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Auto-Bäcklund transformation and exact solutions for compound KdV-type and compound KdV–Burgers-type equations with nonlinear terms of any order

Biao Li *, Yong Chen, Hongqing Zhang

Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, People's Republic of China

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Abstract

In this Letter, based on the idea of homogeneous balance (HB) method and with help of MATHEMATICA, we obtain a new auto-Bäcklund transformation for compound KdV-type and compound KdV–Burgers-type equations with nonlinear terms of any order. Then based on the Bäcklund transformation, some solutions for these two equations are derived.

Keywords: Bäcklund transformation; HB method; Compound KdV-type equation; Compound KdV–Burgers-type equation; Solitary-wave solution; MATHEMATICA

1. Introduction

In recent years, the homogenous balance method (HB) has been widely applied to derive the nonlinear transformation and exact solutions (especially the solitary wave solutions) [8–18], and auto-Bäcklund transformations [10,13,14] as well as the similarity reductions [13,14] of nonlinear partial differential equations (PDEs) in mathematical physics. The Bäcklund transformations of nonlinear PDEs play an important role in solitary theory, which is an efficient method to obtain exact solutions of nonlinear PDE. The nonlinear iterative principle from Bäcklund transformations converts the problem of solving nonlinear PDE to purely algebraic calculations [1,2]. In Refs. [13, 14], Fan extended HB method to search for Bäcklund transformation and similarity reductions of nonlinear PDE. So more solutions can be obtained by the improved HB method. However, they only dealt with the cases whose balance constants are positive integers. In this Letter, we would further extend the HB method so that it can deal with the other cases whose balance constant is fraction or negative integer. To illustrate the extended HB method,

* Corresponding author.
E-mail addresses: libiao@student.dlut.edu.cn (B. Li), chenyong@dlut.edu.cn (Y. Chen).

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we consider the compound KdV-type equation with nonlinear terms of any order

$$u_t + au^p u_x + bu^{2p} u_x + \delta u_{xxx} = 0, \quad a, b, \delta, p = \text{constants}, \quad p > 0, \tag{1.1}$$

and the compound KdV–Burgers-type equation with nonlinear terms of any order

$$u_t + au^p u_x + bu^{2p} u_x + \gamma u_{xx} + \delta u_{xxx} = 0, \quad a, b, \delta, p = \text{constants}, \quad p > 0. \tag{1.2}$$

These equations include a number of equations which have been studied by many authors [3–7]. The KdV-type equation (1.1) have some application in quantum field theory, plasma physics and solid-state physics. For example, the kink soliton can be used to calculate energy and momentum flow and topological charge in the quantum field. In [5,6], Dey and Coffey considered the kink-profile solitary-wave solutions for Eq. (1.1) with $p = 1$ and $p = 2$. Wadati considered soliton solutions, conservation laws, Bäcklund transformation and other properties for Eq. (1.1) with $p = 1$ [3]. Zhang et al. studied the solitary-wave solutions for Eq. (1.1) [6]. The compound KdV–Burgers-type equation (1.2) with $p \geqslant 1$ is a model for long-wave propagation in nonlinear media with dispersion and dissipation [3–7]. In [6], Zhang et al. obtained kink-profile solitary-wave solutions for Eq. (1.2). The stability of traveling -wave solutions of Eq. (1.2) with $b = 0$, $\gamma \leqslant 0$ and $p \geqslant 1$ are studied by Pego et al. [4].

In this Letter, based on the idea of HB method and with the help of symbolic computation MATHEMATICA, a new Bäcklund transformation for Eqs. (1.1) and (1.2) is derived by use of a proper transformation. To our knowledge, this type of Bäcklund transformation obtained has not been ever seen before in the literature. Then based on this Bäcklund transformation, some solutions for Eqs. (1.1) and (1.2) are found.

This Letter is organized as follows. In Section 2, we derive the Bäcklund transformation for Eqs. (1.1) and (1.2). In Section 3, based on the Bäcklund transformation, a family of solutions for Eq. (1.1) is obtained. In Section 4, some solutions for Eq. (1.1) are found. Conclusions are given in the last section.

2. Bäcklund transformation

Let us consider the compound KdV–Burgers-type equation with nonlinear terms of any order, i.e., Eq. (1.2). According to the idea of HB method [8–18], by balancing the highest order partial derivative term and the nonlinear term in Eq. (1.2), we obtain balance constant $m = 1/p$. It is obvious that m may be arbitrary constants. In order to apply the HB method under this condition, we firstly make the following transformation:

$$u(x,t) = \varphi^{1/p}(x,t), \tag{2.1}$$

then substituting transformation (2.1) into Eq. (1.2) yields

$$\begin{aligned} &p^2\varphi^2\left[\varphi_t + a\varphi\varphi_x + b\varphi^2\varphi_x\right] + \gamma p\varphi\left[(1-p)\varphi_x^2 + p\varphi\varphi_{xx}\right] \\ &\quad + \delta(1-p)\left[(1-2p)\varphi_x^3 + 3p\varphi\varphi_x\varphi_{xx} + p^2\varphi^2\varphi_{xxx}\right] = 0. \end{aligned} \tag{2.2}$$

Then by balancing the highest order partial derivative term and the nonlinear term in Eq. (2.2), we get the value of the balance constant $m = 1$. Therefore, we seek for the Bäcklund transformation of Eq. (2.2) in the form

$$\varphi = f'\phi_x + \bar{\varphi}. \tag{2.3}$$

Here and in the following context 'prime' $:= \partial/\partial\phi$, $f^{(r)} = \partial^r/\partial\phi^r$, and $f = f(\phi)$, $\phi = \phi(x,t)$ is undetermined function and $\bar{\varphi}$ is a solution of Eq. (2.2).

With the help of MATHEMATICA, substituting (2.3) into (2.2) yields (because the formula is so long, just one part of it is shown here)

$$\left[bp^2 f'^4 f'' + (1-p)(1-2p)\delta f''^3 + 3(1-p)p\delta f' f'' f^{(3)} + p^2\delta f'^2 f^{(4)}\right]\phi_x^6 + \cdots = 0. \tag{2.4}$$

To simplify Eq. (2.4), setting the coefficient of ϕ_x^6 to zero yields an ordinary differential equation for f

$$bp^2 f'^4 f'' + (1-p)(1-2p)\delta f''^3 + 3(1-p)p\delta f' f'' f^{(3)} + p^2\delta f'^2 f^{(4)} = 0. \tag{2.5}$$

Solving (2.5) we obtain a solution

$$f = \pm\sqrt{-\frac{(1+p)(1+2p)\delta}{bp^2}}\,\ln\phi. \tag{2.6}$$

Setting $\beta = \pm\sqrt{-\frac{(1+p)(1+2p)\delta}{bp^2}}$ (note: in the rest of this Letter β denotes $\pm\sqrt{-(1+p)(1+2p)\delta/(bp^2)}$), then substituting (2.6) into (2.4), formula (2.4) can be simplified to a polynomial of $1/\phi^i$ $(i=0,\ldots,5)$, then setting the coefficients of $1/\phi^i$ $(i=0,\ldots,5)$, to zero yields a set of partial differential equations for $\phi(x,t)$

$$\big[bp\beta\gamma + a\delta + 2ap\delta + 2b(2+p)\bar\varphi\delta\big]\phi_x + b(2+p)\beta\delta\phi_{xx} = 0, \tag{2.7}$$

$$\begin{aligned}
&p^2(1+3p+2p^2)\delta\phi_t\phi_x + 6bp^2(1+3p+p^2)\bar\varphi^2\delta\phi_x^2\\
&\quad + p^2\bar\varphi\phi_x\big[\big(bp(1+3p)\beta\gamma + 3a(1+3p+2p^2)\delta\big)\phi_x + b(-4+3p+p^2)\beta\delta\phi_{xx}\big]\\
&\quad + \delta\big[-2bp^2(-1+3p+p^2)\beta\bar\varphi_x\phi_x^2\\
&\quad - (1+3p+2p^2)\big[-3\delta\phi_{xx}^2 + p\phi_x\big(\big(ap\beta-(2+p)\gamma\big)\phi_{xx} - (3+p)\delta\phi_{xxx}\big)\big]\big] = 0,
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
&4b^2p^4\bar\varphi^3\beta\phi_x^3 + bp^2\bar\varphi^2\phi_x^2\big[p^2(3a\beta-2\gamma)\phi_x + 6(1+3p)\delta\phi_{xx}\big]\\
&\quad + p^2\bar\varphi_x\phi_x^2\big[\big[-2b(p-1)p\beta\gamma + a(1+3p+2p^2)\delta\big]\phi_x + 3b(2-3p+p^2)\beta\delta\phi_{xx}\big]\\
&\quad + p^2\bar\varphi\phi_x\big[2bp^2\beta\phi_t\phi_x + 2b(2+3p+7p^2)\delta\bar\varphi_x\phi_x^2 + 2bp\beta\gamma\phi_x\phi_{xx} + 4bp^2\beta\gamma\phi_x\phi_{xx} + 3a\delta\phi_x\phi_{xx}\\
&\quad + 9ap\delta\phi_x\phi_{xx} + 6ap^2\delta\phi_x\phi_{xx} + 9bp\beta\delta\phi_{xx}^2 - 3bp^2\beta\delta\phi_{xx}^2 + 3bp\beta\delta\phi_x\phi_{xxx} + 5bp^2\beta\delta\phi_x\phi_{xxx}\big]\\
&\quad + \delta\big[-3b(p-1)p^3\beta\bar\varphi_{xx}\phi_x^3 + (1-5p^2+4p^4)\delta\phi_{xx}^3 - p(-1-2p+p^2+2p^3)\phi_x\phi_{xx}(\gamma\phi_{xx}+3\delta\phi_{xxx})\\
&\quad + p^2(1+3p+2p^2)\phi_x^2(\phi_{xt}+\gamma\phi_{xxx}+\delta\phi_{xxxx})\big] = 0,
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
&-p^2\bar\varphi^2\phi_t\phi_x + \big[p^2\beta\bar\varphi_t + p\bar\varphi\big(3ap\beta + 6bp\bar\varphi\beta + 2(p-1)\gamma\big)\bar\varphi_x\\
&\quad - 3(1-3p+2p^2)\delta\bar\varphi_x^2 + p\big[(p\beta\gamma - 3\bar\varphi\delta + 3p\bar\varphi\delta)\bar\varphi_{xx} - p\big(\bar\varphi^3(a+b\bar\varphi) - \beta\delta\bar\varphi_{xxx}\big)\big]\big]\phi_x^2\\
&\quad + \phi_{xx}\big[\big[p\bar\varphi(\beta\gamma - p\beta\gamma - 3p\bar\varphi\delta) + 3(1-3p+2p^2)\beta\delta\bar\varphi_x\big]\phi_{xx} - 3(p-1)p\bar\varphi\beta\delta\phi_{xxx}\big]\\
&\quad + p\phi_x\big[2p\bar\varphi\beta\phi_{xt} + \big[p\bar\varphi^2(3a\beta + 4b\bar\varphi\beta - 3\gamma) - (p-1)\big((2\beta\gamma - 9\bar\varphi\delta)\bar\varphi_x - 3\beta\delta\bar\varphi_{xx}\big)\big]\phi_{xx}\\
&\quad + \big(2p\bar\varphi\beta\gamma - 4p\bar\varphi^2\delta + 3\beta\delta\bar\varphi_x(1-p)\big)\phi_{xxx} + 2p\bar\varphi\beta\delta\phi_{xxxx}\big] = 0,
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
&2p^2\bar\varphi\bar\varphi_t\phi_x - (p-1)\bar\varphi_x^2\big(p\gamma\phi_x + 3(1-2p)\delta\phi_{xx}\big)\\
&\quad + p\bar\varphi_x\big[\big(p\bar\varphi^2(3a+4b\bar\varphi) - 3(p-1)\delta\bar\varphi_{xx}\big)\phi_x - (p-1)\bar\varphi(2\gamma\phi_{xx} + 3\delta\phi_{xxx})\big]\\
&\quad + p\bar\varphi\big[\bar\varphi_{xx}\big(2p\gamma\phi_x - 3(p-1)\delta\phi_{xx}\big) + p\big(2\delta\bar\varphi_{xxx}\phi_x + \bar\varphi\big(\phi_{xt} + \bar\varphi(a+b\bar\varphi)\phi_{xx} + \gamma\phi_{xxx} + \delta\phi_{xxxx}\big)\big)\big] = 0,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
&ap^2\bar\varphi^3\bar\varphi_x + bp^2\bar\varphi^4\bar\varphi_x + (1-3p+2p^2)\delta\bar\varphi_x^3\\
&\quad - (p-1)p\bar\varphi\bar\varphi_x(\gamma\bar\varphi_x + 3\delta\bar\varphi_{xx}) + p^2\bar\varphi^2(\bar\varphi_t + \gamma\bar\varphi_{xx} + \delta\bar\varphi_{xxx}) = 0.
\end{aligned} \tag{2.12}$$

From (2.3) and (2.6), we obtain desired Bäcklund transformation of Eq. (1.2)

$$u = \left[\pm\sqrt{-\frac{(1+p)(1+2p)\delta}{bp^2}}\frac{\partial}{\partial x}\ln\phi + \bar\varphi\right]^{1/p}, \tag{2.13}$$

where ϕ satisfies (2.7)–(2.11), $\bar\varphi$ is a solution of Eq. (2.2).

3. Explicit exact solutions for Eq. (1.2)

Now we use the Bäcklund transformation consisted of (2.13) and (2.7)–(2.12) to exploit some explicit exact solutions for Eqs. (1.1) and (1.2). If we take initial solution of Eq. (2.2) as $\bar{\varphi}=0$, then (2.7)–(2.12) reduce to

$$\big(bp\beta\gamma+a(1+2p)\delta\big)\phi_x+b(2+p)\beta\delta\phi_{xx}=0, \tag{3.1}$$

$$bp^2\beta\phi_t\phi_x+3b\beta\delta\phi_{xx}^2+\phi_x\big[\big(bp(2+p)\beta\gamma+a\big(1+3p+2p^2\big)\delta\big)\phi_{xx}+bp(3+p)\beta\delta\phi_{xxx}\big]=0, \tag{3.2}$$

$$\big(1-3p+2p^2\big)\delta\phi_{xx}^3-(-1+p)p\phi_x\phi_{xx}(\gamma\phi_{xx}+3\delta\phi_{xxx})+p^2\phi_x^2(\phi_{xt}+\gamma\phi_{xxx}+\delta\phi_{xxxx})=0. \tag{3.3}$$

Then we discuss the solutions of Eqs. (3.1)–(3.3). If ϕ is taken as a function with respect to x only, Eq. (3.1) is changed into an ordinary differential equation. Solving the ordinary differential equation (3.1), we can obtain the following solution

$$\phi=-\frac{b(2+p)\beta\delta}{bp\beta\gamma+a(1+2p)\delta}\exp\left[-\frac{(bp\beta\gamma+a(1+2p)\delta)x}{b(2+p)\beta\delta}\right]c_1+c_2, \tag{3.4}$$

where c_1, c_2 are arbitrary constants. Therefore, Eq. (3.1) has the following formal solution:

$$\phi=-\frac{b(2+p)\beta\delta}{bp\beta\gamma+a(1+2p)\delta}\exp\left[-\frac{(bp\beta\gamma+a(1+2p)\delta)x}{b(2+p)\beta\delta}\right]c_1(t)+c_2(t), \tag{3.5}$$

where $c_1(t)$, $c_2(t)$ are undetermined arbitrary functions. Then substituting (3.5) into (3.2), (3.3), we can determine the functions $c_1(t)$, $c_2(t)$ as follows:

$$c_1(t)=c_1\exp\left[\frac{[bp(1+p)\beta\gamma-a(1+2p)\delta][bp\beta\gamma+a(1+2p)\delta]^2}{b^2(2+p)^3(1+p)(1+2p)\beta\delta^3}t\right],\qquad c_2(t)=c_2, \tag{3.6}$$

where c_1, c_2 are arbitrary constants.

From (2.13), (3.5) and (3.6), a family of solution of Eq. (1.2) is obtained as follows:

$$u=\left\{\frac{c_1\beta\exp[k(x-\lambda t)]}{-\frac{b(2+p)\beta\delta c_1}{bp\beta\gamma+a(1+2p)\delta}\exp[k(x-\lambda t)]+c_2}\right\}^{1/p}, \tag{3.7}$$

where

$$k=-\frac{bp\beta\gamma+a(1+2p)\delta}{b(2+p)\beta\delta},\qquad \lambda=\frac{(bp(1+p)\beta\gamma-a(1+2p)\delta)(bp\beta\gamma+a(1+2p)\delta)}{b(2+p)^2(1+p)(1+2p)\delta^2}.$$

If setting

$$c_2=-\frac{b(2+p)\beta\delta c_1}{bp\beta\gamma+a(1+2p)\delta},$$

from (3.7) we can obtain the kink-profile solitary-wave solutions for Eq. (1.2):

$$u=\left\{-\frac{a(1+2p)\delta+bp\beta\gamma}{2b(2+p)\delta}\big[1+\tanh\big[k(x-\lambda t)\big]\big]\right\}^{1/p}, \tag{3.8}$$

where

$$\lambda=\frac{(bp(1+p)\beta\gamma-a(1+2p)\delta)(bp\beta\gamma+a(1+2p)\delta)}{b(2+p)^2(1+p)(1+2p)\delta^2},\qquad k=-\frac{bp\beta\gamma+a(1+2p)\delta}{2b(2+p)\beta\delta},$$

$$\beta=\pm\sqrt{-\frac{(1+p)(1+2p)\delta}{bp^2}}.$$

Remark 1. It is not difficult to verify that the solutions (3.8) of Eq. (1.2) are the same as the solutions (4.7) and (4.8) in [6]. Therefore, the solutions (4.7) and (4.8) given in [6] are special cases of our solutions (3.7).

4. Explicit exact solutions for Eq. (1.1)

From (3.7), we can obtain the following solutions for Eq. (1.1)

$$u = \left\{ \frac{c_1 \beta \exp[k(x+\lambda t)]}{-\frac{b(2+p)\beta c_1}{a(1+2p)} \exp[k(x+\lambda t)] + c_2} \right\}^{1/p}, \tag{4.1}$$

where

$$k = -\frac{a(1+2p)}{b(2+p)\beta}, \qquad \lambda = \frac{a^2(1+2p)}{b(2+p)^2(1+p)}, \qquad \beta = \pm\sqrt{-\frac{(1+p)(1+2p)\delta}{bp^2}}.$$

When setting

$$c_2 = -\frac{b(2+p)\beta c_1}{a(1+2p)},$$

the above solutions become into

$$u = \left\{ -\frac{a(1+2p)}{2b(2+p)} \left[1 + \tanh\left[-\frac{a(1+2p)}{b(2+p)\beta} \left[x + \frac{a^2(1+2p)}{b(2+p)^2(1+p)} t \right] \right] \right] \right\}^{1/p}, \tag{4.2}$$

where $\beta = \pm\sqrt{-(1+p)(1+2p)\delta/(bp^2)}$. So we obtain a family of kink-profile solitary-wave solutions (4.2) for Eq. (1.1).

Remark 2. It is easy to see that: (i) the solutions (4.9) given in [7] are just the solutions (4.2) obtained by us; (ii) the results given in [5] are the special cases of (4.2) with $p = 1$ and $p = 2$, $\delta \geqslant 0$; (iii) the solutions (3.10) and (3.13) in [3] are the special cases of (4.2) with $p = 1$, $b = 6\beta$, $\delta = 1$ and $p = 1/2$, $b = 6\alpha$, $\delta = 1$, respectively. Therefore, our results for Eq. (1.1) include not only many previous results but also many new exact solutions.

5. Conclusions

We have found a new Bäcklund transformation for compound KdV-type and compound KdV–Burgers-type equations with nonlinear terms of any order by use of the extended HB method. To our knowledge, this type of Bäcklund transformation obtained has not been ever seen before in the literature. Based on this Bäcklund transformation, several families of exact solutions for Eqs. (1.1) and (1.2) are derived. This method can also apply to other PDEs. In addition, this method is also computerizable, which allow us to perform complicated and tedious symbolic algebraic calculation on a computer.

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