Available online at www.sciencedirect.com
SCIENCE DIRECT
PHYSICS LETTERS A
ELSEVIER
Physics Letters A 311 (2003) 145–157
www.elsevier.com/locate/pla

Generalized extended tanh-function method and its application to (1 + 1)-dimensional dispersive long wave equation

Xuedong Zheng, Yong Chen *, Hongqing Zhang

Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, PR China

Received 9 September 2002; received in revised form 25 February 2003; accepted 11 March 2003

Communicated by A.R. Bishop

Abstract

Making use of a new generalized ansatzes, we present the generalized extended tanh-function method for constructing the exact solutions of nonlinear partial differential equations (NPDEs) in a unified way. Applying the generalized method, with the aid of MAPLE, we consider the Wu–Zhang equation (which describes $(1+1)$-dimensional dispersive long wave). As a result, we can successfully obtain the solitary wave solutions that can be found by the extended tanh-function method and the modified extended tanh-function method. More importantly, for the equation, we also obtain other new and more general solutions at the same time. The results include kink-profile solitary wave solutions, bell-profile solitary wave solutions, periodic wave solutions, rational solutions, singular solutions and other new formal solutions. As an illustrative sample, the properties of some soliton solutions for Wu–Zhang equation are shown by some figures.

Keywords: Extended tanh-function method; Nonlinear partial differential equation; Wu–Zhang equation; Solitary wave solutions; Symbolic computation; Wu-elimination

1. Introduction

As is well known, the nonlinear partial differential equations (NPDEs) are widely used to describe many important phenomena and dynamic processes in physics, mechanics, chemistry, biology, etc. Many efforts have been made on the study of NPDEs. Long wave in shallow water is a subject of broad interests and has a long colorful history. Physically, it has a rich variety of phenomenological manifestation, especially the existence of waves permanent in form and robust in maintaining their entities through mutual interaction and collision as well as the remarkable property of exhibiting recurrences of initial data when circumstances should prevail. These characteristics are due to the intimate interplay between the roles of nonlinearity and dispersion. Mathematically, it has been noted that validity of theoretical models critically depends on the domain of underlying key parameters which characterize the specific motions to be modelled. In [1], Wu and Zhang derived three sets of model equations

* Corresponding author.
E-mail address: chenyong@dlut.edu.en (Y. Chen).

0375-9601/03/$ – see front matter © 2003 Elsevier Science B.V. All rights reserved.
doi:10.1016/S0375-9601(03)00451-1

for modelling nonlinear and dispersive long gravity waves travelling in two horizontal directions on shallow waters of uniform depth. Their comparative study of these models is directed to explore the intrinsic properties in physical and mathematical terms that these models possess. Omitting the higher order terms, one of these equations, Wu–Zhang (WZ) equation, can be written as [1,2]

$$u_t + uu_x + vu_y + w_x = 0, \tag{1}$$

$$v_t + uv_x + vv_y + w_y = 0, \tag{2}$$

$$w_t + (uw)_x + (vw)_y + \frac{1}{3}(u_{xxx} + u_{xyy} + v_{xxy} + v_{yyy}) = 0, \tag{3}$$

where w is the elevation of the water, u is the surface velocity of water along x-direction, v is the surface velocity of water along y-direction. By scaling transformation and symmetry reduction, Eqs. (1)–(3) can be reduced to the $(1+1)$-dimensional dispersive long wave equation [1,2]

$$v_t + vv_x + w_x = 0, \tag{4}$$

$$w_t + (wv)_x + \frac{1}{3}v_{xxx} = 0. \tag{5}$$

A good understanding of all solutions of Eqs. (1)–(3) and Eqs. (4), (5) is very helpful for coastal and civil engineers to apply the nonlinear water wave model in harbor and coastal design. In [2], some special type soliton solutions for Eqs. (1)–(3) are derived directly by using the standard and nonstandard truncation of the WTC's approach and the modified Conte's invariant Painlevé expansion for the WZ equation [2–8]. The present work is motivated by the desire to find new and more formal solutions with the use of some proper transformations and the generalized method in the hope that the method will lead to a deeper and more comprehensive understanding of the complex structure of the NPDEs, and we consider WZ equations (4) and (5). On the one hand, to seek more formal solutions of NPDEs is needed from mathematical point of view; on the other hand, the more formal solutions of NPDEs may provide a useful help for physicist in studying more complicated physical phenomena. Our approach stems mainly from the extended tanh-function method presented by Fan [9,10]. The tanh-function method has been proposed to construct exact solution to NPDEs [11–16]. Recently, based on the well-known Riccati equation, Fan presented a useful extended tanh-function method to find exact solutions of given NPDEs [9,10]. More recently Fan [17,18] and Yan [19] further develop this idea and made it much more lucid and straightforward for a class of NPDEs. Most recently, Elwakil presented the modified extended tanh-function method [20]. By introducing more general proper transformation we improve the above extended tanh-function method. As a result, we can successfully recover the previously known solitary wave solutions that can be found by the extended tanh-function method and modified extended tanh-function method. More importantly, we also obtain other new and more general solutions of Eqs. (4), (5) at the same time. The results include kink-profile solitary-wave solutions, bell-profile solitary-wave solutions, periodic wave solutions, rational solutions, singular solutions and other new formal solutions.

This Letter is arranged as follows. In Section 2, we present the generalized extended tanh-function method. In Section 3, the method is used to deal with the system of Wu-Zhang equations. In addition, as an illustrative sample, the properties of some type of soliton solutions for Eqs. (4), (5) are shown by some figures. Finally, some summaries and conclusions are given.

2. Summary of the generalized method

Now, we simply describe the generalized extended tanh-function method. Consider a given system of NPDEs, say, in two variables, x, t,

$$H_j(u_i, u_{i,t}, u_{i,x}, u_{i,xt}, u_{i,tt}, \ldots) = 0 \quad (i, j = 1, 2, 3, \ldots, n), \tag{6}$$

where j means the ith equation, n denotes the number of the equations. We first consider the following formal travelling wave solutions $u_i(x,t) = \phi_i(\xi)$, $\xi = x - \lambda t$, where λ is a constant to be determined later. Then Eq. (6) becomes a system of nonlinear ordinary differential equations (NODEs)

$$F_j(\phi_i, \phi_i', \phi_i'', \phi_i''', \ldots) = 0, \tag{7}$$

where $'$ denotes $\frac{d}{d\xi}$. In order to seek the travelling wave solutions of Eq. (7), we introduce the following new ansatzes

$$\phi_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \left\{ a_{ij}\omega^j + b_{ij}\omega^{-j} + c_{ij}\omega^{j-1}\sqrt{R+\omega^2} + d_{ij}\frac{\sqrt{R+\omega^2}}{\omega^j} \right\}, \tag{8}$$

and the new variable $\omega = \omega(\xi)$ satisfying

$$\omega' = \frac{d\omega}{d\xi} = R + \omega^2, \tag{9}$$

where $a_{i0}, a_{ij}, b_{ij}, c_{ij}, d_{ij}$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m_i)$ and R are constants to be determined later. The value of m_i in Eq. (8) can be determined by balancing the highest-order derivative term with the nonlinear term [9,10] in Eq. (6) or Eq. (7). After that, substitute Eq. (8) into Eq. (7), the corresponding NODEs, and then let all coefficients of $\omega^p(\sqrt{R+\omega^2})^q$ $(q = 0, 1;\ p = 0, 1, 2, \ldots)$ to be zero to get an over-determined system of nonlinear algebraic equations with respect to $a_{i0}, a_{ij}, b_{ij}, c_{ij}, d_{ij}, R, \lambda$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m_i)$. With the aid of MAPLE, we apply Wu-elimination method [21,22] to solve the above mentioned over-determined system of nonlinear algebraic equations to yield the values of $a_{i0}, a_{ij}, b_{ij}, c_{ij}, d_{ij}, R, \lambda$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m_i)$. Because the Riccati equation, Eq. (9), has the following general solutions,

(i) if $R < 0$,

$$\omega(\xi) = -\sqrt{-R}\tanh\big(\sqrt{-R}\,\xi\big), \tag{10}$$

$$\omega(\xi) = -\sqrt{-R}\coth\big(\sqrt{-R}\,\xi\big), \tag{11}$$

(ii) if $R = 0$,

$$\omega(\xi) = -\frac{1}{\xi}, \tag{12}$$

(iii) if $R > 0$,

$$\omega(\xi) = \sqrt{R}\tan\big(\sqrt{R}\,\xi\big), \tag{13}$$

$$\omega(\xi) = -\sqrt{R}\cot\big(\sqrt{R}\,\xi\big), \tag{14}$$

then combined the values of $a_{i0}, a_{ij}, b_{ij}, c_{ij}, d_{ij}, R, \lambda$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m_i)$ with Eq. (8), and Eqs. (10)–(14), more travelling wave solutions of Eq. (6) are obtained.

The algorithm is more powerful than the typical tanh method [11–16], the extended tanh-function method [9,10,17–19] and the modified extended tanh-function method [20]. When $b_{ij} = 0$, $c_{ij} = 0$, $d_{ij} = 0$ $(i = 1, 2, \ldots, n;$ $j = 1, 2, \ldots, m_i)$ in Eq. (8), Eq. (8) becomes the transformation proposed by Fan [10]. When $b_{ij} = 0$, $d_{ij} = 0$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m_i)$ in Eq. (8), Eq. (8) becomes the transformation proposed by Yan [19]. When $c_{ij} = 0$, $d_{ij} = 0$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m_i)$ in Eq. (8), Eq. (8) becomes the transformation proposed by Elwakil [20]. So we would find many new exact solutions of system (6) by the generalized extended tanh-function method.

In what follows we would like to apply the generalized method to WZ equation (4) and (5) to illustrate our algorithm which is more powerful than the typical tanh method and other sophisticated tanh-function method.

Note. Since tan- and cot-type solutions appear in pairs with tanh- and coth-type solutions, respectively, they are omitted in this Letter.

3. Exact solutions of WZ equation

In this section, we consider the WZ equation, i.e., Eqs. (4) and (5). Firstly, we let

$$v(x,t)=\phi(\xi), \qquad w(x,t)=\theta(\xi), \qquad \xi=x-\lambda t, \tag{15}$$

where λ is constant to be determined later. Then the system of Eqs. (4) and (5) reduces to a system of NODEs

$$-\lambda\phi+\frac{1}{2}\phi^2+\theta=0, \tag{16}$$

$$-\lambda\theta+\phi\theta+\frac{1}{3}\phi''=0. \tag{17}$$

According to description in Section 2, by balancing the highest-order derivative term with the nonlinear term in Eqs. (4) and (5), we support that system of Eqs. (16) and (17) has the following formal solutions

$$\phi=a_{10}+a_{11}\omega+\frac{b_{11}}{\omega}+c_{11}\sqrt{R+\omega^2}+\frac{d_{11}\sqrt{R+\omega^2}}{\omega}, \tag{18}$$

$$\theta=a_{20}+a_{21}\omega+\frac{b_{21}}{\omega}+c_{21}\sqrt{R+\omega^2}+a_{22}\omega^2+\frac{b_{22}}{\omega^2}+c_{22}\omega\sqrt{R+\omega^2} + \frac{d_{21}\sqrt{R+\omega^2}}{\omega}+\frac{d_{22}\sqrt{R+\omega^2}}{\omega^2}, \tag{19}$$

and $\omega=\omega(\xi)$ satisfying Eq. (9), where a_{10}, a_{11}, b_{11}, c_{11}, d_{11}, a_{20}, a_{21}, b_{21}, c_{21}, d_{21}, a_{22}, b_{22}, c_{22}, d_{22}, R,λ are constants to be determined later. With the aid of MAPLE, substitute Eqs. (18) and (19) into Eqs. (16) and (17) along with Eq. (9), and let the coefficients to be zero of $\omega^p(\sqrt{R+w^2})^q$ $(q=0,1;\ p=0,1,2,\ldots)$ with the same power, we get the over-determined system of algebraic equations. With the aid of MAPLE and applying Wu-elimination method [21,22], we can obtain the solutions of the over-determined system. To avoid the tediousness, we omit the over-determined system obtained and its solutions' cases.

So, combining Eqs. (10) and (11) along with cases of solutions of the over-determined system, we can obtain the travelling wave solutions of WZ equations (4) and (5) as follows.

Case 1.

$$v_{11}=\pm\frac{2}{3}\sqrt{-3R}+\frac{2}{3}\frac{\sqrt{3}R\coth(\sqrt{-R}\,\xi)}{\sqrt{-R}}, \tag{20}$$

$$w_{11}=-\frac{2}{3}R+\frac{2}{3}R\coth^2\bigl(\sqrt{-R}\,\xi\bigr), \tag{21}$$

$$v_{12}=\pm\frac{2}{3}\sqrt{-3R}+\frac{2}{3}\frac{\sqrt{3}R\tanh(\sqrt{-R}\,\xi)}{\sqrt{-R}}, \tag{22}$$

$$w_{12}=-\frac{2}{3}R+\frac{2}{3}R\tanh^2\bigl(\sqrt{-R}\,\xi\bigr), \tag{23}$$

where $\xi=x\mp\frac{2}{3}\sqrt{-3R}\,t$. The properties of the solitary solutions v_{12}, w_{12} are shown in Fig. 1.

Case 2.

$$v_{21} = -\frac{1}{3}\sqrt{6}\sqrt{R} \pm \frac{2}{3}\sqrt{3}\sqrt{-R}\,\mathrm{sech}\big(\sqrt{-R}\,\xi\big), \tag{24}$$

$$w_{21} = -\frac{1}{3}R + \frac{2}{3}R\tanh^2\big(\sqrt{-R}\,\xi\big), \tag{25}$$

$$v_{22} = -\frac{1}{3}\sqrt{6}\sqrt{R} \pm \frac{2}{3}\sqrt{3}\sqrt{-R}\,\mathrm{csch}\big(\sqrt{-R}\,\xi\big), \tag{26}$$

$$w_{22} = -\frac{1}{3}R + \frac{2}{3}R\coth^2\big(\sqrt{-R}\,\xi\big), \tag{27}$$

where $\xi = x + \frac{1}{3}\sqrt{6}\sqrt{R}\,t$.

Case 3.

$$v_{31} = \pm\frac{2}{3}\sqrt{-3R} - \frac{2}{3}\frac{\sqrt{3}R\coth(\sqrt{-R}\,\xi)}{\sqrt{-R}}, \tag{28}$$

$$w_{31} = -\frac{2}{3}R + \frac{2}{3}R\coth^2\big(\sqrt{-R}\,\xi\big), \tag{29}$$

$$v_{32} = \pm\frac{2}{3}\sqrt{-3R} - \frac{2}{3}\frac{\sqrt{3}R\tanh(\sqrt{-R}\,\xi)}{\sqrt{-R}}, \tag{30}$$

$$w_{32} = -\frac{2}{3}R + \frac{2}{3}R\tanh^2\big(\sqrt{-R}\,\xi\big), \tag{31}$$

where $\xi = x \mp \frac{2}{3}\sqrt{-3R}\,t$.

Case 4.

$$v_{41} = \pm\frac{2}{3}\sqrt{-3R} + \frac{2}{3}\sqrt{3}\sqrt{-R}\tanh\big(\sqrt{-R}\,\xi\big), \tag{32}$$

$$w_{41} = -\frac{2}{3}R + \frac{2}{3}R\tanh^2\big(\sqrt{-R}\,\xi\big), \tag{33}$$

$$v_{42} = \pm\frac{2}{3}\sqrt{-3R} + \frac{2}{3}\sqrt{3}\sqrt{-R}\coth\big(\sqrt{-R}\,\xi\big), \tag{34}$$

$$w_{42} = -\frac{2}{3}R + \frac{2}{3}R\coth^2\big(\sqrt{-R}\,\xi\big), \tag{35}$$

where $\xi = x \mp \frac{2}{3}\sqrt{-3R}\,t$.

Case 5.

$$v_{51} = \pm\frac{2}{3}\sqrt{-3R} - \frac{2}{3}\sqrt{3}\sqrt{-R}\tanh\big(\sqrt{-R}\,\xi\big), \tag{36}$$

$$w_{51} = -\frac{2}{3}R + \frac{2}{3}R\tanh^2\big(\sqrt{-R}\,\xi\big), \tag{37}$$

$$v_{52} = \pm\frac{2}{3}\sqrt{-3R} - \frac{2}{3}\sqrt{3}\sqrt{-R}\coth\big(\sqrt{-R}\,\xi\big), \tag{38}$$

$$w_{52} = -\frac{2}{3}R + \frac{2}{3}R\coth^2\big(\sqrt{-R}\,\xi\big), \tag{39}$$

where $\xi = x \mp \frac{2}{3}\sqrt{-3R}\,t$.

Case 6.

$$v_{61} = -\frac{1}{3}\sqrt{-3R} \mp \frac{1}{3}\sqrt{3}\sqrt{-R}\tanh(\sqrt{-R}\,\xi) - \frac{1}{3}i\sqrt{3}\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\,\xi), \tag{40}$$

$$w_{61} = -\frac{1}{3}R + \frac{1}{3}R\tanh^2(\sqrt{-R}\,\xi) \pm \frac{1}{3}iR\tanh(\sqrt{-R}\,\xi)\,\mathrm{sech}(\sqrt{-R}\,\xi), \tag{41}$$

$$v_{62} = -\frac{1}{3}\sqrt{-3R} \mp \frac{1}{3}\sqrt{3}\sqrt{-R}\coth(\sqrt{-R}\,\xi) - \frac{1}{3}\sqrt{3}\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi), \tag{42}$$

$$w_{62} = -\frac{1}{3}R + \frac{1}{3}R\coth^2(\sqrt{-R}\,\xi) \pm \frac{1}{3}R\coth(\sqrt{-R}\,\xi)\,\mathrm{csch}(\sqrt{-R}\,\xi), \tag{43}$$

where $\xi = x + \frac{1}{3}\sqrt{-3R}\,t$.

Case 7.

$$v_{71} = \pm\frac{1}{3}\sqrt{-3R} - \frac{1}{3}\sqrt{3}\sqrt{-R}\tanh(\sqrt{-R}\,\xi) \mp \frac{1}{3}i\sqrt{3}\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\,\xi), \tag{44}$$

$$w_{71} = -\frac{1}{3}R + \frac{1}{3}R\tanh^2(\sqrt{-R}\,\xi) \pm \frac{1}{3}iR\tanh(\sqrt{-R}\,\xi)\,\mathrm{sech}(\sqrt{-R}\,\xi), \tag{45}$$

$$v_{72} = \pm\frac{1}{3}\sqrt{-3R} - \frac{1}{3}\sqrt{3}\sqrt{-R}\coth(\sqrt{-R}\,\xi) \mp \frac{1}{3}\sqrt{3}\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi), \tag{46}$$

$$w_{72} = -\frac{1}{3}R + \frac{1}{3}R\coth^2(\sqrt{-R}\,\xi) \pm \frac{1}{3}R\coth(\sqrt{-R}\,\xi)\,\mathrm{csch}(\sqrt{-R}\,\xi), \tag{47}$$

where $\xi = x \mp \frac{1}{3}\sqrt{-3R}\,t$.

Case 8.

$$v_{81} = \frac{1}{3}\sqrt{6}\sqrt{R} \pm \frac{2}{3}i\sqrt{3}\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\,\xi), \tag{48}$$

$$w_{81} = -\frac{1}{3}R + \frac{2}{3}R\tanh^2(\sqrt{-R}\,\xi), \tag{49}$$

$$v_{82} = \frac{1}{3}\sqrt{6}\sqrt{R} \pm \frac{2}{3}\sqrt{3}\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi), \tag{50}$$

$$w_{82} = -\frac{1}{3}R + \frac{2}{3}R\coth^2(\sqrt{-R}\,\xi), \tag{51}$$

where $\xi = x - \frac{1}{3}\sqrt{6}\sqrt{R}\,t$.

Case 9.

$$v_{91} = \pm\frac{1}{3}\sqrt{-3R} + \frac{1}{3}\sqrt{3}\sqrt{-R}\tanh(\sqrt{-R}\,\xi) + \frac{1}{3}i\sqrt{3}\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\,\xi), \tag{52}$$

$$w_{91} = -\frac{1}{3}R + \frac{1}{3}R\tanh^2(\sqrt{-R}\,\xi) + \frac{1}{3}iR\tanh(\sqrt{-R}\,\xi)\,\mathrm{sech}(\sqrt{-R}\,\xi), \tag{53}$$

$$v_{92} = \pm\frac{1}{3}\sqrt{-3R} + \frac{1}{3}\sqrt{3}\sqrt{-R}\coth(\sqrt{-R}\,\xi) + \frac{1}{3}\sqrt{3}\sqrt{-R}\,\mathrm{csch}(\sqrt{-R}\,\xi), \tag{54}$$

$$w_{92} = -\frac{1}{3}R + \frac{1}{3}R\coth^2(\sqrt{-R}\,\xi) + \frac{1}{3}R\coth(\sqrt{-R}\,\xi)\,\mathrm{csch}(\sqrt{-R}\,\xi), \tag{55}$$

where $\xi = x \mp \frac{1}{3}\sqrt{-3R}\,t$. The properties of the solitary solutions v_{91}, w_{91} are shown in Figs. 2 and 3.

Case 10.

$$v_{101}=\frac{1}{3}\sqrt{-3R}\mp\frac{1}{3}\sqrt{3}\sqrt{-R}\tanh\big(\sqrt{-R}\,\xi\big)\pm\frac{1}{3}i\sqrt{3}\sqrt{-R}\,\mathrm{sech}\big(\sqrt{-R}\,\xi\big), \tag{56}$$

$$w_{101}=-\frac{1}{3}R+\frac{1}{3}R\tanh^2\big(\sqrt{-R}\,\xi\big)-\frac{1}{3}iR\tanh\big(\sqrt{-R}\,\xi\big)\,\mathrm{sech}\big(\sqrt{-R}\,\xi\big), \tag{57}$$

$$v_{102}=\frac{1}{3}\sqrt{-3R}\mp\frac{1}{3}\sqrt{3}\sqrt{-R}\coth\big(\sqrt{-R}\,\xi\big)\pm\frac{1}{3}\sqrt{3}\sqrt{-R}\,\mathrm{csch}\big(\sqrt{-R}\,\xi\big), \tag{58}$$

$$w_{102}=-\frac{1}{3}R+\frac{1}{3}R\coth^2\big(\sqrt{-R}\,\xi\big)-\frac{1}{3}R\coth\big(\sqrt{-R}\,\xi\big)\,\mathrm{csch}\big(\sqrt{-R}\,\xi\big), \tag{59}$$

where $\xi=x-\frac{1}{3}\sqrt{-3R}\,t$.

Case 11.

$$v_{111}=\mp\frac{2}{3}\sqrt{6}\sqrt{R}-\frac{2}{3}\sqrt{3}\sqrt{-R}\tanh\big(\sqrt{-R}\,\xi\big)-\frac{2}{3}\frac{\sqrt{3}\,R\coth(\sqrt{-R}\,\xi)}{\sqrt{-R}}, \tag{60}$$

$$w_{111}=\frac{2}{3}R\tanh^2\big(\sqrt{-R}\,\xi\big)+\frac{2}{3}R\coth^2\big(\sqrt{-R}\,\xi\big), \tag{61}$$

where $\xi=x\pm\frac{2}{3}\sqrt{6}\sqrt{R}\,t$. The properties of the solitary solutions v_{111}, w_{111} are shown in Figs. 4 and 5.

Case 12.

$$v_{121}=\pm\frac{2}{3}\sqrt{6}\sqrt{R}+\frac{2}{3}\sqrt{3}\sqrt{-R}\tanh\big(\sqrt{-R}\,\xi\big)+\frac{2}{3}\frac{\sqrt{3}\,R\coth(\sqrt{-R}\,\xi)}{\sqrt{-R}}, \tag{62}$$

$$w_{121}=\frac{2}{3}R\tanh^2\big(\sqrt{-R}\,\xi\big)+\frac{2}{3}R\coth^2\big(\sqrt{-R}\,\xi\big), \tag{63}$$

where $\xi=x\mp\frac{2}{3}\sqrt{6}\sqrt{R}\,t$.

Case 13.

$$v_{131}=\pm\frac{4}{3}\sqrt{-3R}\pm\frac{2}{3}\sqrt{3}\sqrt{-R}\tanh\big(\sqrt{-R}\,\xi\big)\pm\frac{2}{3}\sqrt{3}\sqrt{-R}\coth\big(\sqrt{-R}\,\xi\big), \tag{64}$$

$$w_{131}=-\frac{4}{3}R+\frac{2}{3}R\tanh^2\big(\sqrt{-R}\,\xi\big)+\frac{2}{3}R\coth^2\big(\sqrt{-R}\,\xi\big), \tag{65}$$

where $\xi=x\mp\frac{4}{3}\sqrt{-3R}\,t$.

Case 14.

$$v_{141}=\mp\frac{4}{3}\sqrt{-3R}\pm\frac{2}{3}\sqrt{3}\sqrt{-R}\tanh\big(\sqrt{-R}\,\xi\big)\pm\frac{2}{3}\sqrt{3}\sqrt{-R}\coth\big(\sqrt{-R}\,\xi\big), \tag{66}$$

$$w_{141}=-\frac{4}{3}R+\frac{2}{3}R\tanh^2\big(\sqrt{-R}\,\xi\big)+\frac{2}{3}R\coth^2\big(\sqrt{-R}\,\xi\big), \tag{67}$$

where $\xi=x\mp\frac{4}{3}\sqrt{-3R}\,t$.

Case 15.

$$v_{151}=\pm\frac{1}{2}\sqrt{2}\,d_{11}-i d_{11}\operatorname{csch}\left(\frac{1}{2}\sqrt{-3d_{11}^2}\,\xi\right), \tag{68}$$

$$w_{151}=-\frac{1}{4}d_{11}^2+\frac{1}{2}d_{11}^2\coth^2\left(\frac{1}{2}\sqrt{-3d_{11}^2}\,\xi\right), \tag{69}$$

$$v_{152}=\pm\frac{1}{2}\sqrt{2}\,d_{11}-d_{11}\operatorname{sech}\left(\frac{1}{2}\sqrt{-3d_{11}^2}\,\xi\right), \tag{70}$$

$$w_{152}=-\frac{1}{4}d_{11}^2+\frac{1}{2}d_{11}^2\tanh^2\left(\frac{1}{2}\sqrt{-3d_{11}^2}\,\xi\right), \tag{71}$$

where $\xi=x\mp\frac{1}{2}\sqrt{2}\,d_{11}t$. The properties of the solitary solutions v_{152}, w_{152} are shown in Figs. 6–8.

Case 16.

$$v_{161}=\pm i d_{11}-\frac{\sqrt{3}\,d_{11}^2\coth(\sqrt{-3d_{11}^2}\,\xi)}{\sqrt{-3d_{11}^2}}-i d_{11}\operatorname{csch}\left(\sqrt{-3d_{11}^2}\,\xi\right), \tag{72}$$

$$w_{161}=\Theta+d_{11}^2\coth^2\left(\sqrt{-3d_{11}^2}\,\xi\right)+i\Psi\cosh\left(\sqrt{-3d_{11}^2}\,\xi\right)\operatorname{csch}^2\left(\sqrt{-3d_{11}^2}\,\xi\right), \tag{73}$$

$$v_{162}=\pm i d_{11}-\frac{\sqrt{3}\,d_{11}^2\tanh(\sqrt{-3d_{11}^2}\,\xi)}{\sqrt{-3d_{11}^2}}-d_{11}\operatorname{sech}\left(\sqrt{-3d_{11}^2}\,\xi\right), \tag{74}$$

$$w_{162}=\Theta+d_{11}^2\tanh^2\left(\sqrt{-3d_{11}^2}\,\xi\right)+\Psi\sinh\left(\sqrt{-3d_{11}^2}\,\xi\right)\operatorname{sech}^2\left(\sqrt{-3d_{11}^2}\,\xi\right), \tag{75}$$

where $\xi=x\mp i d_{11}t$, $\Theta=-d_{11}^2$, $\Psi=\frac{\sqrt{3}d_{11}}{3}\sqrt{-3d_{11}^2}$.

Case 17.

$$v_{171}=\pm i d_{11}+\frac{\sqrt{3}\,d_{11}^2\coth(\sqrt{-3d_{11}^2}\,\xi)}{\sqrt{-3d_{11}^2}}-i d_{11}\operatorname{csch}\left(\sqrt{-3d_{11}^2}\,\xi\right), \tag{76}$$

$$w_{171}=\Theta+d_{11}^2\coth^2\left(\sqrt{-3d_{11}^2}\,\xi\right)-i\Psi\cosh\left(\sqrt{-3d_{11}^2}\,\xi\right)\operatorname{csch}^2\left(\sqrt{-3d_{11}^2}\,\xi\right), \tag{77}$$

$$v_{172}=\pm i d_{11}+\frac{\sqrt{3}\,d_{11}^2\tanh(\sqrt{-3d_{11}^2}\,\xi)}{\sqrt{-3d_{11}^2}}-d_{11}\operatorname{sech}\left(\sqrt{-3d_{11}^2}\,\xi\right), \tag{78}$$

$$w_{172}=\Theta+d_{11}^2\tanh^2\left(\sqrt{-3d_{11}^2}\,\xi\right)-\Psi\sinh\left(\sqrt{-3d_{11}^2}\,\xi\right)\operatorname{sech}^2\left(\sqrt{-3d_{11}^2}\,\xi\right), \tag{79}$$

where $\xi=x\mp i d_{11}t$, $\Theta=-d_{11}^2$, $\Psi=\frac{\sqrt{3}d_{11}}{3}\sqrt{-3d_{11}^2}$. The properties of the solitary solutions v_{172}, w_{172} are shown in Figs. 9–12.

Remark. Using MAPLE, we have verified all solutions we obtained by putting them back into the original NPDEs, WZ equation (4) and (5). And it is worthy to point out that the new forms of solutions in Cases 11–17 cannot be obtained by the extended tanh-function method [9,10,17–19] and the forms in Cases 15–17 cannot be obtained by the modified extended tanh-function method [20]. In addition, we obtain some new complex formal solutions in the paper, and we suppose that these solutions are not just only the extension from the mathematical meaning, but in the hope that they will lead to a deeper and more comprehensive understanding of the complex structures resulted from the nonlinearity of soliton equations. To compare the new formal solutions for Eqs. (4) and (5) with the known formal solutions, as an illustrative sample, we draw some plots for some formal solutions of WZ equation (4) and (5). The properties of some formal solution are shown in the following figures, respectively.

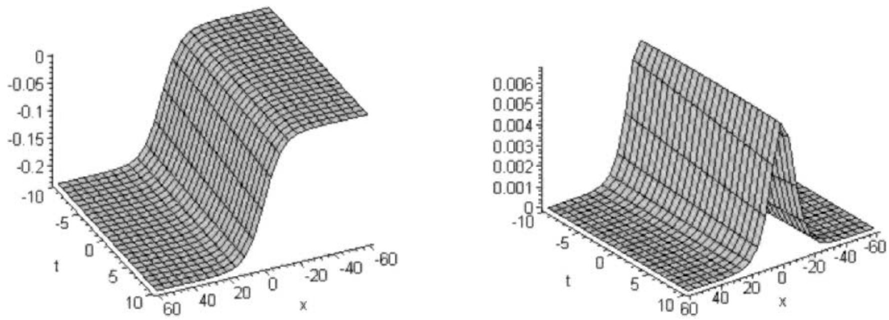

Fig. 1. The solitary solution v_{12} and w_{12}, where $R = -0.01$.

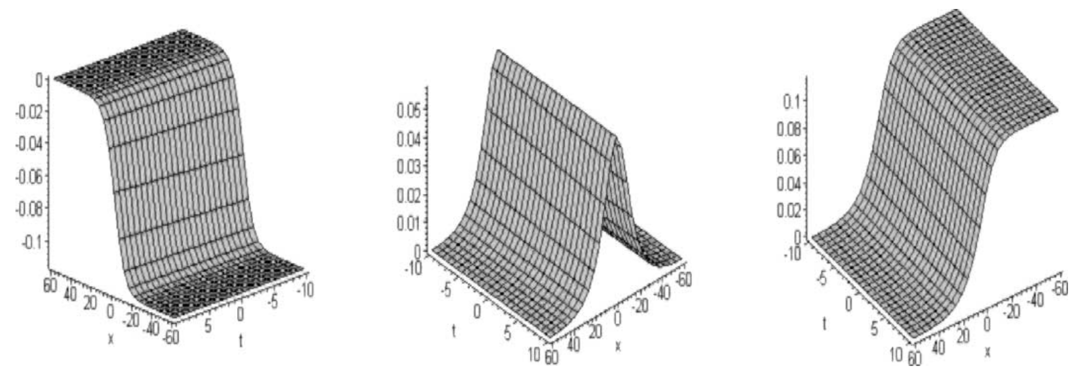

Fig. 2. The solitary solution v_{91}, the real part, imaginary part and the modulus, where $R = -0.01$.

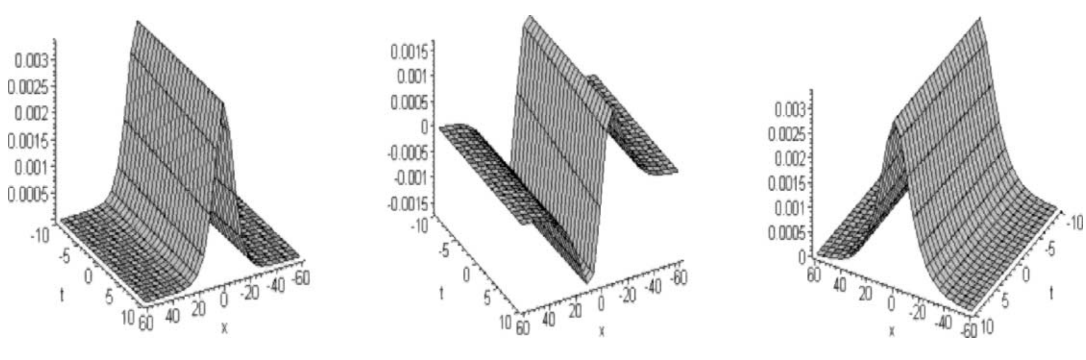

Fig. 3. The solitary solution w_{91}, the real part, imaginary part and the modulus, where $R = -0.01$.

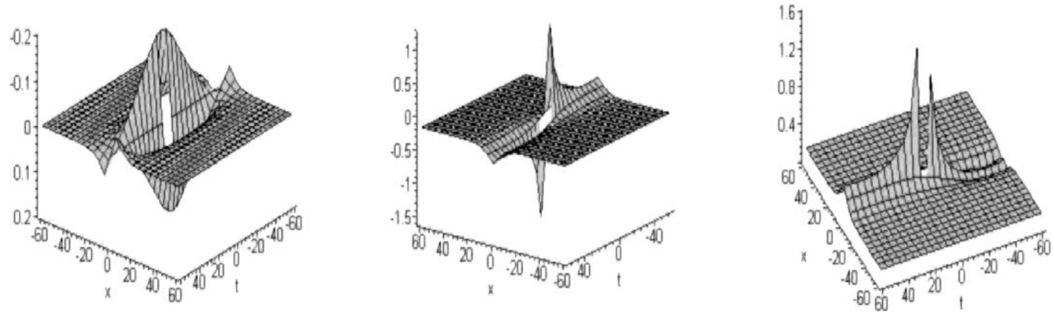

Fig. 4. The solitary solution v_{111}, the real part, imaginary part and the modulus, where $R = -0.01$.

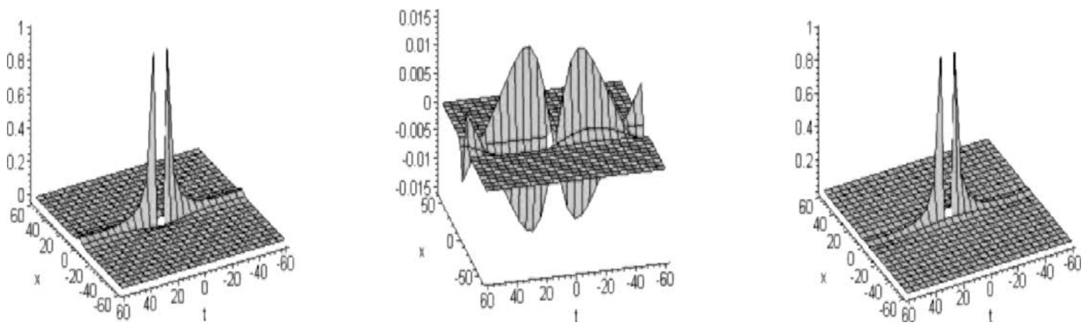

Fig. 5. The solitary solution w_{111}, the real part, imaginary part and the modulus, where $R = -0.01$.

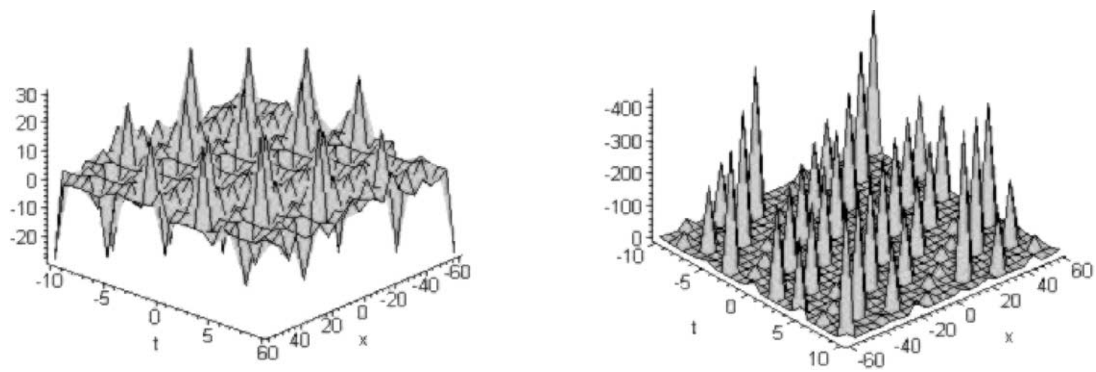

Fig. 6. The solitary solution v_{152} and w_{152}, where $d_{11} = 1$.

4. Conclusions

In this Letter, making use of a new more general ansatze, the extended tanh-function method is generalized, called generalized extended tanh-function method, for finding the exact solutions of NPDEs, and applied to WZ equation. As a result, seventeen families of soliton solutions of WZ equation are found. It's necessary to point out, to our knowledge, that some of the new formal solutions obtained here be not found by known tanh method before. The properties of the some soliton solutions for WZ equation are shown by some figures. The method used in this letter can also be applied to many other NPDEs.

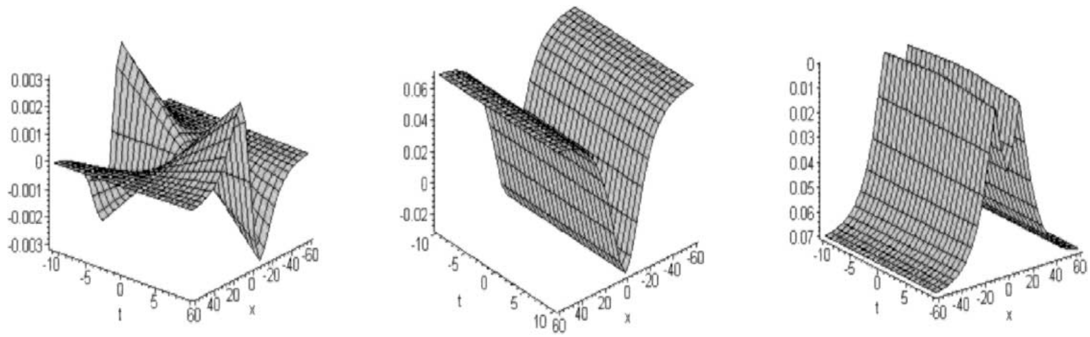

Fig. 7. The solitary solution v_{152}, the real part, imaginary part and the modulus, where $d_{11} = 0.1i$.

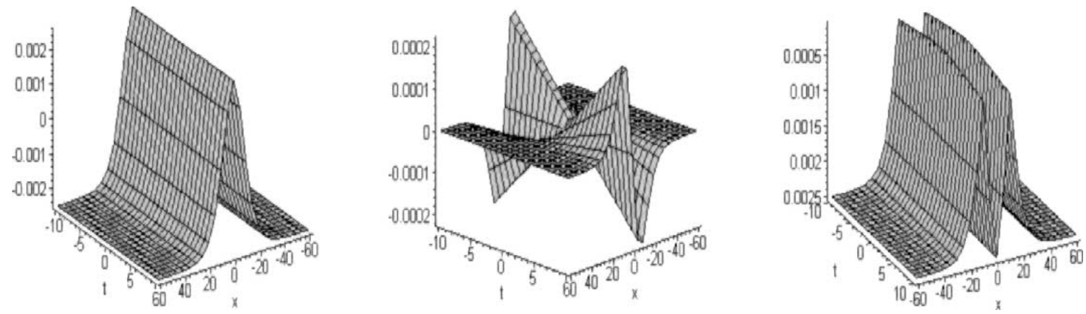

Fig. 8. The solitary solution w_{152}, the real part, imaginary part and the modulus, where $d_{11} = 0.1i$.

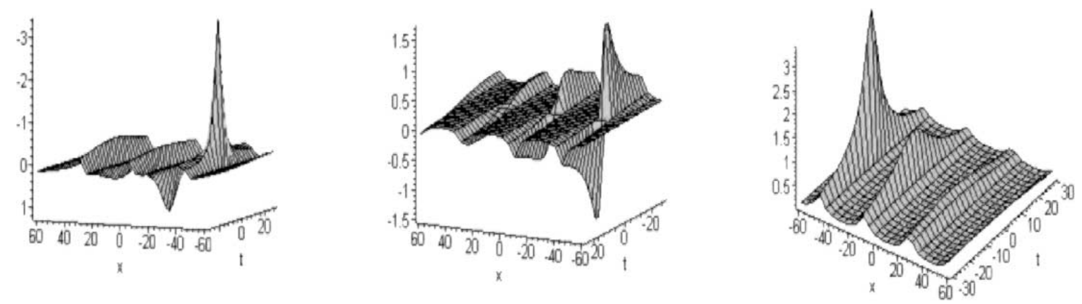

Fig. 9. The solitary solution v_{172}, the real part, imaginary part and the modulus, where $d_{11} = 0.1$.

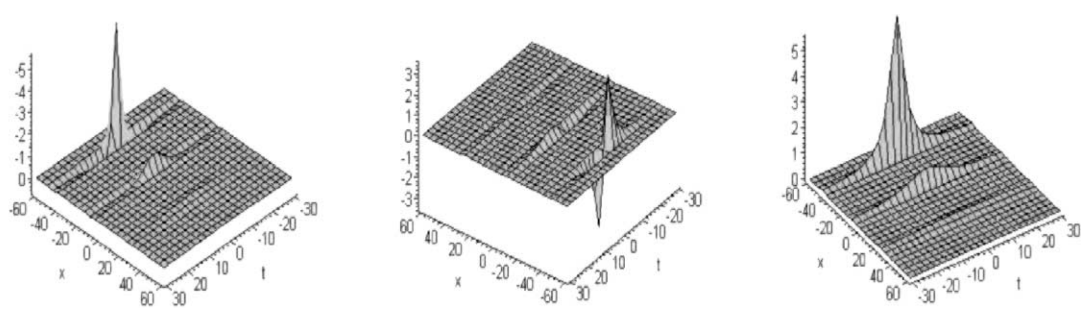

Fig. 10. The solitary solution w_{172}, the real part, imaginary part and the modulus, where $d = 0.1$.

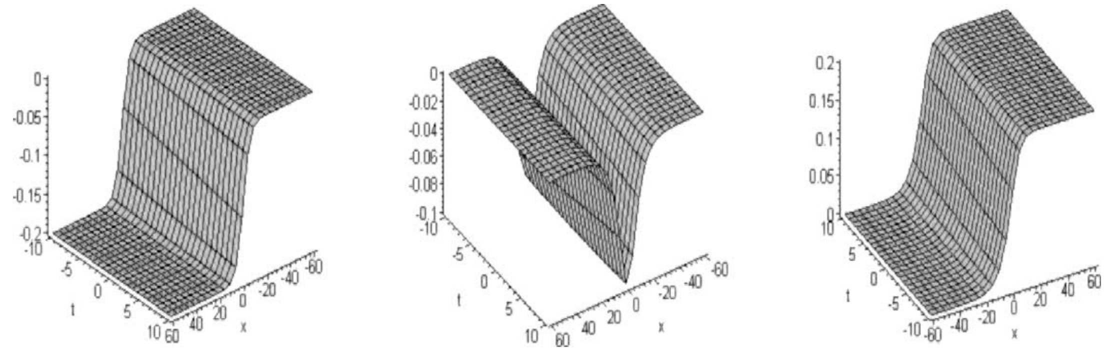

Fig. 11. The solitary solution v_{172}, the real part, imaginary part and the modulus, where $d_{11} = 0.1i$.

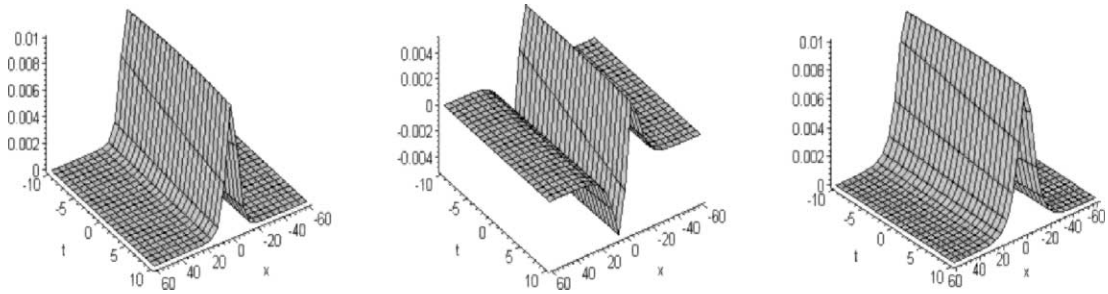

Fig. 12. The solitary solution w_{172}, the real part, imaginary part and the modulus, where $d = 0.1i$.

Acknowledgements

The authors would like to express their thanks to Professor Lou S.Y. for providing references [1,2] to us. The work is supported by the National Natural Science Foundation of China under the Grant No. 1007201, the National Key Basic Research Development Project Program under the Grant No. G1998030600 and Doctoral Foundation of China under the Grant No. 98014119.

References

[1] T.Y. Wu, J.E. Zhang, On modeling nonlinear long wave, in: L.P. Cook, V. Roytbhurd, M. Tulin (Eds.), Mathematics is for Solving Problems, SIAM, 1996, p. 233.
[2] C.L. Chen, X.Y. Tang, S.Y. Lou, Exact solutions of $(2+1)$-dimensional dispersive long wave equation, in preparation.
[3] J. Weiss, M. Tabor, G. Carnevale, J. Math. Phys. 24 (1983) 522.
[4] A. Pickering, J. Phys. A: Math. Gen. 26 (1993) 4395.
[5] S.Y. Lou, Z. Naturforsch. 53 (1998) 251.
[6] J. Weiss, J. Math. Phys. 32 (1991) 1450.
[7] R. Conte, M. Musette, J. Phys. A: Math. Gen. 22 (1989) 169.
[8] F. Cariello, M. Tabor, Physica D 39 (1989) 77.
[9] E. Fan, H.Q. Zhang, Phys. Lett. A 245 (1998) 389.
[10] E. Fan, H.Q. Zhang, Phys. Lett. A 246 (1998) 403.
[11] W. Malkfliet, Am. J. Phys. 60 (1992) 650.
[12] E.J. Parkes, B.R. Duffy, Comput. Phys. Commun. 98 (1996) 288.
[13] B.R. Duffy, E.J. Parkes, Phys. Lett. A 214 (1996) 271.
[14] E.J. Parkes, B.R. Duffy, Phys. Lett. A 299 (1997) 217.

[15] Y.T. Gao, B. Tian, Comput. Math. Appl. 33 (1997) 115.
[16] V.V. Gadkov, J. Math. Phys. 38 (1997) 4794.
[17] E. Fan, Phys. Lett. A 277 (2000) 212.
[18] E. Fan, Z. Naturforsch. 56a (2001) 312.
[19] Z.Y. Yan, Phys. Lett. A 292 (2001) 100.
[20] S.A. Elwakil, S.K. El-labany, M.A. Zahran, R. Sabry, Phys. Lett. A 299 (2002) 179.
[21] W. Wu, in: D.Z. Du, et al. (Eds.), Algorithms and Computation, Springer, Berlin, 1994, p. 1.
[22] W. Wu, Kexue Tongbao 31 (1986) 1 (in Chinese).