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Explicit exact solutions for new general two-dimensional KdV-type and two-dimensional KdV–Burgers-type equations with nonlinear terms of any order

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Abstract

In this paper, we have improved the tanh-method by means of a proper transformation and a general ansatz. Applying the improved method and direct assumption method with symbolic computation, we consider two kinds of equations, general two-dimensional KdV-type equations with nonlinear terms of any order, $\left(u_t + au^p u_x + bu^{2p} u_x + \delta u_{xxx}\right)_x + su_{yy} = 0$, and general two-dimensional KdV–Burgers-type equations with nonlinear terms of any order, $\left(u_t + au^p u_x + bu^{2p} u_x + \gamma u_{xx} + \delta u_{xxx}\right)_x + su_{yy} = 0$. As a result, rich explicit exact solutions for these two equations, which contain kink-profile solitary-wave solutions, bell-profile solitary-wave solutions, rational solutions, periodic solutions and combined formal solutions, are obtained.

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1. Introduction

The KdV-type equations and KdV–Burgers-type equations arise in a variety of physical contexts and have been studied by many authors [1–5, 13–21]. The KdV-type equations have applications in quantum field theory, plasma physics and solid-state physics. For example, the kink soliton can be used to calculate energy and momentum flow and topological charge in quantum fields. Recently, equations of these two types have received much attention. It is well known that the KdV–Burgers equation takes the form

$$u_t + uu_x - \alpha u_{xx} + \beta u_{xxx} = 0 \tag{1.1}$$

and its two-dimensional generalization is

$$(u_t + uu_x - \alpha u_{xx} + \beta u_{xxx})_x + \gamma u_{yy} = 0 \tag{1.2}$$

where α, β and γ are constants. Due to the nonintegrability of equations (1.1) and (1.2), recently much attention has been given to the study of their exact solutions by direct methods such as the tanh-method and homogeneous balance method [1–3, 5, 7]. In [1–3], the soliton solution of equation (1.2) is

$$u = \frac{3\alpha^2}{25\beta}(\operatorname{sech}^2\xi \pm 2\tanh\xi \pm 2) \qquad \xi = \mp\frac{\alpha}{10\beta}x + \frac{1}{2}ly + \left(\frac{3\alpha^3}{125\beta^2} \mp \frac{5l^2\beta\gamma}{2\alpha}\right)t \tag{1.3}$$

where l is an arbitrary constant. Besides above solution, Fan *et al* [5] found another type of soliton solution for equation (1.2)

$$u = -(d + c^2\gamma) \pm \frac{6\alpha^2}{25\beta}\tanh\xi + \frac{6\alpha^2}{25\beta}\operatorname{sech}^2\xi + \mathrm{i}\frac{6\alpha^2}{25\beta}(1 \mp \tanh\xi)\operatorname{sech}\xi \tag{1.4}$$

where $\xi = \mp\frac{\alpha}{5\beta}(x + cy + dt)$. More recently, Zhang *et al* [4] considered explicit exact solitary-wave solutions for the compound KdV-type equation with nonlinear terms of any order

$$u_t + au^p u_x + bu^{2p}u_x + \delta u_{xxx} = 0 \qquad a, b, \delta, p = \text{const} \quad p > 0 \tag{1.5}$$

and the compound KdV–Burgers-type equation with nonlinear terms of any order

$$u_t + au^p u_x + bu^{2p}u_x + \gamma u_{xx} + \delta u_{xxx} = 0 \qquad a, b, \gamma, \delta, p = \text{const} \quad \gamma \neq 0 \quad p > 0. \tag{1.6}$$

For equation (1.5), they obtained the following bell-profile solitary-wave solutions:

$$u_{1,2}(\xi) = \left[\frac{\pm v(1+p)(2+p)\sqrt{\frac{1+2p}{a^2(1+2p)+bv(1+p)(2+p)^2}}\operatorname{sech}^2\frac{p}{2}\sqrt{\frac{v}{\delta}}(x - vt + \xi_0)}{2 + \left(-1 \pm a\sqrt{\frac{1+2p}{a^2(1+2p)+bv(1+p)(2+p)^2}}\right)\operatorname{sech}^2\frac{p}{2}\sqrt{\frac{v}{\delta}}(x - vt + \xi_0)}\right]^{\frac{1}{p}}. \tag{1.7}$$

For equation (1.6), they obtained the following kink-profile solitary-wave solutions:

$$u_{1,2}(x,t) = \left\{\frac{A}{2}\left[1 \pm \tanh\frac{1}{2}\sqrt{-\frac{bp^2}{\delta(1+p)(1+2p)}}\right.\right.$$

$$\left.\left.\times A\left(x - \left(\frac{b}{1+2p}A^2 + \frac{a}{1+p}A\right)t + \xi_0\right)\right]\right\}^{\frac{1}{p}} \tag{1.8}$$

where $A = -\frac{(1+p)(1+2p)}{pb(2+p)}\left[\frac{pa}{1+p} \pm \gamma\sqrt{-\frac{p^2 b}{\delta(1+p)(1+2p)}}\right]$. In (1.8), when setting $\gamma = 0$, the kink-profile solitary-wave solutions for equation (1.5) were obtained.

In this paper, firstly we present two equations, a general two-dimensional KdV-type equation with nonlinear terms of any order (G2DKdV for short)

$$\left(u_t + au^p u_x + bu^{2p}u_x + \delta u_{xxx}\right)_x + su_{yy} = 0 \qquad a, b, \delta, s, p = \text{const} \tag{I}$$

and a general two-dimensional KdV–Burgers-type equation with nonlinear terms of any order (G2DKdV–Burgers for short)

$$\left(u_t + au^p u_x + bu^{2p}u_x + \gamma u_{xx} + \delta u_{xxx}\right)_x + su_{yy} = 0 \qquad a, b, \gamma, \delta, p = \text{const}. \tag{II}$$

Besides equations (1.1), (1.2), (1.5) and (1.6), the above two equations also include many KdV-type and KdV–Burgers-type equations in (1 + 1)-dimensional and (2 + 1)-dimensional cases (see [13–21] for details). Secondly, we would seek explicit exact solutions for G2DKdV and G2DKdV–Burgers equations. Making use of the improved method described in section 2 and the direct assumption method in section 4, many explicit exact solutions, which contain bell-profile solitary-wave solutions, kink-profile solitary-wave solutions, periodic solutions, combined formal solitary-wave solutions and rational solutions, are obtained. The solutions obtained in [1–5] are included in our obtained solutions.

This paper is organized as follows. In section 2, we describe the improved method. In section 3, we apply the improved method to G2DKdV and G2DKdV–Burgers equations and bring out many solutions. In section 4, the bell-profile solitary-wave solutions of G2DKdV equation are found. Conclusions will be presented in section 5.

2. Summary of the improved method

In this section, we will improve the tanh-method developed by other authors [5–11]. For given nonlinear evolution equations, say, in two variables, x, t

$$F(u, u_t, u_x, u_{xt}, u_{tt}, \ldots) = 0 \tag{2.1}$$

we seek the following formal travelling wave solutions:

$$u(x,t) = u(\xi) \qquad \xi = x - \lambda t \tag{2.2}$$

where λ is a constant to be determined later. Then equation (2.1) reduces to a nonlinear ordinary differential equation

$$G(u, u', u'', u''', \ldots) = 0 \tag{2.3}$$

where a prime denotes $\frac{\mathrm{d}}{\mathrm{d}\xi}$. In order to seek the travelling wave solutions of equation (2.3), we take the following improved transformations (see [5, 9] for details):

$$u(\xi) = \sum_{i=1}^{m} \omega^{i-1}(\xi)\left[A_i\omega(\xi) + B_i\sqrt{R + \omega^2(\xi)}\right] + A_0 \tag{2.4}$$

and the new variable $\omega = \omega(\xi)$ satisfies

$$\omega' - (R + \omega^2) = \frac{\mathrm{d}\omega}{\mathrm{d}\xi} - (R + \omega^2) = 0 \tag{2.5}$$

where A_0, A_i, B_i $(i = 1, 2, \ldots, m)$ and R are constants to be determined later, and m is a positive integer. However, when we balance the highest order partial derivative term and the nonlinear term in equation (2.1) or (2.3), we find that the constant m need not be a positive integer. In order to apply the method described in [5–11] when m is equal to a fraction or a negative integer, we make the following transformation:

(1) When $m = \frac{q}{p}$ (where $m = \frac{q}{p}$ is a fraction in lowest terms), we let

$$u(\xi) = \varphi^{q/p}(\xi) \tag{2.6}$$

then substitute (2.6) into equation (2.3) and return to determine the value of m by balancing the highest order partial derivative term and the nonlinear term in the new equation (2.3).

(2) When m is a negative integer, we let

$$u(\xi) = \varphi^{m}(\xi) \tag{2.7}$$

then substitute (2.7) into equation (2.3) and return to determine the value of m again.

In general, the constant m can be changed into a positive integer by means of the above proper transformation. Otherwise we have to seek other proper transformations.

We summarize the extended method as follows:

Step 1. Determine the values of m in (2.4) by balancing the highest order partial derivative term and the nonlinear term in (2.1) (or (2.3)).

(i) If m is a positive integer then step 2.
(ii) If $m = \frac{q}{p}$, we make the transformation (2.6) and then return to step 1.
(iii) If m is a negative integer, we make the transformation (2.7) and then return to step 1.

Step 2. With the aid of Mathematica, substituting (2.4) along with the condition (2.5) into equation (2.3) yields a system of algebraic equations with respect to $\omega^i(R+\omega^2)^{j/2}$ $(j = 0, 1; i = 0, 1, 2, \ldots)$.
Step 3. Collect all terms with the same power in $\omega^i(R+\omega^2)^{j/2}$ $(j = 0, 1; i = 0, 1, 2, \ldots)$. Set the coefficients of the terms $\omega^i(R+\omega^2)^{j/2}$ $(j = 0, 1; i = 0, 1, 2, \ldots)$ to zero to get an over-determined system of nonlinear algebraic equations with respect to the unknown variables $\lambda, R, A_0, A_i, B_i$ $(i = 1, 2, \ldots, m)$.
Step 4. With the aid of Mathematica, we apply the Wu-elimination method [12] to solve the above over-determined system of nonlinear algebraic equations obtained in step 3, which yields the values of $\lambda, R, A_0, A_i, B_i$ $(i = 1, 2, \ldots, m)$.
Step 5. It is well known that the general solutions of equation (2.5) are

(1) when $R < 0$,
$$\omega(\xi) = -\sqrt{-R}\tanh(\sqrt{-R}\,\xi) \qquad \omega(\xi) = -\sqrt{-R}\coth(\sqrt{-R}\,\xi) \tag{2.8}$$
(2) when $R = 0$,
$$\omega(\xi) = -\frac{1}{\xi} \tag{2.9}$$
(3) when $R > 0$,
$$\omega(\xi) = \sqrt{R}\tan(\sqrt{R}\,\xi) \qquad \omega(\xi) = -\sqrt{R}\cot(\sqrt{R}\,\xi). \tag{2.10}$$

Thus according to equations (2.2), (2.4), (2.6) or (2.7)–(2.10) and the conclusions in step 4, we can obtain many travelling wave solutions of equation (2.1).

3. Explicit exact solutions for the general two-dimensional KdV-type and the general two-dimensional KdV–Burgers-type equations with nonlinear term of any order

Let us consider the G2DKdV–Burgers equation, i.e. equation (II). Firstly we take the form of the required solution as follows:
$$u(x, y, t) = v(\xi) \qquad \xi = kx + ny - \lambda t \tag{3.1}$$
where k, n and λ are constants to be determined, and thus equation (II) becomes
$$\left(-\lambda k v_\xi + ak^2 v^p v_\xi + bk^2 v^{2p} v_\xi + \gamma k^3 v_{\xi\xi} + \delta k^4 v_{\xi\xi\xi}\right)_\xi + sn^2 v_{\xi\xi} = 0. \tag{3.2}$$
Integrating the above equation twice with regard to ξ, we obtain
$$\delta k^4 v''(\xi) + \gamma k^3 v'(\xi) + (sn^2 - \lambda k)v(\xi) + \frac{ak^2}{p+1}v^{p+1}(\xi) + \frac{bk^2}{2p+1}v^{2p+1}(\xi) = 0 \tag{3.3}$$
with the integration constants taken to be zero. According to step 1 in section 2, if $\delta \neq 0$, $b \neq 0$ and $p \neq 0, \frac{1}{2}$, by balancing $v''(\xi)$ and $v^{2p+1)}(\xi)$ in equation (3.3), we get the value of m, $m = 1/p$. Therefore we make the following transformation:
$$v(\xi) = \varphi^{\frac{1}{p}}(\xi) \tag{3.4}$$
then substituting (3.4) into equation (3.3) reads
$$\delta k^4[p\varphi(\xi)\varphi''(\xi) + (1-p)\varphi'^2(\xi)] + \gamma k^3 p\varphi(\xi)\varphi'(\xi) + p^2\left[(sn^2 - \lambda k)\varphi^2(\xi) + \frac{ak^2}{p+1}\varphi^3(\xi) + \frac{bk^2}{2p+1}\varphi^4(\xi)\right] = 0. \tag{3.5}$$

For ease of computing, we reduce (3.5) and get the following equation:

$$(1+p)(1+2p)\delta k^4[p\varphi(\xi)\varphi''(\xi)+(1-p)\varphi'^2(\xi)]+(1+p)(1+2p)\gamma k^3 p\varphi(\xi)\varphi'(\xi)$$
$$+p^2[(1+p)(1+2p)(sn^2-\lambda k)\varphi^2(\xi)$$
$$+(1+2p)ak^2\varphi^3(\xi)+(1+p)bk^2\varphi^4(\xi)]=0. \tag{3.6}$$

According to step 1 in section 2, by balancing $\varphi(\xi)\varphi''(\xi)$ (or $\varphi'^2(\xi)$) and $\varphi^4(\xi)$ in equation (3.6), we get the value of $m, m = 1$. Therefore we suppose that (3.6) has the following formal solutions:

$$\varphi(\xi) = A_0 + A_1\omega + B_1\sqrt{R+\omega^2} \tag{3.7}$$

and $\omega = \omega(\xi)$ satisfies (2.5), where A_0, A_1, B_1 are constants to be determined later.

With the aid of Mathematica, substituting (3.7) into (3.6) along with (2.5) and collecting all terms with the same power in $\omega^i(R+\omega^2)^{j/2}$ $(j = 0, 1; i = 0, 1, 2, 3, 4)$, yields a system of equations w.r.t. $\omega^i(R+\omega^2)^{j/2}$. Setting the coefficients of $\omega^i(R+\omega^2)^{j/2}$ $(j = 0, 1; i = 0, 1, 2, 3, 4)$ in the obtained system of equations to zero, we can deduce the following set of over-determined algebraic polynomials with respect to the unknowns A_0, A_1, B_1, R, k, n and λ:

$$A_0{}^2p^2\big(A_0k^2(a+2ap+A_0b(1+p))\big)+(1+p)(1+2p)(n^2s-k\lambda))$$
$$+pR\big(B_1{}^2p\big(3A_0k^2(a+2ap+2A_0b(1+p))$$
$$+(1+p)(1+2p)(n^2s-k\lambda)\big)+A_0A_1k^3(1+p)(1+2p)\gamma\big)$$
$$+k^2(1+p)R^2\big(bB_1{}^4p^2+k^2(1+2p)\big(-A_1{}^2(-1+p)+B_1{}^2p\big)\delta\big)=0 \tag{3.8}$$

$$B_1p\big(A_0p\big(A_0k^2(4A_0b(1+p)+a(3+6p))+2(1+p)(1+2p)(n^2s-k\lambda)\big)$$
$$+k^2R\big(B_1{}^2p(a+2ap+4A_0b(1+p))$$
$$+A_1k(1+p)(1+2p)\gamma+A_0k^2(1+p)(1+2p)\delta\big)\big)=0 \tag{3.9}$$

$$p(A_0A_1p(A_0k^2(4A_0b(1+p)+a(3+6p))+2(1+p)(1+2p)(n^2s-k\lambda))$$
$$+k^2R\big(3aA_1B_1{}^2p(1+2p)+\big(\big(A_1{}^2+B_1{}^2\big)k(1+2p)\gamma$$
$$+2A_0A_1\big(6bB_1{}^2p+k^2(1+2p)\delta\big)\big)\big)\big)=0 \tag{3.10}$$

$$A_0B_1k^3p(1+p)(1+2p)\gamma+A_1B_1(2k^4(1+p)(1+2p)R\delta$$
$$+p\big(2p\big(3A_0k^2(a+2ap+2A_0b(1+p))+2bB_1{}^2k^2(1+p)R$$
$$+(1+p)(1+2p)(n^2s-k\lambda)\big)+k^4(1+p)(1+2p)R\delta)\big)=0 \tag{3.11}$$

$$\big(A_1{}^2+B_1{}^2\big)p^2\big(3A_0k^2(a+2ap+2A_0b(1+p))\big)+(1+p)(1+2p)(n^2s-k\lambda))$$
$$+A_0A_1k^3p(1+p)(1+2p)\gamma+k^2(1+p)R\big(2bB_1{}^2\big(3A_1{}^2+B_1{}^2\big)p^2$$
$$+k^2(1+2p)\big(2A_1{}^2+B_1{}^2(1+2p)\big)\delta\big)=0 \tag{3.12}$$

$$B_1k^2p\left(\big(3A_1{}^2+B_1{}^2\big)p(a+2ap+4A_0b(1+p))+2k(1+p)(1+2p)(A_1\gamma+A_0k\delta)\right)=0 \tag{3.13}$$

$$k^2p\left(A_1\big(A_1{}^2+3B_1{}^2\big)p(a+2ap+4A_0b(1+p))+A_1{}^2k(1+p)(1+2p)\gamma\right.$$
$$\left.+B_1{}^2k(1+p)(1+2p)\gamma+2A_0A_1k^2(1+p)(1+2p)\delta\right)=0 \tag{3.14}$$

$$2A_1B_1k^2(1+p)\big(2A_1{}^2bp^2+2bB_1{}^2p^2+k^2(1+3p+2p^2)\delta\big)=0 \tag{3.15}$$

$$k^2(1+p)\left(b\big(A_1{}^4+6A_1{}^2B_1{}^2+B_1{}^4\big)p^2+\big(A_1{}^2+B_1{}^2\big)k^2(1+p)(1+2p)\delta\right)=0. \tag{3.16}$$

To solve equations (3.8)–(3.16) by using Wu-elimination method [12], which is a sufficient method to solve the systems of algebraic polynomial equations with more unknowns and with the aid of Mathematica, we get the following conclusions from the system of equations (3.8)–(3.16).

Case 1.

$$B_1 = 0 \qquad A_0 = -\frac{a(1+2p)}{2b(2+p)} \pm \frac{\gamma\sqrt{-(1+p)(1+2p)b\delta}}{2b(2+p)\delta}$$
$$A_1 = \pm\sqrt{-\frac{k^2(1+p)(1+2p)\delta}{bp^2}} \qquad R = \frac{A_0^2bp^2}{k^2(1+p)(1+2p)\delta}$$
$$\lambda = \frac{4bk}{1+2p}A_0^2 + \frac{2ak}{1+p}A_0 + \frac{sn^2}{k}. \tag{3.17}$$

Case 2.

$$A_0 = R = 0 \qquad \lambda = \frac{sn^2}{k} \qquad A_1 = B_1 = \pm\sqrt{-\frac{k^2(1+p)(1+2p)\delta}{4bp^2}}$$
$$\gamma^2 = -\frac{a^2\delta(1+2p)}{b(1+p)}. \tag{3.18}$$

Case 3.

$$A_0 = -\frac{a(1+2p)}{2b(2+p)} \pm \frac{\gamma\sqrt{-(1+p)(1+2p)b\delta}}{2b(2+p)\delta} \qquad A_1 = \pm B_1 = \pm\sqrt{-\frac{k^2(1+p)(1+2p)\delta}{4bp^2}}$$
$$R = \frac{4A_0^2bp^2}{k^2(1+p)(1+2p)\delta} \qquad \lambda = \frac{4bk}{1+2p}A_0^2 + \frac{2ak}{1+p}A_0 + \frac{sn^2}{k}. \tag{3.19}$$

Case 4.

$$\gamma = A_0 = B_1 = a = 0 \qquad p = 1 \qquad A_1 = \pm\sqrt{\frac{3(sn^2-\lambda k)}{bk^2R}} \qquad R = \frac{\lambda k - sn^2}{2\delta k^4}. \tag{3.20}$$

Case 5.

$$\gamma = A_0 = A_1 = a = 0 \qquad B_1 = \pm\sqrt{-\frac{(1+p)(1+2p)(sn^2-\lambda k)}{bk^2R}}$$
$$R = \frac{(sn^2-\lambda k)p^2}{\delta k^4}. \tag{3.21}$$

Case 6.

$$\gamma = A_0 = a = 0 \qquad p = 1 \qquad A_1 = \pm B_1 = \pm\sqrt{\frac{3(sn^2-\lambda k)}{bk^2R}} \qquad R = -\frac{2(sn^2-\lambda k)}{\delta k^4}. \tag{3.22}$$

Case 7.

$$\gamma = A_1 = 0 \qquad p = 1 \qquad A_0 = \pm\sqrt{\frac{a^2}{4b}} \qquad B_1 = \pm\sqrt{\frac{a^2}{2bR}} \qquad R = -\frac{a^2}{12\delta k^2}$$
$$\lambda = -\frac{a^2k}{6} + \frac{sn^2}{k}. \tag{3.23}$$

Therefore, according to step 5 in section 2, some exact travelling wave solutions, which contain solitary-wave solutions, periodic wave solutions, rational solutions and combined formal solitary-wave solutions, are found for equations (I) and (II). Next, we describe the solutions for equations (II) and (I), respectively.

3.1. Explicit exact solutions for the general two-dimensional KdV–Burgers-type equation with nonlinear terms of any order

In this section, we consider the solutions for equation (II). From equations (2.8)–(2.10), (3.1), (3.4), (3.7) and (3.17)–(3.19), we obtain the explicit exact solutions for equation (II) as follows:

Case 1. From (3.17), equation (II) has the following solutions:

(1) when $R < 0$, i.e. $(1+p)(1+2p)b\delta < 0$,

$$u_{11} = \left[A_0\left[1 \pm \tanh\left(\pm\sqrt{-\frac{A_0^2bp^2}{k^2(1+p)(1+2p)\delta}}(kx+ny-\lambda t+\xi_0)\right)\right]\right]^{1/p} \tag{3.24}$$

$$u_{12} = \left[A_0\left[1 \pm \coth\left(\pm\sqrt{-\frac{A_0^2bp^2}{k^2(1+p)(1+2p)\delta}}(kx+ny-\lambda t+\xi_0)\right)\right]\right]^{1/p} \tag{3.25}$$

(2) when $R > 0$, i.e. $(1+p)(1+2p)b\delta > 0$,

$$u_{13} = \left[A_0\left[1 \pm \mathrm{i}\tan\left(\pm\sqrt{\frac{A_0^2bp^2}{k^2(1+p)(1+2p)\delta}}(kx+ny-\lambda t+\xi_0)\right)\right]\right]^{1/p} \tag{3.26}$$

$$u_{14} = \left[A_0\left[1 \pm \mathrm{i}\cot\left(\pm\sqrt{\frac{A_0^2bp^2}{k^2(1+p)(1+2p)\delta}}(kx+ny-\lambda t+\xi_0)\right)\right]\right]^{1/p} \tag{3.27}$$

where $A_0 = -\frac{a(1+2p)}{2b(2+p)} \pm \frac{\gamma\sqrt{-(1+p)(1+2p)b\delta}}{2b(2+p)\delta}$, $\lambda = \frac{4bk}{1+2p}A_0^2 + \frac{2ak}{1+p}A_0 + \frac{sn^2}{k}$, ξ_0 is an arbitrary constant (Note: in the rest of this paper ξ_0 denotes an arbitrary constant).

Case 2. From (3.18), equation (II) has the following rational solutions:

$$u_3 = \left[\pm\sqrt{-\frac{k^2(1+P)(1+2p)\delta}{bp^2}}\frac{1}{kx+ny-\frac{sn^2}{k}t+\xi_0}\right]^{\frac{1}{p}} \tag{3.28}$$

where γ satisfies $\gamma^2 = -\frac{a^2\delta(1+2p)}{b(1+p)}$.

Case 3. From (3.19), equation (II) has the following solutions:

$$u_{41} = \{A_0[1 \pm \tanh[\sqrt{-R}(kx+ny-\lambda t)] \pm \mathrm{i}\,\mathrm{sech}[\sqrt{-R}(kx+ny-\lambda t+\xi_0)]]\}^{\frac{1}{p}} \tag{3.29}$$

$$u_{42} = \{A_0[1 \pm \coth[\sqrt{-R}(kx+ny-\lambda t)] \pm \mathrm{cosech}[\sqrt{-R}(kx+ny-\lambda t+\xi_0)]]\}^{\frac{1}{p}} \tag{3.30}$$

$$u_{43} = \{A_0[1 \pm \mathrm{i}\tan[\sqrt{R}(kx+ny-\lambda t)] \pm \mathrm{i}\sec[\sqrt{R}(kx+ny-\lambda t+\xi_0)]]\}^{\frac{1}{p}} \tag{3.31}$$

$$u_{44} = \{A_0[1 \pm \mathrm{i}\cot[\sqrt{R}(kx+ny-\lambda t)] \pm \mathrm{i}\,\mathrm{cosec}[\sqrt{R}(kx+ny-\lambda t+\xi_0)]]\}^{\frac{1}{p}} \tag{3.32}$$

where $A_0 = -\frac{a(1+2p)}{2b(2+p)} \pm \frac{\gamma\sqrt{-(1+p)(1+2p)b\delta}}{2b(2+p)\delta}$, $R = \frac{4A_0^2bp^2}{k^2(1+p)(1+2p)\delta}$, $\lambda = \frac{4bk}{1+2p}A_0^2 + \frac{2ak}{1+p}A_0 + \frac{sn^2}{k}$.

Remark 1. Our obtained solutions include the solutions obtained in [1–5]. By simple calculation, it is not difficult to verify that

(1) when setting $a = 0, b = 1, p = \frac{1}{2}, \gamma = -\alpha, \delta = \beta, s = \gamma$ and $n = \frac{5\beta lk}{\alpha}$, the solutions (3.24) are just the solution (1.3).
(2) when setting $n = 0$ and $s = 0$, the solutions (3.24) are just the solutions (1.8).
(3) when setting $a = 0, b = 1, p = \frac{1}{2}, \gamma = -\alpha, \delta = \beta, s = \gamma, n = ck$ and $\lambda = -dk$, the solutions (3.29) are just the solution (1.4) for equation (1.2).

3.2. Explicit exact solutions for the general two-dimensional KdV-type equation with nonlinear terms of any order

In this section, we consider the general two-dimensional KdV-type equation with nonlinear terms of any order, i.e. equation (I). From equations (2.8)–(2.10), (3.1), (3.4), (3.7) and (3.17)–(3.23), we obtain the explicit exact solutions for equation (I) as follows:

Case 1. From (3.17), we can obtain the following solutions for equation (I):

(1) when $R < 0$, i.e. $(1+p)(1+2p)\,b\delta < 0$

$$u_{11} = \left[-\frac{a(1+2p)}{2b(2+p)}\left(1 \pm \tanh\left(\sqrt{-\frac{a^2p^2(1+2p)}{4bk^2(1+p)(2+p)^2\delta}}(kx+ny-\lambda t+\xi_0)\right)\right)\right]^{1/p} \tag{3.33}$$

$$u_{12} = \left[-\frac{a(1+2p)}{2b(2+p)}\left(1 \pm \coth\left(\sqrt{-\frac{a^2p^2(1+2p)}{4bk^2(1+p)(2+p)^2\delta}}(kx+ny-\lambda t+\xi_0)\right)\right)\right]^{1/p} \tag{3.34}$$

(2) when $R > 0$, i.e. $(1+p)(1+2p)b\delta > 0$,

$$u_{13} = \left[-\frac{a(1+2p)}{2b(2+p)}\left(1 \pm \mathrm{i}\tan\left(\sqrt{\frac{a^2p^2(1+2p)}{4bk^2(1+p)(2+p)^2\delta}}(kx+ny-\lambda t+\xi_0)\right)\right)\right]^{1/p} \tag{3.35}$$

$$u_{14} = \left[-\frac{a(1+2p)}{2b(2+p)}\left(1 \pm \mathrm{i}\cot\left(\sqrt{\frac{a^2p^2(1+2p)}{4bk^2(1+p)(2+p)^2\delta}}(kx+ny-\lambda t+\xi_0)\right)\right)\right]^{1/p} \tag{3.36}$$

where $\lambda = -\frac{a^2k(1+2p)}{b(1+p)(2+p)^2} + \frac{sn^2}{k}$.

Case 2. Note that $\gamma = 0$ if and only if $a = 0$ in (3.18). Therefore the equation $\left(u_t + bu^{2p}u_x + \delta_{xxx}\right)_x + su_{yy} = 0$ has the following rational solutions:

$$u_2 = \left[\pm\sqrt{-\frac{k^2(1+p)(1+2p)\delta}{bp^2}}\frac{1}{\left(kx+ny-\frac{sn^2}{k}t+\xi_0\right)}\right]^{\frac{1}{p}}. \tag{3.37}$$

Case 3. From (3.19), the equation $\left(u_t + au^pu_x + bu^{2p}u_x + \delta u_{xxx}\right)_x + su_{yy} = 0$ has the following formal solutions:

(1) when $R < 0$, i.e. $\delta(sn^2 - \lambda k) < 0$,

$$u_{31} = \{A_0[1 \pm \tanh[\sqrt{-R}(kx+ny-\lambda t+\xi_0)] \pm \mathrm{i}\,\mathrm{sech}\,[\sqrt{-R}(kx+ny-\lambda t+\xi_0)]]\}^{\frac{1}{p}} \tag{3.38}$$

$$u_{32} = \{A_0[1 \pm \coth[\sqrt{-R}(kx+ny-\lambda t+\xi_0)] \pm \mathrm{cosech}[\sqrt{-R}(kx+ny-\lambda t+\xi_0)]]\}^{\frac{1}{p}} \tag{3.39}$$

(2) when $R > 0$, i.e. $\delta(sn^2 - \lambda k) > 0$,

$$u_{33} = \{A_0[1 \pm \mathrm{i}\tan[\sqrt{R}(kx + ny - \lambda t + \xi_0)] \pm \mathrm{i}\sec[\sqrt{R}(kx + ny - \lambda t + \xi_0)]]\}^{\frac{1}{p}} \quad (3.40)$$

$$u_{34} = \{A_0[1 \pm \mathrm{i}\cot[\sqrt{R}(kx + ny - \lambda t + \xi_0)] \pm \mathrm{i}\,\mathrm{cosec}[\sqrt{R}(kx + ny - \lambda t + \xi_0)]]\}^{\frac{1}{p}} \quad (3.41)$$

where $A_0 = -\frac{a(1+2p)}{2b(2+p)}$, $R = \frac{a^2p^2(1+2p)}{bk^2(1+p)(2+p)^2\delta}$, $\lambda = -\frac{a^2k(1+2p)}{b(1+p)(2+p)^2} + \frac{sn^2}{k}$.

Case 4. From (3.20), the equation $(u_t + bu^2u_x + \delta u_{xxx})_x + su_{yy} = 0$ has the following solutions:

(1) when $R < 0$, i.e. $\delta(sn^2 - \lambda k) > 0$,

$$u_{41} = \pm\sqrt{-\frac{3(sn^2 - \lambda k)}{bk^2}}\tanh\left[\sqrt{\frac{sn^2 - \lambda k}{2\delta k^4}}(kx + ny - \lambda t + \xi_0)\right] \quad (3.42)$$

$$u_{42} = \pm\sqrt{-\frac{3(sn^2 - \lambda k)}{bk^2}}\coth\left[\sqrt{\frac{sn^2 - \lambda k}{2\delta k^4}}(kx + ny - \lambda t + \xi_0)\right] \quad (3.43)$$

(2) when $R > 0$, i.e. $\delta(sn^2 - \lambda k) < 0$,

$$u_{43} = \pm\sqrt{\frac{3(sn^2 - \lambda k)}{bk^2}}\tan\left[\sqrt{-\frac{sn^2 - \lambda k}{2\delta k^4}}(kx + ny - \lambda t + \xi_0)\right] \quad (3.44)$$

$$u_{44} = \pm\sqrt{\frac{3(sn^2 - \lambda k)}{bk^2}}\cot\left[\sqrt{-\frac{sn^2 - \lambda k}{2\delta k^4}}(kx + ny - \lambda t + \xi_0)\right]. \quad (3.45)$$

Case 5. From (3.21), the solutions of the equation, $(u_t + bu^{2p}u_x + \delta u_{xxx})_x + su_{yy} = 0$, are as follows:

(1) when $R < 0$, i.e. $\delta(sn^2 - \lambda k) < 0$,

$$u_{51} = \left\{\pm\sqrt{-\frac{(1+p)(1+2p)(sn^2 - \lambda k)}{bk^2}} \times \mathrm{sech}\left[\sqrt{-\frac{(sn^2 - \lambda k)p^2}{\delta k^4}}(kx + ny - \lambda t + \xi_0)\right]\right\}^{\frac{1}{p}} \quad (3.46)$$

$$u_{52} = \left\{\pm\sqrt{\frac{(1+p)(1+2p)(sn^2 - \lambda k)}{bk^2}} \times \mathrm{cosech}\left[\sqrt{-\frac{(sn^2 - \lambda k)p^2}{\delta k^4}}(kx + ny - \lambda t + \xi_0)\right]\right\}^{\frac{1}{p}} \quad (3.47)$$

(2) when $R > 0$, i.e. $\delta(sn^2 - \lambda k) > 0$,

$$u_{53} = \left\{\pm\sqrt{-\frac{(1+p)(1+2p)(sn^2 - \lambda k)}{bk^2}} \times \sec\left[\sqrt{\frac{(sn^2 - \lambda k)p^2}{\delta k^4}}(kx + ny - \lambda t + \xi_0)\right]\right\}^{\frac{1}{p}} \quad (3.48)$$

$$u_{54} = \left\{ \pm\sqrt{-\frac{(1+p)(1+2p)(sn^2-\lambda k)}{bk^2}} \times \operatorname{cosec}\left[\sqrt{\frac{(sn^2-\lambda k)p^2}{\delta k^4}}(kx+ny-\lambda t+\xi_0)\right]\right\}^{\frac{1}{p}}. \tag{3.49}$$

Case 6. From (3.22), the equation, $\left(u_t + bu^2u_x + \delta u_{xxx}\right)_x + su_{yy} = 0$, has the following formal solutions:

(1) when $R < 0$, i.e. $\delta(sn^2 - \lambda k) > 0$,

$$u_{61} = \pm M\{\tanh[\sqrt{-R}(kx+ny-\lambda t+\xi_0)] \pm \mathrm{i}\,\mathrm{sech}[\sqrt{-R}(kx+ny-\lambda t+\xi_0)]\} \tag{3.50}$$

$$u_{62} = \pm M\{\coth[\sqrt{-R}(kx+ny-\lambda t+\xi_0)] \pm \mathrm{cosech}[\sqrt{-R}(kx+ny-\lambda t+\xi_0)]\} \tag{3.51}$$

where $M = A_1\sqrt{-R} = \sqrt{-\frac{3(sn^2-\lambda k)}{bk^2}}$, $R = -\frac{2(sn^2-\lambda k)}{\delta k^4}$;

(2) when $R > 0$, $\delta(sn^2 - \lambda k) < 0$,

$$u_{63} = \pm M\{\tan[\sqrt{R}(kx+ny-\lambda t+\xi_0)] \pm \sec[\sqrt{R}(kx+ny-\lambda t+\xi_0)]\} \tag{3.52}$$

$$u_{64} = \pm M\{\cot[\sqrt{R}(kx+ny-\lambda t+\xi_0)] \pm \operatorname{cosec}[\sqrt{R}(kx+ny-\lambda t+\xi_0)]\} \tag{3.53}$$

$M = A_1\sqrt{R} = \sqrt{\frac{3(sn^2-\lambda k)}{bk^2}}$, $R = -\frac{2(sn^2-\lambda k)}{\delta k^4}$.

Case 7. From (3.23), the equation, $\left(u_t + auu_x + bu^2u_x + \delta_{xxx}\right)_x + su_{yy} = 0$, has the following formal solutions:

(1) when $R < 0$, i.e. $\delta > 0$,

$$u_{71} = \pm\sqrt{\frac{a^2}{4b}} \pm \sqrt{-\frac{a^2}{2b}}\,\mathrm{sech}\left[\sqrt{\frac{a^2}{12\delta k^2}}(kx+ny-\lambda t+\xi_0)\right] \tag{3.54}$$

$$u_{72} = \pm\sqrt{\frac{a^2}{4b}} \pm \sqrt{\frac{a^2}{2b}}\,\mathrm{cosech}\left[\sqrt{\frac{a^2}{12\delta k^2}}(kx+ny-\lambda t+\xi_0)\right] \tag{3.55}$$

(2) when $R > 0$, i.e. $\delta < 0$,

$$u_{73} = \pm\sqrt{\frac{a^2}{4b}} \pm \sqrt{\frac{a^2}{2b}}\sec\left[\sqrt{-\frac{a^2}{12\delta k^2}}(kx+ny-\lambda t+\xi_0)\right] \tag{3.56}$$

$$u_{74} = \pm\sqrt{\frac{a^2}{4b}} \pm \sqrt{\frac{a^2}{2b}}\operatorname{cosec}\left[\sqrt{-\frac{a^2}{12\delta k^2}}(kx+ny-\lambda t+\xi_0)\right] \tag{3.57}$$

where $\lambda = -\frac{a^2k}{6} + \frac{sn^2}{k}$.

Remark 2.

(1) It is easy to see that, when $n = s = 0$, the solutions (3.33) are the same as the solutions (1.8) with $\gamma = 0$, i.e. the results in [4].

(2) From our results, the solutions of some well-known equations such as the KdV equation, mKdV equation, combined KdV–mKdV equation, KdV–Burgers equation in (1 + 1)-dimensional cases and KP equation, mKP equation and various generalized KP equations in (2 + 1)-dimensional cases [13–20], can be recovered. For example, taking $p = 1, n = s = 0, \lambda = vk$ in (3.46), we obtain the solutions

$$u_{(x,t)} = \pm\sqrt{\frac{6v}{b}}\,\mathrm{sech}\left[\sqrt{\frac{v}{\delta}}(x-vt+\xi_0)\right]$$

which are the bell-profile solitary-wave solutions of the mKdV equation.

4. The bell-profile solitary-wave solutions to the general two-dimensional KdV-type equation with nonlinear terms of any order

In section 3, the bell-profile solitary-wave solutions (3.46) for equation (I) with $a = 0$ are obtained. In this section, we consider the bell-profile solitary-wave solutions for equation (I) under the condition $a \neq 0$. By using the same deduction as for formula (3.6), we know that equation (I) changes into the following equation:

$$(1+p)(1+2p)\delta k^4[p\varphi(\xi)\varphi''(\xi) + (1-p)\varphi'^2(\xi)] + p^2[(1+p)(1+2p)(sn^2 - \lambda k)\varphi^2(\xi) + (1+2p)ak^2\varphi^3(\xi) + (1+p)bk^2\varphi^4(\xi)] = 0. \tag{4.1}$$

Now, we assume that the solution of equation (4.1) has the following form:

$$\varphi(\xi) = \frac{A\,\mathrm{e}^{\alpha(\xi+\xi_0)}}{(1+\mathrm{e}^{\alpha(\xi+\xi_0)})^2 + B\,\mathrm{e}^{\alpha(\xi+\xi_0)}} = \frac{A\,\mathrm{sech}^2(\alpha/2)(\xi+\xi_0)}{4 + B\,\mathrm{sech}^2(\alpha/2)(\xi+\xi_0)} \tag{4.2}$$

where A, B and α are constants to be determined, and ξ_0 is an arbitrary phase shift.

With the aid of Mathematica, substituting (4.2) into (4.1), we obtain

$$A^2(1+p)(1+2p)(p^2(n^2s - k\lambda) + k^4\alpha^2\delta) = 0 \tag{4.3}$$

$$A^2p(1+2p)(aAk^2p + 2(2+B)p(1+p)(n^2s - k\lambda) - (2+B)k^4(1+p)\alpha^2\delta) = 0 \tag{4.4}$$

$$A^2(A^2bk^2p^2(1+p) + aA(2+B)k^2p^2(1+2p) + (6+4B+B^2)p^2(1+p)(1+2p)(n^2s - k\lambda) - 2k^4(1+p)(\alpha+2p\alpha)^2\delta) = 0 \tag{4.5}$$

$$A^2p(1+2p)(aAk^2p + 2(2+B)p(1+p)(n^2s - k\lambda) - (2+B)k^4(1+p)\alpha^2\delta) = 0 \tag{4.6}$$

$$A^2(1+p)(1+2p)(p^2(n^2s - k\lambda) + k^4\alpha^2\delta) = 0. \tag{4.7}$$

By solving equations (4.3)–(4.7) with the aid of Mathematica, we get the following two groups of solutions:

$$\alpha = \pm\sqrt{\frac{p^2(-sn^2 + \lambda k)}{\delta k^4}} \tag{4.8}$$

$$A_{1.2} = \mp(1+p)(2+p)(sn^2 - k\lambda)L \tag{4.9}$$

$$B_{1.2} = -2 \pm 2ak^2L \tag{4.10}$$

where $L = \frac{\sqrt{1+2p}}{\sqrt{k^2[a^2k^2(1+2p) - b(1+p)(2+p)^2(sn^2 - \lambda k)]}}$.

Therefore, there are two solutions of the form (4.2) to equation (4.1):

$$\varphi_1(\xi) = \frac{-(1+p)(2+p)(sn^2 - k\lambda)L\,\mathrm{sech}^2\left[\pm\sqrt{\frac{p^2(-sn^2+\lambda k)}{4\delta k^4}}(\xi+\xi_0)\right]}{2 + (-1 + ak^2L)\,\mathrm{sech}^2\left[\pm\sqrt{\frac{p^2(-sn^2+\lambda k)}{4\delta k^4}}(\xi+\xi_0)\right]} \tag{4.11}$$

$$\varphi_2(\xi) = \frac{(1+p)(2+p)(sn^2 - k\lambda)L\,\mathrm{sech}^2\left[\pm\sqrt{\frac{p^2(-sn^2+\lambda k)}{4\delta k^4}}(\xi+\xi_0)\right]}{2 + (-1 - ak^2L)\,\mathrm{sech}^2\left[\pm\sqrt{\frac{p^2(-sn^2+\lambda k)}{4\delta k^4}}(\xi+\xi_0)\right]}. \tag{4.12}$$

From (4.11) and (4.12), equation (I) has two families of solutions as follows:

$$u_1(x, y, t) = v(\xi) = [\varphi_1(\xi)]^{\frac{1}{p}} \tag{4.13}$$

where $\varphi_1(\xi)$ is given by (4.11).

$$u_2(x, y, t) = v(\xi) = [\varphi_2(\xi)]^{\frac{1}{p}} \tag{4.14}$$

where $\varphi_2(\xi)$ is given by (4.12).

Remark 3. It is easy to see that

(1) when $n = y = 0$ and $s = 0$, the solutions (4.13) and (4.14) are just the solutions (1.7)
(2) when $a = 0$, the solutions (4.13) and (4.14) are just the solutions (3.46).

5. Conclusions

In this paper, firstly we present two new equations, the general KdV-type and general KdV–Burgers-type equations with nonlinear terms of any order. Secondly, by means of a proper transformation and a more general ansatz, we improve the tanh-method. Applying the improved method and direct assumption method, many types of exact solutions for these two equations, which contain kink-profile solitary-wave solutions, bell-profile solitary-wave, rational solutions, periodic solutions and combined formal solutions, are obtained. The method can also be easily extended to treat other partial differential equations (PDEs) and is sufficient to seek more solitary-wave solutions and other formal solutions of given PDEs. In addition, this method is also computerizable, which allows us to perform complicated and tedious symbolic algebraic calculations on a computer.

In a recent paper [21], Gao and Tian presented a generalized hyperbolic-function method (HFM) with computerized symbolic computation to construct the solitonic solutions to nonlinear equations of mathematical physics. Using the HFM, one can obtain (a) the non-travelling solitonic solutions, (b) the multi-hyperbolic function solutions, (c) the coefficient function solutions for some NPDEs. However, they only considered the case for which the balancing constant is a positive integer. In [22], Ohta studied stability and instability of standing waves for the nonlinear Schrödinger equation $\mathrm{i}u_t + u_{xx} + f(u) = 0$ $(t \geqslant 0, x \in \mathbb{R})$, where $f(u) = a|u|^{p-1}u + b|u|^{q-1}u$ with $a, b \in \mathbb{R}$ and $1 < p < q < +\infty$. The idea in [22] is very useful for studying the NPDEs with nonlinear terms of any order. In a forthcoming paper, we shall combine our method with the HFM in [21] and the idea in [22] to seek further for the solutions of some NPDEs with nonlinear terms of any order.

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