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An algorithmic method in Painleve analysis of PDE ☆

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Abstract

In this paper, a new algorithm is presented to decide whether the partial differential algebraic equation possesses Painleve property or not. Under the situation of not getting recursion relations, all resonance points are found. The symbolic computation is employed in all process. As a power tool, Wu–Ritt elimination method plays a important role in the process of testing the compatibility of resonance equations. The algorithm improves on the WTC method on the aspect of the computation.

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1. Introduction

Ablowitz et al. [1] stated that when all the ordinary differential equations (ODE) obtained by exact similarity transforms from a given partial differential equation (PDE) have the Painleve property, then the PDE is "integrable". The definition of "Painleve property" of the ODE was extended to the case of PDE by Weiss et al. [2]. Briefly, a partial differential equation has the Painleve property when the solutions of the PDE are "single-valued" about the movable, and the singularity manifold is "non-characteristic". To be precise, if the singularity manifold is determined by

$$\phi(z_1, z_2, \ldots, z_n) = 0, \tag{1}$$

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and $u = u(z_1, z_2, \ldots, z_n)$ is a solution of the PDE, then we assume that

$$u(z) = \phi^{-p}(z) \sum_{j=0}^{\infty} u_j(z)\phi^j(z), \tag{2}$$

where $u_j(z), \phi(z)$ are analytic functions of $z = (z_1, z_2, \ldots, z_n)$ in a neighborhood of the manifold (1), and $u_0(z) \neq 0$, p is an integer. The requirement that the manifold (1) be non-characteristic insures that the expansion (2) will be well defined, in the sense of Cauchy–Kowalevskaya theorem. Substitution (2) into the PDE determines the value of p, and defines the recursion relations for u_j, $j = 1, 2, \ldots$. When the ansatz (2) is correct, the PDE is said to possess the Painleve property and is conjectured to be integrable [2–4,10]. The Backlund transformation, Lax pair and soliton solutions of the PDE can be obtained by truncating the expansion at the constant level term if the PDE possesses the Painleve property, and the other good properties were studied [3–15].

There are two key steps in the process given in Ref. [1]. One is how to get the recursion relations, the other is how to test the compatibility of resonance equations. In order to achieve it, various skills are employed and a great lot of computation is done. In this paper, a new algorithm is presented. It is different from truncating the expansion at the constant level term although we expand (2) in finite terms too. We do not need seek the recursion relations but find all resonance points. As for the compatibility of resonance equations, we use the Wu–Ritt elimination method [16–18] to deal with it and no skills can be found in the process. In the next section, an algorithm is developed by using of symbolic computation and Wu–Ritt elimination theory. In Section 3, we apply this new algorithm to Burgers equation, the mKdV equation and the Caudey–Dodd–Gibbon equation. The last section is devoted to some conclusions.

2. A new algorithm

Substitution (2) into the given PDE and by using of lead-term analysis or homogeneous balance method [19] determine the possible values of p. We know that there exists recursion relation as follows although it is difficult to be found.

$$(n+1)(n-\alpha_1)(n-\alpha_2)\cdots(n-\alpha_k)u_n - F(u_0, u_1, \ldots, u_{n-1}, \phi, z) = 0,$$

where $-1, \alpha_1, \alpha_2, \ldots, \alpha_k$ are called "resonance point". $j = -1$ corresponds to the arbitrary singularity manifold $(\phi = 0)$. On the other hand, the resonance point α_j $(j = 1, 2, \ldots, k)$ introduces an arbitrary function u_j and the compatibility conditions on the functions $(\phi, u_0, u_1, \ldots, u_{j-1}, z)$ are required. If all of the compatibility conditions are satisfied, then we say that the PDE possesses Painleve property.

For convenience, we set $u = u(x,t)$, $u_j = u_j(x,t)$, $\phi = \phi(x,t)$, and expand finite terms in (2),

$$u = \phi^{-p} \sum_{j=0}^{n} u_j \phi^j, \tag{3}$$

where n is an arbitrary integer.

Substituting (3) into PDE and multiply a suitable power of ϕ such that the lowest power of ϕ is equal to zero in the expansion, we get

$$P_0 + P_1\phi + P_2\phi^2 + \cdots + P_n\phi^n + \cdots = 0, \tag{4}$$

where P_i $(i = 0, 1, 2, \ldots, n)$ is the polynomial function of $(u_0, u_1, \ldots, \phi_x, \phi_t, \ldots)$, and the biggest value of subscript u_i is not exceeded j $(j = 0, 1, 2, \ldots, n)$ in P_j, n is an integer. The terms $P_{n+1}, \ldots$ are omitted because the biggest value of subscript u_j in them is n and it is useless for us to investigate this question.

We can generally solve u_0 in P_0, then substitute it into P_j $(j > 0)$, and divide P_i $(i = 1, \ldots, n)$ into two parts, and set

$$C_1 = \{P_j \mid u_j \text{ is effectively occurred in } P_j\},$$
$$C_2 = \{P_j \mid u_j \text{ is not occurred in } P_j\},$$

where the set C_2 is the set of compatibility conditions.

Let the set R be the set constituted by j which is the subscript of P_j in the set C_2, i.e. $R = \{j \mid P_j \in C_2\}$. Obviously, the set of resonance points R is obtained. Certainly, if n is enough big, then the set R includes all the positive resonance points, but the work of computation is heavy. The question is whether the set R is covered all the resonance points for a fixed n (for example, $n = 10$). The answer is yes.

All the integer coefficients of u_j in P_j $(j \neq 0)$ appeared in C_1 with it subscript j compose the set A.

Set

$$A = \{(\beta_{j_1}, j_1), (\beta_{j_2}, j_2), \ldots, (\beta_{j_s}, j_s)\}, \qquad R = \{\alpha_1, \alpha_2, \ldots, \alpha_k\},$$

where k, s are two integers.

At first, we determine whether -1 can be put into the set R or not.

Set

$$r'_{j_i} = \frac{\beta_{j_i}}{j_i + 1} \quad (i = 1, 2, \ldots, s). \tag{5}$$

If all values of (5) are integers, we unite $\{-1\}$ with R, and then get a new set, still denoted it by R. Otherwise, we say that the PDE don't possess Painleve property.

We now assume that -1 is in the set R, and let

$$r_{j_i} = \frac{\beta_{j_i}}{b(j_i + 1)(j_i - \alpha_1) \cdots (j_i - \alpha_k)} \quad (i = 1, 2, \ldots, s), \tag{6}$$

where b is the coefficient of the highest order derivation in PDE.

Case 1. If r_{j_i} $(i = 1, 2, \ldots, s)$ is equal to 1, then we say R covers all resonance points. Since $\beta_{j_i} = b(i + 1)(i - \alpha_1) \cdots (i - \alpha_k)$, although we don't get the recursion relations indeed existed in fact. It is well known that the PDE possesses Painleve property if the set C_2 can be reduced to zero with respect to the set C_1, that is to say the system $\{P_1, \ldots, P_n\}$ is compatible. Wu–Ritt differential elimination method is just the powerful tool dealing with this problem. Here, we omit the definitions and formulas related to reduction in differential algebra (detail for [16–18,20,21]). So the question whether the PDE possesses Painleve property is now transformed the one whether the set C_2 is reduced to zero with respect to C_1. The function order is $u_n \succ u_{n-1} \succ \cdots \succ u_1 \succ \phi$. The independent variable order is $x \succ t$. If the set C_2 is reduced to zero with respect to C_1, it is shown that the system is compatible and then we say PDE possesses Painleve property. Otherwise, the PDE do not pass the Painleve test.

Case 2. $r_{j_i} \neq 1$. Let $|R| = \mathrm{card}(R)$ be the cardinality of the set. Here, $|R|$ denotes the number of the elements in R since the elements in R are finite. Ord(PDE) denotes the order of the PDE. If $|R| = \mathrm{Ord(PDE)}$, it is shown that the PDE has no Painleve property. If $|R| < \mathrm{Ord(PDE)}$, this implies that the set R don't includes all resonance point. Obviously, the values of this points are less than -1 or grater than n.

Let $\gamma_1, \ldots, \gamma_q$ be this points. The following equations can be obtained from (6):

$$r_{j_i} = (j_i - \gamma_1)(j_i - \gamma_2) \cdots (j_i - \gamma_q) \quad (i = 1, 2, \ldots, s).$$

Since the j_i, γ_d are integers, we have

$$|\gamma_d| - |j_i| \leqslant |\gamma_d - j_i| \leqslant |r_{j_i}|.$$

That is to say that the absolute value of γ_d is not greater than $|r_{j_i}| + j_i$. So we can find the γ_d by the following algorithm.

Algorithm A

Input: r_{j_i}, $i = 1, 2, \ldots, s$.
Output: the set B of resonance points.
Step 1. Set $B_{j_i} = \emptyset$, $i = 1, 2, \ldots, s$, $B = \emptyset$.
Step 2. Set $r = \min\{r_{j_i} \mid i = 1, 2, \ldots, s\}$, w denotes the subscript of r.
Step 3. for i from 1 to s
    for h from $-r - w$ to $r + w$
      $r_{j_i} \equiv a \bmod (j_i - h)$ $(h \neq j_i,\ h \neq 0)$.
      if $a = 0$ then $B_{j_i} = B_{j_i} \cup \{h\}$,
    next h
  next i
Step 4. $B = \bigcap B_{j_i}$
Step 5. if $B = \emptyset$ then output "The PDE has no Painleve property".
  else output B.

In what follows, the whole algorithm for testing the Painleve property of the PDE is given.

Algorithm B

Input the partial differential algebraic equation PDE.
Output Ture (the PDE has the Painleve property), False (the PDE don't pass the Painleve test).
Step 1. Set $u = \phi^{-p} \sum_{j=0}^{n} u_j \phi^j$.
Step 2. Computer the set C_1, C_2, A, and R.
Step 3. for i from 1 to s
  $k = 0$, $\beta_{j_i} \equiv a \bmod (j_i + 1)$
  $k = k + a$
  next i
Step 4. if $k = 0$ then $R = R \cup \{-1\}$
  else output False, goto Step 10
Step 5. Computer r_{j_i}
Step 6. if all $r_{j_i} = 1$ then goto Step 9
  else execute Algorithm A, $R = R \cup B$.
Step 7. Computer $|R|$ and $\mathrm{Ord(PDE)}$.
Step 8. if $|R| = \mathrm{Ord(PDE)}$ and all $r_{j_i} = 1$ then goto Step 9
  else output False, goto Step 10.
Step 9. Computer the differential remainder of C_2 with respect to C_1.
  if diff-remaind$(C_2, C_1) = 0$ then output True
  else output False
Step 10. the algorithm is over.

Remark. (1) In this algorithm, we generally set $m = \mathrm{Ord(PDE)} + 1$ so that the equations for resonance points are over-determined. Because if the $\mathrm{Ord(PDE)}$ is equal to k and the PDE has Painleve property, there is exactly k resonance points. If there are h resonance points in $[1, k+1]$, $k - h$ resonance points are to be determined. But the number of equations of resonance points is $k - h + 1$. So there are sufficient information to determine the rest.

(2) If the greatest value of the resonance point is greater than $\mathrm{Ord(PDE)} + 1$, we only expand the expression (3) from 0 to the value of the one. The all resonance equations are obviously obtained.

(3) If there are two or more than two branches of u_0 solved from P_0, the algorithm is applied to each of the branches.

3. Examples

1. Burgers equation

$$u_t + uu_x - u_{xx} = 0. \tag{7}$$

We get $p = 1$ by leading-order analysis or homogeneous balance method, and expand (3) as follows

$$u(x,t) = \phi^{-1} \sum_{j=0}^{3} u_j \phi^j. \tag{8}$$

Substituting (8) into Eq. (7) and multiplying ϕ^3, we get the following expression with the aid of Maple 7

$$P_0 + P_1\phi + P_2\phi^2 + P_3\phi^3 + \cdots = 0. \tag{9}$$

Set $u_0 \neq 0$, we can get $u_0 = -2\phi_x$ and substitute it to the other P_i, the following expressions can be obtained

$$\begin{aligned}
&P_1 = 2u_1\phi_x^2 - 2\phi_x\phi_{xx} + 2\phi_x\phi_t,\\
&P_2 = -2u_1\phi_{xx} - 2u_{1x}\phi_x - 2\phi_{xt} + \phi_{xxx},\\
&P_3 = -4u_3\phi_x^2 + u_2\phi_t + u_{1t} - 4u_{2x}\phi_x + u_1u_2\phi_x + u_1u_{1x} - 3u_2\phi_{xx} - u_{1xx}.
\end{aligned}$$

So the sets C_1, C_2, A, R are as follows respectively

$$\begin{aligned}
&C_1 = \{P_1, P_3\}, \qquad && C_2 = \{P_2\},\\
&A = \big\{(2,1), (-4,3)\big\}, \qquad && R = \{2\}.
\end{aligned}$$

It is easily shown that -1 can be put into the set R. So the new set $R = \{-1, 2\}$ is obtained. In what follows, we would like to know whether the all results in (6) are equal to 1 or not (where $b = -1$). Computed r_{j_i} in (6), we know that r_1 and r_2 are all equal to 1.

It is well known that Eq. (7) possesses Painleve property if the set C_2 can be reduced to zero with respect to C_1. We set the function order is $u_3 \succ u_2 \succ u_1$ and the independent variable order is $x \succ t$. Because P_2 is reduced with respect to P_3, the only work is test whether P_2 is reduced to zero with respect to P_1 or not. With the soft-shell Diffalg in Maple 7, we get diff-remainder$(P_2, P_1) = 0$.

So the Burgers equation possesses Painleve property.

2. mKdV equation

$$2u_t - 3u^2u_x + 2u_{xxx} = 0. \tag{10}$$

For Eq. (10), it is found that

$$u(x,t) = \phi^{-1} \sum_{j=0}^{4} u_j \phi^j, \tag{11}$$

where $u_0 = \pm 2\phi_x$.

The P_i $(i = 1,2,3,4)$ are as follows

$$\begin{aligned}
&P_1 = 24\phi^3 u_1 - 24\phi_x^2\phi_{xx},\\
&P_2 = 12\phi_x^3 u_2 - 24\phi_x u_1\phi_{xx} + 16\phi_x\phi_{xxx} + 4\phi_t\phi_x - 12\phi_x^2 u_{1x} - 6u_1^2\phi_x^2 + 12\phi_{xx}^2,\\
&P_3 = 6u_1^2\phi_{xx} - 4\phi_{xxxx} - 12\phi_x^2 u_{2x} + 12\phi_x u_1 u_{1x} - 24\phi_x u_2\phi_{xx} - 4\phi_{tx},\\
&P_4 = 6\phi_{xx}u_{2x} - 12\phi_x\phi_{xx}u_3 + 6\phi_x u_{2xx} - 3u_1^2 u_{1x} + 2u_{1xxx} + 12\phi_x u_2 u_{1x}\\
&\qquad + 12\phi_x u_1 u_{2x} - 3u_1^2\phi_x u_2 + 2u_{1t} + 2\phi_{xxx}u_2 + 6\phi_x^2 u_2^2 + 2\phi_t u_2 + 12u_1u_3\phi_x^2 + 12u_1u_2\phi_{xx}.
\end{aligned}$$

The sets C_1, C_2, A, R are respectively

$$C_1 = \{P_1, P_2\}, \qquad C_2 = \{P_3, P_4\},$$
$$A = \{(24,1), (12,2)\}, \qquad R = \{3,4\}.$$

The new set $R = \{-1, 3, 4\}$ is easily got. We know that R covers all resonance points by computing r_{j_i} in (6). Now, we test the compatibility of the system $\{P_1, P_2, P_3, P_4\}$.

$$\begin{aligned} r_{32} &= \text{diff-remainder}(P_3, P_2) \\ &= \phi_x^2\phi_{xxxx} + 2\phi_x\phi_{xx}\phi_{xxx} - 3\phi_x^2 u_{1x}\phi_{xx} - 2\phi_x^2 u_1\phi_{xxx} - \phi_x^3 u_{1xx}\phi_{xx}^3, \end{aligned}$$

$$r_{321} = \text{diff-remainder}(r_{32}, P_1) = 0,$$

$$\begin{aligned} r_{42} &= \text{diff-remainder}(P_4, P_2) \\ &= 3\phi_x^3 u_{1xx}\phi_{xx} - 3\phi_x^3 u_{1x}\phi_t - 3\phi_x\phi_t\phi_{xx}^2 + 24u_1\phi_x\phi_{xx}^3 \\ &\quad + 3\phi_x^3 u_1^3\phi_{xx} + 2\phi_x^3 u_1\phi_{xt} + 33u_1\phi_x^3 u_{1x}\phi_{xx} - 6\phi_x^5\phi_{xx}u_3 + 9u_1^2\phi_x^3\phi_{xxx} \\ &\quad + 9u_1\phi_x^4 u_{1xx} - 6\phi_x^3 u_{1x}\phi_{xxx} + 6\phi_x^6 u_1 u - 3\phi_x^2\phi_t\phi_{xxx} + 3\phi_x^2\phi_{xt}\phi_{xx} \\ &\quad + 6\phi_x^2\phi_{xx}\phi_{xxxx} + 30\phi_x\phi_{xx}^2\phi_{xxx} - 2\phi_x^3 u_1\phi_{xxxx} + 9\phi_x^4 u_1^2 u_{1x} \\ &\quad - 48\phi_x^2\phi_{xx}u_1\phi_{xxx} - 24\phi_{xx}^4 + 12u_{1x}^2\phi_x^4 + \phi_x^4 u_{1t} - 27\phi_x^2\phi_{xx}^2 u_{1x} \\ &\quad + 4\phi_x^4 u_{1xxx} + 6\phi_x^2\phi_{xxx}^2 - 4\phi_x^3\phi_{xxxxx} - \phi_{xxt}\phi_x^3, \end{aligned}$$

$$r_{421} = \text{diff-remainder}(r_{42}, P_1) = 0.$$

It is shown that the system $\{P_4, P_3, P_2, P_1\}$ is compatible. The mKdV equation has the Painleve property.

3. Caudrey–Dodd–Gibbon equation

$$u_t + \frac{\partial}{\partial x}(u_{xxxx} + 30uu_x + 60u^3) = 0. \tag{12}$$

For Eq. (12), we have

$$u(x,t) = \phi^{-2}\sum_{j=0}^{6} u_j\phi^j,$$

and $P_0 = -1080u_0^2\phi_x^3 - 720u_0\phi_x^5 - 360u_0^3\phi_x$.

So $u_0 = -2\phi_x^2$ and $u_0 = -\phi_x^2$ can be solved.
For $u_0 = -2\phi_x^2$, P_i $(i = 1, \ldots, 6)$ are obtained

$$\begin{aligned} P_1 &= -1320u_1\phi_x^5 + 2640\phi_x^5\phi_{xx}, \\ P_2 &= -1440u_2\phi_x^5 + 240\phi_x^4\phi_{xxx} + 1200u_1^2\phi_x^3 + \cdots - 4\phi_x\phi_{xxx}, \\ P_3 &= -1080u_3\phi_x^5 + 1980u_2u_1\phi_x^3 + 120u_1\phi_x^2\phi_{xxx} + \cdots - 360\phi_x\phi_{xx}^2, \\ P_4 &= -480u_4\phi_x^5 + 1320u_3u_1\phi_x^3 + 720u_2\phi_x\phi_{xx}^2 + \cdots + 720u_2^2\phi_x^3, \\ P_5 &= 480u_4\phi_x^3\phi_{xx} + 720u_3u_2\phi_x^3 - 60u_{2x}u_{1x}\phi_x + \cdots + 180u_1^2u_{2x}, \\ P_6 &= -240u_{4x}\phi_x^2\phi_{xx} - 144u_3u_2\phi_x\phi_{xx} + \cdots + 180u_1^2u_{3x}, \end{aligned}$$

here, some terms are too long to omit.

The sets C_1, C_2, A, R are respectively

$$C_1 = \{P_1, P_2, P_3, P_4\}, \qquad C_2 = \{P_5, P_6\},$$
$$A = \big\{(-1320, 1), (-1440, 2), (1080, 3), (-480, 4)\big\}, \qquad R = \{5, 6\}.$$

Obviously, -1 cab be enter the set R. Computing r_{j_i} in (7), we get

$$r_1 = -33, \quad r_2 = -40, \quad r_3 = -45, \quad r_4 = -48,$$

the B_i $(i = 1, 2, 3, 4)$ is obtained by using of Algorithm A

$$B_1 = \{-32, -10, -2, 2, 4, 12, 34\},$$
$$B_2 = \{-18, -8, -6, -3, -2, 1, 3, 4, 6, 7, 10, 12, 22\},$$
$$B_3 = \{-12, -6, -2, 2, 4, 6, 8, 12, 18\},$$
$$B_4 = \{-2, -12, -8, -4, -2, 1, 2, 3, 5, 6, 7, 8, 10, 12, 16, 20, 28\},$$

so, $B = B_1 \cap B_2 \cap B_3 \cap B_4 = \{-2, 12\}$. The new resonance points set R is gained, i.e. $R = \{-2, -1, 5, 6, 12\}$. We know that R covers all resonance points by computing r_{j_i} in (7).

Repeating the above procedure, the other branch $u_0 = -\phi_x^2$ has the set of the resonance points $R = \{-1, 2, 3, 6, 10\}$.

Both of them can pass Painleve test since the set C_2 can be reduced to zero with respect to C_1.

4. Conclusion

It's well known whether the given PDE possesses Painleve property can be determinate by analyzing resonance by using of WTC-method. But it is generally difficult to obtain the recursion relations for the sake of expansive computation and fine skills. Based on symbolic computation and differential algebra, we give a new truncation method to decide whether the given equation possesses Painleve property under no the recursion relation. The procedure is the one of the pure computation and no skill is appeared.

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