \frac{6\rho^2 - 8R^4\alpha^4}{6} \pm \sqrt{6}\,\rho\,R^2\alpha^2\mathrm{csch}\Big[R\Big(\alpha x - \frac{-5R^2\alpha^3}{2}t\Big)\Big],$$
$$u_{43} = -R^2\alpha^2\cot^2\Big[R\Big(\alpha x - \frac{5R^2\alpha^3}{2}t\Big)\Big]\,,$$
$$v_{43} = \rho \pm \sqrt{\frac{3}{2}}\,R^2\alpha^2\csc\Big[R\Big(\alpha x - \frac{5R^2\alpha^3}{2}t\Big)\Big]\,,$$
$$w_{43} = \frac{6\rho^2 - 8R^4\alpha^4}{6} \pm \sqrt{6}\,\rho R^2\alpha^2\csc\Big[R\Big(\alpha x - \frac{5R^2\alpha^3}{2}t\Big)\Big],$$
$$u_{44} = -R^2\alpha^2\tan^2\Big[R\Big(\alpha x - \frac{5R^2\alpha^3}{2}t\Big)\Big]\,,$$
$$v_{44} = \rho \pm \sqrt{\frac{3}{2}}\,R^2\alpha^2\sec\Big[R\Big(\alpha x - \frac{5R^2\alpha^3}{2}t\Big)\Big]\,,$$
$$w_{44} = \frac{6\rho^2 - 8R^4\alpha^4}{6} \pm \sqrt{6}\,\rho\,R^2\alpha^2\sec\Big[R\Big(\alpha x - \frac{5R^2\alpha^3}{2}t\Big)\Big]\,;$$

Case 5

$$u_{51} = -\frac{2\lambda + 5R^2\alpha^3}{6\alpha} - R^2\alpha^2\tanh^2\Big[R\Big(\alpha x - \frac{5R^2\alpha^3}{2}t\Big)\Big]\,,$$
$$v_{51} = \pm\mathrm{i}\sqrt{\frac{3}{2}}\,R^2\alpha^2\,\mathrm{sech}\,\Big[R\Big(\alpha x - \frac{5R^2\alpha^3}{2}t\Big)\Big]\,,$$
$$w_{51} = \frac{4R^4\alpha^6}{3}\,,$$
$$u_{52} = -\frac{2\lambda + 5R^2\alpha^3}{6\alpha} - R^2\alpha^2\coth^2\Big[R\Big(\alpha x - \frac{5R^2\alpha^3}{2}t\Big)\Big]\,,$$
$$v_{52} = \pm\sqrt{\frac{3}{2}}\,R^2\alpha^2\,\mathrm{csch}\,\Big[R\Big(\alpha x - \frac{5R^2\alpha^3}{2}t\Big)\Big]\,,$$
$$w_{53} = \frac{4R^4\alpha^6}{3}\,,$$
$$u_{53} = \frac{2\lambda - 5R^2\alpha^3}{6\alpha} - R^2\alpha^2\cot^2\Big[R\Big(\alpha x - \frac{5R^2\alpha^3}{2}t\Big)\Big]\,,$$
$$v_{53} = \pm\sqrt{\frac{3}{2}}\,R^2\alpha^2\csc\Big[R\Big(\alpha x - \frac{5R^2\alpha^3}{2}t\Big)\Big]\,,$$
$$w_{53} = \frac{4R^4\alpha^6}{3}\,,$$
$$u_{54} = -\frac{2\lambda + 5R^2\alpha^3}{6\alpha} - R^2\alpha^2\tan^2\Big[R\Big(\alpha x - \frac{5R^2\alpha^3}{2}t\Big)\Big]\,,$$
$$v_{54} = \pm\sqrt{\frac{3}{2}}\,R^2\alpha^2\sec\Big[R\Big(\alpha x - \frac{5R^2\alpha^3}{2}t\Big)\Big]\,,$$
$$w_{54} = \frac{4R^4\alpha^6}{3}\,;$$

Case 6

$$u_{61} = \frac{-R^2\alpha^2}{2}\tanh^2[R(\alpha x + R^2\alpha^3 t)] \mp \mathrm{i}\,\frac{R^2\alpha^2}{2}\tanh[R(\alpha x + R^2\alpha^3 t)]\,\mathrm{sech}\,[R(\alpha x + R^2\alpha^3 t)]\,,$$

$$v_{61} = \frac{\rho}{2\delta} + \mathrm{i}\delta\ \tanh[R(\alpha x + R^2\alpha^3 t)] \mp \delta\ \mathrm{sech}\,[R(\alpha x + R^2\alpha^3 t)]\,,$$

$$w_{61} = \frac{1}{12}\Big(R^4\alpha^4 + \frac{3(4\delta^4+\rho^2)}{\delta^2}\Big) + \mathrm{i}\rho\ \tanh[R(\alpha x + R^2\alpha^3 t)] \mp \rho\ \mathrm{sech}\,[R(\alpha x + R^2\alpha^3 t)]\,,$$

$$u_{62} = \frac{-R^2\alpha^2}{2}\coth^2[R(\alpha x + R^2\alpha^3 t)] \mp \frac{R^2\alpha^2}{2}\coth[R(\alpha x + R^2\alpha^3 t)]\,\mathrm{csch}\,[R(\alpha x + R^2\alpha^3 t)]\,,$$

$$v_{62} = \frac{\rho}{2\delta} + \delta\ \coth[R(\alpha x + R^2\alpha^3 t)] \mp \delta\,\mathrm{csch}\,[R(\alpha x + R^2\alpha^3 t)]\,,$$

$$w_{62} = \frac{1}{12}\Big(R^4\alpha^4 + \frac{3(4\delta^4+\rho^2)}{\delta^2}\Big) + \rho\ \coth[R(\alpha x + R^2\alpha^3 t)] \mp \rho\,\mathrm{csch}\,[R(\alpha x + R^2\alpha^3 t)]\,,$$

$$u_{63} = \frac{-R^2\alpha^2}{2}\cot^2[R(\alpha x + R^2\alpha^3 t)] \mp \frac{R^2\alpha^2}{2}\cot[R(\alpha x + R^2\alpha^3 t)]\,\csc[R(\alpha x + R^2\alpha^3 t)]\,,$$

$$v_{63} = \frac{\rho}{2\delta} + \delta\ \cot[R(\alpha x + R^2\alpha^3 t)] \mp \delta\ \csc[R(\alpha x + R^2\alpha^3 t)]\,,$$

$$w_{63} = \frac{1}{12}\Big(R^4\alpha^4 + \frac{3(4\delta^4+\rho^2)}{\delta^2}\Big) + \rho\ \cot[R(\alpha x + R^2\alpha^3 t)] \mp \rho\ \csc[R(\alpha x + R^2\alpha^3 t)]\,,$$

$$u_{64} = \frac{-R^2\alpha^2}{2}\tan^2[R(\alpha x + R^2\alpha^3 t)] \mp \frac{R^2\alpha^2}{2}\tan[R(\alpha x + R^2\alpha^3 t)]\,\csc[R(\alpha x + R^2\alpha^3 t)]\,,$$

$$v_{64} = \frac{\rho}{2\delta} + \delta\ \tan[R(\alpha x + R^2\alpha^3 t)] \mp \delta\,\sec[R(\alpha x + R^2\alpha^3 t)]\,,$$

$$w_{64} = \frac{1}{12}\Big(R^4\alpha^4 + \frac{3(4\delta^4+\rho^2)}{\delta^2}\Big) + \mathrm{i}\rho\ \tan[R(\alpha x + R^2\alpha^3 t)] \mp \rho\sec[R(\alpha x + R^2\alpha^3 t)]\,,$$

where δ, ρ are arbitrary constants.

4 Conclusions

In summary, based on the well-known Riccati equation, many new types of exact solutions for the generalized Hirota–Satsuma coupled KdV system have been derived by a generalized transformation. The method can easily be extended to other NEEs and is sufficient to seek more new solitary wave solutions of NEEs. It not only uses a more generalized transformation to produce an over-determined system of nonlinear algebraic equation but also can look for more solutions. In addition, this method is also computerizable, which allows us to perform complicated and tedious algebraic calculation on a computer.

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