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2010 Commun. Theor. Phys. 53 1027

(http://iopscience.iop.org/0253-6102/53/6/06)

Commun. Theor. Phys. (Beijing, China) **53** (2010) pp. 1027–1034

Vol. 53, No. 6, June 15, 2010

Analytic Approximations for Soliton Solutions of Short-Wave Models for Camassa–Holm and Degasperis–Procesi Equations*

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(Received May 15, 2009; revised manuscript received October 30, 2009)

Abstract *In this paper, the short-wave model equations are investigated, which are associated with the Camassa–Holm (CH) and Degasperis–Procesi (DP) shallow-water wave equations. Firstly, by means of the transformation of the independent variables and the travelling wave transformation, the partial differential equation is reduced to an ordinary differential equation. Secondly, the equation is solved by homotopy analysis method. Lastly, by the transformations back to the original independent variables, the solution of the original partial differential equation is obtained. The two types of solutions of the short-wave models are obtained in parametric form, one is one-cusp soliton for the CH equation while the other one is one-loop soliton for the DP equation. The approximate analytic solutions expressed by a series of exponential functions agree well with the exact solutions. It demonstrates the validity and great potential of homotopy analysis method for complicated nonlinear solitary wave problems.*

PACS numbers: 03.40.Kf, 03.65.Ge, 05.45.Yv
Key words: Homotopy analysis method, Camassa–Holm equation, Degasperis–Procesi equation, soliton

1 Introduction

Nonlinear phenomena play an important role in various fields of science and engineering. However, it is usually difficult to solve them either theoretically or numerically. Up to now, many powerful methods for nonlinear differential equations have been developed. For instance, Bäcklund transformation method,[1] Inverse scattering method,[2] Darboux transformation method,[3–4] tanh-function method,[5–6] homogeneous balance method,[7–8] similarity reduction method,[9–10] Adomian decomposition method,[11–12] homotopy analysis method (HAM).[13–19]

HAM is one of the most effective methods to construct approximate analytic solutions of nonlinear differential equations. Different from the traditional analytical approximation methods, HAM is independent of small or large physical parameters and allows us to adjust the region and rate of convergence of series solution via control of an initial approximation, an auxiliary linear operator, an auxiliary function and an auxiliary parameter.

Recently, Camassa–Holm (CH) and Degasperis–Procesi (DP) equations have attracted a lot of attention. Zhang and Qiao explored all possible single peakon solutions for the DP equation[20] and all possible explicit single soliton solutions for the CH equation.[21] Qiao obtained a class of a new algebro-geometric solution of the CH equation by solving the parametric representation of the solution on the symplectic submanifold.[22] Qiao derived the parametric solution of the DP equation which is the first member in the new negative order hierarchy.[23] Beals obtained explicit formulas for the peakon-antipeakon solutions of the CH equation by using classical results of stieltjes.[24] Lundmark computed multi-peakon solutions of the DP equation by an inverse scattering approach.[25] Fan constructed algebro-geometric solutions of the CH-γ equation by using standard Jacobi inversion technique.[26] Zhang applied homotopy perturbation method to slove the modified CH and DP equations with solitary wave solutions.[27] Wazwaz obtained peakons, kinks, compactons, and solitary pattern solutions for a family of CH equations by using hyperbolic schemes.[28] Yang constructed the one-cusp soliton solution of the CH equation and one-loop soliton solution of the DP equation by a simple approach.[29] By means of HAM, Wu derived approximate analytic solutions of the CH equation for solitary waves with and without continuity at crest[30] and Abbasbandy obtained solitary wave solutions of the CH and modified CH equations.[31–32]

We consider the CH short-wave model equation

$$u_{txx} + 2\kappa^2 u_x + 2u_x u_{xx} + u u_{xxx} = 0\,, \tag{1}$$

and the DP short-wave model equation

$$u_{txx} + 3\kappa^3 u_x + 3u_x u_{xx} + u u_{xxx} = 0\,, \tag{2}$$

where $\kappa > 0$. Both of them stem from the short-wave limit of the PDE[33]

$$u_t + \alpha u_x - u_{txx} + (\beta + 1) u u_x = \beta u_x u_{xx} + u u_{xxx}\,. \tag{3}$$

As is known, when $\beta = 2$, it becomes the CH equation,[34–36] and when $\beta = 3$, it reduces to the DP

*Supported by the National Natural Science Foundation of China under Grant Nos. 10771072, 10735030, and 90718041 and Shanghai Leading Academic Discipline Project under Grant No. B412

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equations.[37−38] The recent study reveals that the CH and DP equations can be derived as appropriate asymptotic model equations describing the unidirectional propagation of nonlinear shallow-water waves.[39−41] Hence, Eqs. (1) and (2) may be used to describe the long-term dynamics of short surface waves. Matsuno constructed multi-soliton solutions of the CH and DP short-wave model equations, i.e. cusp and loop solitons.[33] He emphasized that the application of the obtained results to the short-wave dynamics in real fluid systems would be an interesting issue. In this paper, the two short-wave model equations are investigated. Firstly, by means of the transformation of the independent variables and the travelling wave transformation, the partial differential equation (PDE) is reduced to an ordinary differential equation. Secondly, the equation is solved by HAM. Lastly, by the transformations back to the original independent variables, the solution of the original partial differential equation is obtained. As a result, the two types of solutions of the short-wave models are obtained in parametric form, one is one-cusp soliton for the CH equation while the other one is one-loop soliton for the DP equation. The approximate analytic solutions expressed by a series of exponential functions agree well with the exact solutions. It demonstrates the validity and great potential of HAM for complicated nonlinear solitary wave problems.

The paper is organized as follows. In the next section, the CH short-wave model equation is studied. In Sec. 3, the DP short-wave model equation is studied. Finally, some important conclusions are presented in Sec. 4.

2 Application of HAM to CH Short-Wave Model Equation

To obtain the soliton solution of the CH equation, integrating Eq. (1) with respect to x once and setting the integral constant to be zero yield

$$u_{tx}+2\kappa^2u+\frac{u_x^2}{2}+uu_{xx}=0\,. \tag{4}$$

Following Matsuno,[33] we introduce the new independent variables X and T, defined by

$$x=\frac{X}{\kappa}+W(X,T)+x_0,\quad t=T\,, \tag{5}$$

where $u(x,t)=\tilde{U}(X,T)$, and x_0 is a constant. Writing

$$W(X,T)=\int_{-\infty}^{T}\tilde{U}(X,Z)dZ\,, \tag{6}$$

we have

$$u(x,t)=\tilde{U}(X,T)=W_T(X,T)\,. \tag{7}$$

From (5), it follows that

$$\frac{\partial}{\partial X}=\gamma\frac{\partial}{\partial x},\quad \frac{\partial}{\partial T}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\,, \tag{8}$$

where

$$\gamma=\frac{1}{\kappa}+W_X(X,T)\,. \tag{9}$$

We have readily

$$u_x^2=\gamma^{-2}\tilde{U}_X^2\,. \tag{10}$$

So we find

$$\tilde{U}_{TX}+2\kappa^2\gamma\tilde{U}-\frac{1}{2}\gamma^{-1}\tilde{U}_X^2=0\,. \tag{11}$$

Substituting (7) into (11) yields

$$\begin{aligned}&2W_{TTX}+2\kappa W_XW_{TTX}+4\kappa W_T+4\kappa^3W_X^2W_T\\&+8\kappa^2W_XW_T-\kappa W_{TX}^2=0\,.\end{aligned} \tag{12}$$

We introduce the travelling wave transformation

$$W(X,T)=W(\xi),\quad \xi=X-cT\,, \tag{13}$$

where c is a constant and Eq. (12) becomes

$$\begin{aligned}&2cW'''(\xi)+2\kappa cW'(\xi)W'''(\xi)-4\kappa W'(\xi)-4\kappa^3[W'(\xi)]^3\\&-8\kappa^2[W'(\xi)]^2-\kappa c[W''(\xi)]^2=0\,.\end{aligned} \tag{14}$$

Write

$$W(\xi)\approx B\exp(-a\xi)\quad\text{as}\quad \xi\to+\infty\,, \tag{15}$$

where $a>0$ and B are constants. Substituting (15) into (14) and balancing the main term yield

$$a=\sqrt{\frac{2\kappa}{c}}\,, \tag{16}$$

where $c>0$. Under the transformation

$$\theta=-a\xi=-\sqrt{\frac{2\kappa}{c}}\xi\,, \tag{17}$$

and setting $\sqrt{\kappa/c}=b$, Eq. (14) becomes

$$\begin{aligned}&4\sqrt{2}W'''(\theta)-8\kappa bW'(\theta)W'''(\theta)-4\sqrt{2}W'(\theta)\\&-8\sqrt{2}\kappa^2b^2[W'(\theta)]^3+16\kappa b[W'(\theta)]^2\\&+4\kappa b[W''(\theta)]^2=0\,.\end{aligned} \tag{18}$$

From (6) and (17), we have

$$W(-\infty)=0\,. \tag{19}$$

Considering the symmetry of $\tilde{U}(X,T)$ in X-T space and the continuation of its 1st-order derivative, from (7) we get

$$W'(\theta)=W'(-\theta)\,, \tag{20}$$

$$W''(0)=0\,. \tag{21}$$

Integrating (20) gives

$$W(\theta)+W(-\theta)=A\,, \tag{22}$$

where A is a constant. Thus from Eqs. (19)–(22), we get the boundary conditions on Eq. (18)

$$W(0)=\frac{1}{2}A,\quad W''(0)=0\,,\quad W(+\infty)=A\,. \tag{23}$$

We apply HAM to obtain $W(\theta)$ on $\theta\geq 0$ because $W(\theta)$ on $\theta<0$ can be obtained from (22) by the symmetry.

Under the transformation

$$W(\theta)=A+\frac{A}{2}f(\theta)\,, \tag{24}$$

Eq. (18) becomes

$$\begin{aligned}&2\sqrt{2}f'''(\theta)-2\kappa bAf'(\theta)f'''(\theta)-2\sqrt{2}f'(\theta)\\&-\sqrt{2}\kappa^2b^2A^2[f'(\theta)]^3+4\kappa bA[f'(\theta)]^2\\&+\kappa bA[f''(\theta)]^2=0\,.\end{aligned} \tag{25}$$

Setting $bA = \lambda$, we have

$$\begin{aligned} &2\sqrt{2}f'''(\theta) - 2\kappa\lambda f'(\theta)f'''(\theta) - 2\sqrt{2}f'(\theta) \\ &- \sqrt{2}\kappa^2\lambda^2[f'(\theta)]^3 + 4\kappa\lambda[f'(\theta)]^2 \\ &+ \kappa\lambda[f''(\theta)]^2 = 0\,, \end{aligned} \tag{26}$$

subject to the boundary conditions

$$f(0) = -1,\quad f''(0) = 0,\quad f(+\infty) = 0\,. \tag{27}$$

According to Eq. (26) and the boundary condition (27), the solution $f(\theta)$ can be expressed by

$$f(\theta) = \sum_{m=1}^{+\infty} \alpha_m \exp(-m\theta)\,, \tag{28}$$

where α_m is a coefficient. This provides us with the so-called rule of solution expression, as mentioned by Liao.[13] Thereafter, we choose

$$f_0(\theta) = -\frac{4}{3}\exp(-\theta) + \frac{1}{3}\exp(-2\theta)\,, \tag{29}$$

as the initial guess of $f(\theta)$. The auxiliary linear operator is

$$L[G(\theta;q)] = \frac{\partial^3 G(\theta;q)}{\partial\theta^3} - \frac{\partial G(\theta;q)}{\partial\theta}\,, \tag{30}$$

which has the property

$$L[C_1\exp(-\theta) + C_2\exp(\theta) + C_3] = 0\,. \tag{31}$$

Then we construct the so-called zero-order deformation equation

$$(1-q)L[G(\theta;q) - f_0(\theta)] = hqN[G(\theta;q),\Gamma(q)]\,, \tag{32}$$

subject to the boundary conditions

$$G(0;q) = -1,\quad G''(0;q) = 0,\quad G(+\infty;q) = 0\,, \tag{33}$$

where q is an embedding parameter, $\hbar$ is a non-zero auxiliary parameter and

$$\begin{aligned} &N[G(\theta;q),\Gamma(q)] \\ &= 2\sqrt{2}G'''(\theta;q), -2\kappa\Gamma(q)G'(\theta;q)G'''(\theta;q) \\ &\quad - 2\sqrt{2}G'(\theta;q) - \sqrt{2}\kappa^2[\Gamma(q)]^2[G'(\theta;q)]^3 \\ &\quad + 4\kappa\Gamma(q)[G'(\theta;q)]^2 + \kappa\Gamma(q)[G''(\theta;q)]^2\,, \end{aligned} \tag{34}$$

with prime denoting derivatives with respect to θ.

When $q = 0$, the solution of Eqs. (32) and (33) is

$$G(\theta;0) = f_0(\theta)\,. \tag{35}$$

When $q = 1$, Eqs. (32) and (33) are equivalent to Eqs. (26) and (27) provided

$$G(\theta;1) = f(\theta),\quad \Gamma(1) = \lambda\,. \tag{36}$$

As q increases from 0 to 1 continually, $G(\theta;q)$ varies from the initial approximation $f_0(\theta)$ to the exact solution $f(\theta)$ of Eqs. (26) and (27), so does $\Gamma(q)$ from its initial approximation λ_0 to the exact value λ.

If h is properly chosen, the zero-deformation Eqs. (32) and (33) have solutions for all $q \in [0,1]$ and the terms

$$f_m(\theta) = \frac{1}{m!}\frac{\partial^m G(\theta;q)}{\partial q^m}\Big|_{q=0}\,,\quad \lambda_m = \frac{1}{m!}\frac{\partial^m \Gamma(q)}{\partial q^m}\Big|_{q=0}\,, \tag{37}$$

exist for $m \geq 1$. Using (29), we can expand $G(\theta,q)$ and $\Gamma(q)$ in power series with respect to q as follows

$$G(\theta,q) = f_0(\theta) + \sum_{m=1}^{+\infty} f_m(\theta)q^m\,, \tag{38}$$

$$\Gamma(q) = \lambda_0 + \sum_{m=1}^{+\infty} \lambda_m q^m\,. \tag{39}$$

Furthermore, if h is properly chosen so that the power series (38) and (39) are convergent at $q = 1$, from (36) we get the solution series

$$f(\theta) = f_0(\theta) + \sum_{m=1}^{+\infty} f_m(\theta)\,, \tag{40}$$

$$\lambda = \lambda_0 + \sum_{m=1}^{+\infty} \lambda_m\,. \tag{41}$$

For simplicity, define the vectors

$$\begin{aligned} &\vec{f}_m = \{f_0(\theta), f_1(\theta), \ldots, f_m(\theta)\}\,, \\ &\vec{\lambda}_m = \{\lambda_0, \lambda_1, \ldots, \lambda_m\}\,. \end{aligned} \tag{42}$$

Differentiating the zero-order deformation equation (32) and (33) m times with respect to q and then dividing them by $m!$ and finally setting $q = 0$, we have the high-order deformation equation

$$L[f_m(\theta) - \chi_m f_{m-1}(\theta)] = hR_m(\vec{f}_{m-1}, \vec{\lambda}_{m-1})\,,\quad m \geq 1\,, \tag{43}$$

subject to the boundary conditions

$$f_m(0) = f''_m(0) = f(+\infty) = 0\,, \tag{44}$$

where

$$\chi_m = \begin{cases} 0, & m = 1\,, \\ 1, & m > 1 \end{cases} \tag{45}$$

$$\begin{aligned} R_m(\vec{f}_{m-1}, \vec{\lambda}_{m-1}) &= \frac{1}{(m-1)!}\frac{\partial^{m-1} N[G(\theta;q),\Gamma(q)]}{\partial q^{m-1}}\Big|_{q=0} \\ &= 2\sqrt{2}f'''_{m-1}(\theta) - 2\kappa\sum_{i=0}^{m-1}\lambda_{m-1-i}\sum_{j=0}^{i} f'_j(\theta)f'''_{i-j}(\theta) - 2\sqrt{2}f'_{m-1}(\theta) \\ &\quad - \sqrt{2}\kappa^2\sum_{i=0}^{m-1}\sum_{j=0}^{m-1-i}\lambda_j\lambda_{m-1-i-j}\sum_{k=0}^{i} f'_{i-k}\sum_{t=0}^{k} f'_t f'_{k-t} \\ &\quad + 4\kappa\sum_{i=0}^{m-1}\lambda_{m-1-i}\sum_{j=0}^{i} f'_j(\theta)f'_{i-j}(\theta) + \kappa\sum_{i=0}^{m-1}\lambda_{m-1-i}\sum_{j=0}^{i} f''_j(\theta)f''_{i-j}(\theta)\,. \end{aligned} \tag{46}$$

The solution of Eq. (43) can be expressed by

$$f_m(\theta) = f_m^*(\theta) + C_1\exp(-\theta) + C_2\exp(\theta) + C_3\,, \qquad (47)$$

where C_1, C_2, and C_3 are the integral constants, $f_m^*(\theta)$ is a special solution of Eq. (43) and contains the unknown λ_{m-1}. According to the boundary conditions (44) at infinity, C_2 and C_3 must be zero. The unknown λ_{m-1} and C_1 are determined by (44) at $\theta = 0$.

With the aid of *Maple*, we solve Eq. (43) one after the other in order $m = 1, 2, 3, \ldots$ and obtain the M-th order of approximate solutions

$$f(\theta) \approx \sum_{m=0}^{M} f_m(\theta)\,, \quad \lambda \approx \sum_{m=0}^{M-1}\lambda_m\,. \qquad (48)$$

From Eqs. (5), (7), and (24), we get

$$u(\theta) = \frac{\sqrt{2}}{2}c\lambda f'(\theta)\,, \qquad (49)$$

$$x(\theta,t) = -\frac{\theta}{a\kappa} + \frac{c}{\kappa}t + A + \frac{A}{2}f(\theta) + x_0\,, \qquad (50)$$

where x_0 is a constant. For the symmetry in x-t space, we have

$$x_0 = -A - \frac{A}{2}f(0) = -\frac{A}{2}\,. \qquad (51)$$

Substituting $\lambda = bA$ into (50) and from Eq. (16), we have

$$x(\theta,t) = -\frac{\theta}{\sqrt{2}\kappa b} + \frac{t}{b^2} + \frac{\lambda}{2b} + \frac{\lambda}{2b}f(\theta)\,. \qquad (52)$$

So when $\theta \geq 0$, the M-th order approximate solutions of the CH short-wave model equation (1) is given by a series as

$$u(X,t) = u(\theta) = \frac{\sqrt{2}}{2}c\sum_{m=0}^{M-1}\lambda_m\Big[\sum_{m=0}^{M}f_m(\theta)\Big]'\,,$$

$$x(X,t) = x(\theta,t) = -\frac{\theta}{\sqrt{2}\kappa b} + \frac{t}{b^2} + \frac{1}{2b}\sum_{m=0}^{M-1}\lambda_m + \frac{1}{2b}\sum_{m=0}^{M-1}\lambda_m\sum_{m=0}^{M}f_m(\theta)\,, \qquad (53)$$

where $b = \sqrt{\kappa/c}$ and $\theta = -\sqrt{2}b(X - ct)$.

Similarly, when $\theta < 0$, we get from (22)

$$u(\theta) = \frac{\sqrt{2}}{2}c\lambda f'(-\theta)\,,$$

$$x(\theta,t) = -\frac{\theta}{\sqrt{2}\kappa b} + \frac{t}{b^2} - \frac{\lambda}{2b} - \frac{\lambda}{2b}f(-\theta)\,, \qquad (54)$$

where $b = \sqrt{\kappa/c}$ and $\theta = -\sqrt{2}b(X - ct)$. The corresponding M-th order approximate solutions of the CH short-wave model equation (1) is given by a series as

$$u(X,t) = u(\theta) = \frac{\sqrt{2}}{2}c\sum_{m=0}^{M-1}\lambda_m\Big[\sum_{m=0}^{M}f_m(-\theta)\Big]'\,,$$

$$x(X,t) = x(\theta,t) = -\frac{\theta}{\sqrt{2}\kappa b} + \frac{t}{b^2} - \frac{1}{2b}\sum_{m=0}^{M-1}\lambda_m - \frac{1}{2b}\sum_{m=0}^{M-1}\lambda_m\sum_{m=0}^{M}f_m(-\theta)\,, \qquad (55)$$

where $b = \sqrt{\kappa/c}$ and $\theta = -\sqrt{2}b(X - ct)$.

From (13), (16), and (17), the exact one-cusp soliton solution of Eq. (1) in Ref. [29] becomes

$$u(X,t) = u(\theta) = \frac{1}{b^2}\mathrm{sech}^2\Big(\frac{\theta}{2}\Big)\,,$$

$$x(X,t) = x(\theta,t) = -\frac{\theta}{\sqrt{2}\kappa b} + \frac{t}{b^2} + \frac{\sqrt{2}}{\kappa b}\tanh\Big(\frac{\theta}{2}\Big)\,, \qquad (56)$$

where $\theta \in (+\infty, -\infty)$, $b = \sqrt{\kappa/c}$, and $\theta = -\sqrt{2}b(X - ct)$.

Note that the two series (40) and (41) contain the auxiliary parameter h, which controls the convergent rate and region of them. To ensure that the two series converge, we focus on choosing a proper value of h. In Ref. [29], we have

$$\lambda = bA = \frac{2\sqrt{2}}{\kappa}\,. \qquad (57)$$

For any fixed value κ ($\kappa > 0$), we can get an exact value λ and thereafter choose a proper value of h to ensure the convergence of the series solution by means of the so-called h-curve (the $\lambda \sim h$ curve). As an example, we consider the approximate solution of Eq. (1) when $\kappa = 1$ and $c = 1$. We plot $\lambda \sim h$ curve at the 7-th order approximations. The valid region of h is about $0.5 < h < 1$ shown in Fig. 1 and we choose $h = 0.6$. Our approximate analytical solutions (53) and (55) converge rapidly to the exact solution (56) shown in Figs. 2 and 3.

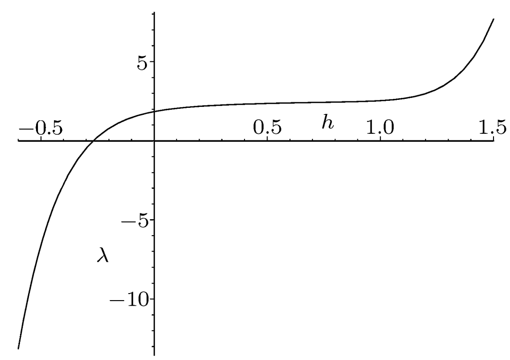

Fig. 1 The $\lambda \sim h$ curve at the 7-th order approximation when $\kappa = 1$ and $c = 1$.

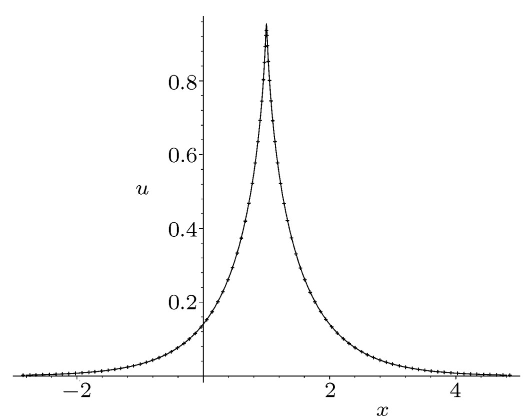

Fig. 2 The approximate analytic solution ($x(X,t)$, $u(X,t)$) of Eq. (1) at $t = 1$ when $\kappa = 1$ and $c = 1$. Solid curve: 8-th order approximation; point curve: 5-th order approximation.

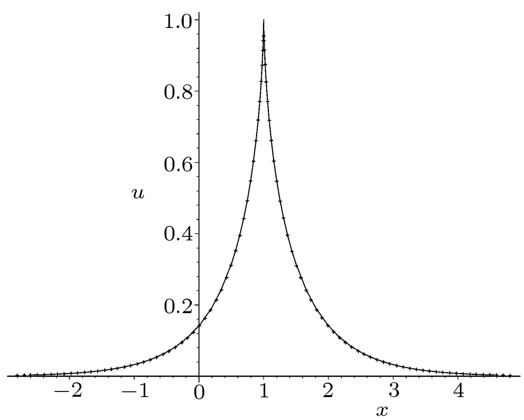

Fig. 3 The approximate analytic solution and exact solution $(x(X,t),u(X,t))$ of Eq. (1) at $t=1$ when $\kappa=1$ and $c=1$. Solid curve: exact solution; point curve: 8-th order approximation.

Figure 4 shows the residual error for different orders of approximation. Figure 5 gives the 8-th order approximate solution of Eq. (1) at different time levels. This clearly indicates that HAM gives the approximate analytic solution with high accuracy and rapid convergence.

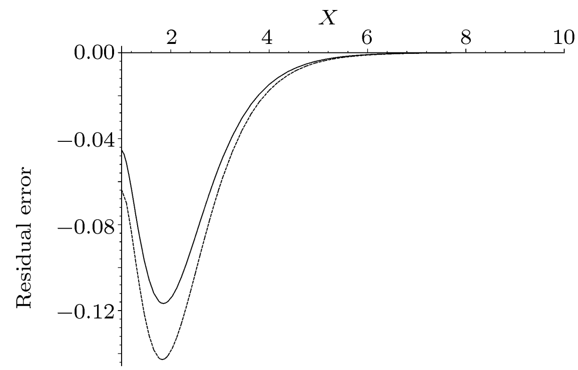

Fig. 4 The residual error for Eq. (1) for $1 \le X \le 10$ at $t=1$ when $\kappa=1$ and $c=1$. Solid curve: 8-th order approximation; dashed curve: 5-th order approximation.

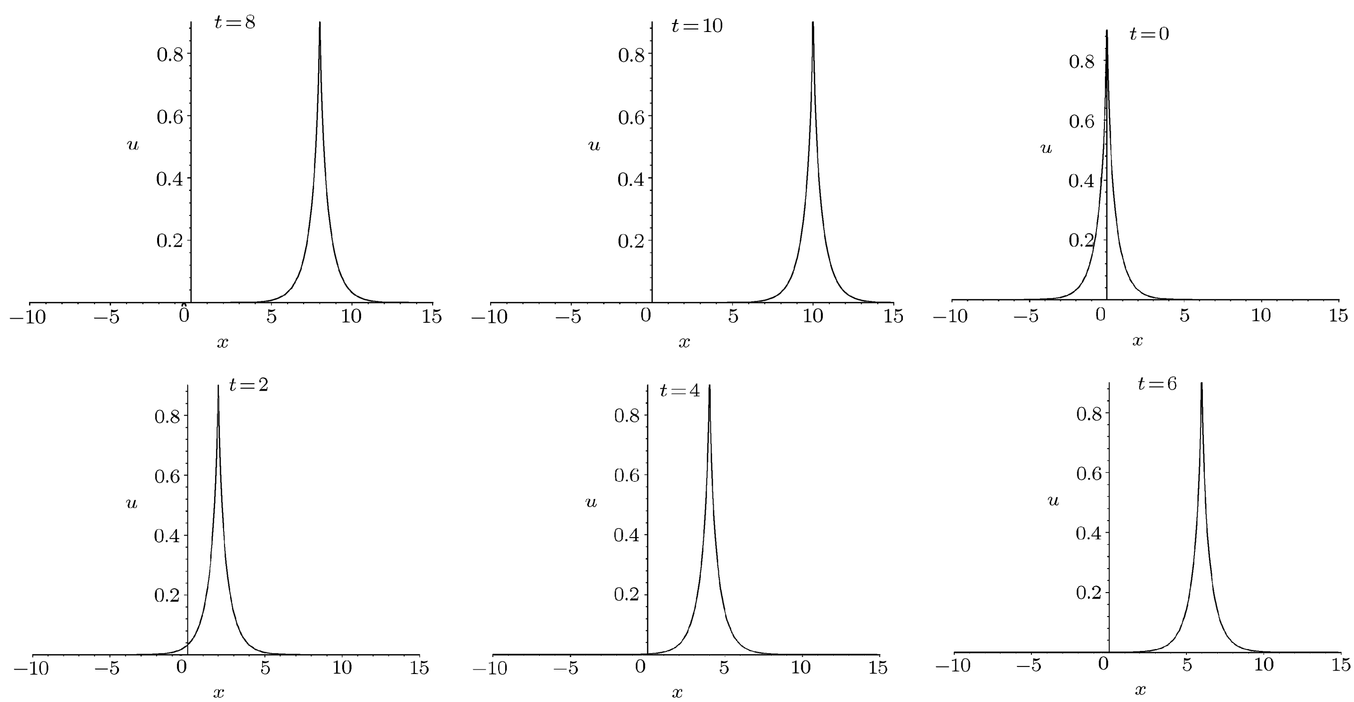

Fig. 5 The HAM 8-th order approximate analytic solution of Eq. (1) for $-10 < X < 15$ at different time levels when $\kappa=1$ and $c=1$.

3 Application of HAM to DP Short-Wave Model Equation

To obtain the soliton solution of the DP equation, integrating Eq. (2) with respect to x once and setting the integral constant to be zero yield

$$u_{tx} + 3\kappa^3 u + u_x^2 + u u_{xx} = 0\,. \tag{58}$$

We introduce the same transformation of the independent variables as (5), we have

$$W_{TTX} + 3\kappa^2 W_T + 3\kappa^3 W_X W_T = 0\,. \tag{59}$$

Under the travelling wave transformation (13), Eq. (59) becomes

$$cW'''(\xi) - 3\kappa^2 W'(\xi) - 3\kappa^3 [W'(\xi)]^2 = 0\,. \tag{60}$$

Write

$$W(\xi) \approx B\exp(-a\xi) \quad \text{as} \quad \xi \to +\infty\,, \tag{61}$$

where $a > 0$ and B are constants. Substituting (61) into (60) and balancing the main term yield

$$a = \kappa\sqrt{\frac{3}{c}}\,, \tag{62}$$

where $c > 0$. Under the transformation

$$\theta = -a\xi = -\kappa\sqrt{\frac{3}{c}}\xi\,, \tag{63}$$

Eq. (60) becomes

$$W'''(\theta) - W'(\theta) + \kappa a[W'(\theta)]^2 = 0\,. \tag{64}$$

The other formulas are the same as those given in Sec. 2. So we get the boundary conditions on Eq. (64)

$$W(0) = \frac{1}{2}A, \quad W''(0) = 0\,, \quad W(+\infty) = A\,. \tag{65}$$

We apply HAM to obtain $W(\theta)$ on $\theta \geq 0$ because $W(\theta)$ on $\theta < 0$ can be obtained from (22) by the symmetry.

Under the transformation (24), Eq. (64) becomes

$$f'''(\theta) - f'(\theta) + \lambda[f'(\theta)]^2 = 0\,, \tag{66}$$

where $\lambda = \kappa a A/2$, subject to the boundary conditions

$$f(0) = -1, \quad f''(0) = 0, \quad f(+\infty) = 0\,. \tag{67}$$

We process the procedure similar to the CH equation in Sec. 2 to get the M-th order approximate solutions

$$f(\theta) \approx \sum_{m=0}^{M} f_m(\theta)\,, \quad \lambda \approx \sum_{m=0}^{M-1} \lambda_m\,. \tag{68}$$

From (5), (7), and (24), we have

$$u(\theta) = \frac{c}{\kappa}\lambda f'(\theta)\,, \tag{69}$$

$$x(\theta, t) = -\frac{\theta}{a\kappa} + \frac{c}{\kappa}t + A + \frac{A}{2}f(\theta) + x_0\,, \tag{70}$$

where x_0 is a constant. For the symmetry in x–t space, we have

$$x_0 = -A - \frac{A}{2}f(0) = -\frac{A}{2}\,. \tag{71}$$

Substituting $\lambda = \kappa a A/2$ into (70), we have

$$x(\theta, t) = -\frac{\theta}{\kappa a} + \frac{c}{\kappa}t + \frac{\lambda}{\kappa a} + \frac{\lambda}{\kappa a}f(\theta)\,, \tag{72}$$

where $a = \kappa\sqrt{3/c}$.

So when $\theta \geq 0$, the M-th order approximate solutions of the DP short-wave model equation (2) is given by a series as

$$u(X,t) = u(\theta) = \frac{c}{\kappa}\sum_{m=0}^{M-1}\lambda_m\Big[\sum_{m=0}^{M} f_m(\theta)\Big]'\,,$$

$$x(X,t) = x(\theta,t) = -\frac{\theta}{\kappa a} + \frac{c}{\kappa}t + \frac{1}{\kappa a}\sum_{m=0}^{M-1}\lambda_m + \frac{1}{\kappa a}\sum_{m=0}^{M-1}\lambda_m\sum_{m=0}^{M} f_m(\theta)\,, \tag{73}$$

where $\theta = -a(X - ct)$.

Similarly, when $\theta < 0$, from (22) we have

$$u(\theta) = \frac{c}{\kappa}\lambda f'(-\theta)\,,$$

$$x(\theta,t) = -\frac{\theta}{\kappa a} + \frac{c}{\kappa}t - \frac{\lambda}{\kappa a} - \frac{\lambda}{\kappa a}f(-\theta)\,, \tag{74}$$

where $a = \kappa\sqrt{3/c}$. The corresponding M-th order approximate solutions of the DP short-wave model equation (2) is given by a series as

$$u(X,t) = u(\theta) = \frac{c}{\kappa}\sum_{m=0}^{M-1}\lambda_m\Big[\sum_{m=0}^{M} f_m(-\theta)\Big]'\,,$$

$$x(X,t) = x(\theta,t) = -\frac{\theta}{\kappa a} + \frac{c}{\kappa}t - \frac{1}{\kappa a}\sum_{m=0}^{M-1}\lambda_m - \frac{1}{\kappa a}\sum_{m=0}^{M-1}\lambda_m\sum_{m=0}^{M} f_m(-\theta)\,, \tag{75}$$

where $\theta = -a(X - ct)$.

From (13), (62), and (63), the exact one-loop soliton solution of Eq. (2) in Ref. [29] becomes

$$u(X,t) = u(\theta) = \frac{3}{2}\frac{c}{\kappa}\mathrm{sech}^2\Big(\frac{\theta}{2}\Big),$$

$$x(X,t) = x(\theta,t) = -\frac{\theta}{\kappa a} + \frac{c}{\kappa}t + \frac{3}{\kappa a}\tanh\Big(\frac{\theta}{2}\Big)\,, \tag{76}$$

where $\theta \in (-\infty, +\infty)$, $a = \kappa\sqrt{3/c}$, and $\theta = -a(X - ct)$.

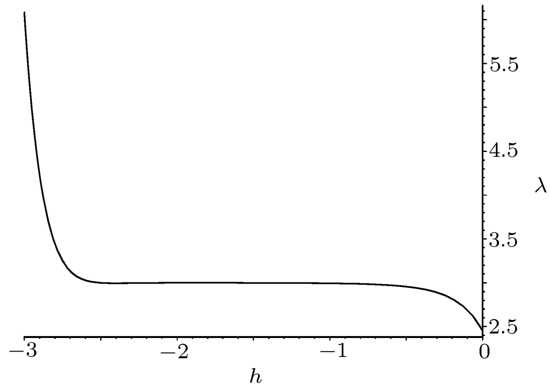

Fig. 6 The $\lambda \sim h$ curve at the 14-th order approximation when $\kappa = 1$ and $c = 1$.

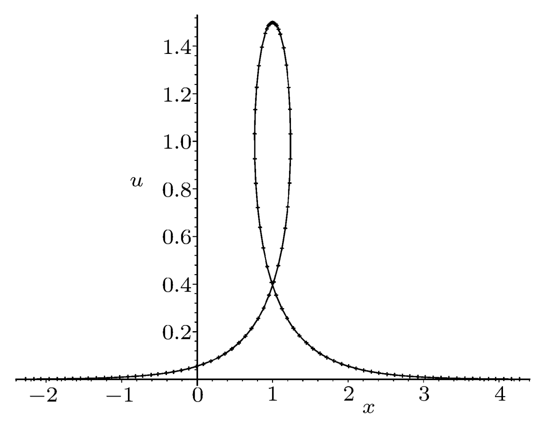

Fig. 7 The approximate analytic solution $(x(X,t), u(X,t))$ of Eq. (2) at $t = 1$ when $\kappa = 1$ and $c = 1$. Solid curve: 15-th order approximation; point curve: 10-th order approximation.

Note that the two series (68) contain the auxiliary parameter h, which controls the convergent rate and region of them. To ensure that the two series converge, we focus on choosing a proper value of h. In Ref. [29], we have

$$\lambda = \kappa a \frac{A}{2} = 3\,. \tag{77}$$

For any fixed value κ ($\kappa > 0$), we can choose a proper value of h to ensure the convergence of the series solution by means of the so-called h-curve (the $\lambda \sim h$ curve).

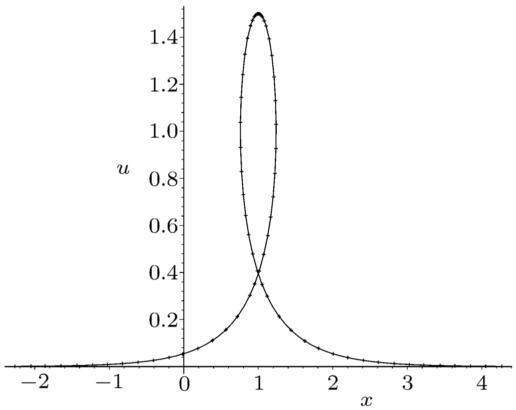

Fig. 8 The approximate analytic solution and exact solution $(x(X,t), u(X,t))$ of Eq. (2) at $t = 1$ when $\kappa = 1$ and $c = 1$. Solid curve: exact solution; point curve: 15-th order approximation.

As an example, we consider the approximate solution of Eq. (2) when $\kappa = 1$ and $c = 1$. We plot $\lambda \sim h$ curve at the 14-th order approximations. The valid region of h is about $-2.5 < h < -0.5$ in Fig. 6 and we choose $h = -1.5$. Our approximate analytic solutions (73) and (75) converge rapidly to the exact solution (76) shown in Figs. 7 and 8. Figure 9 shows the residual error for different orders of approximation. Figure 10 gives the 15-th order approximate solution of Eq. (2) at different time levels. This clearly indicates that HAM gives the approximate analytic solution with high accuracy and rapid convergence.

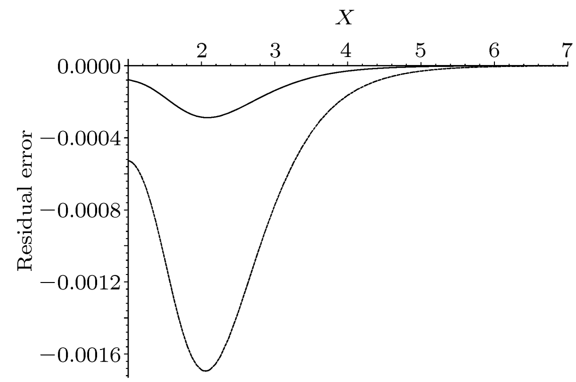

Fig. 9 The residual error for Eq. (2) for $1 \le X \le 7$ at $t = 1$ when $\kappa = 1$ and $c = 1$. Solid curve: 15-th order approximation; dashed curve: 10-th order approximation.

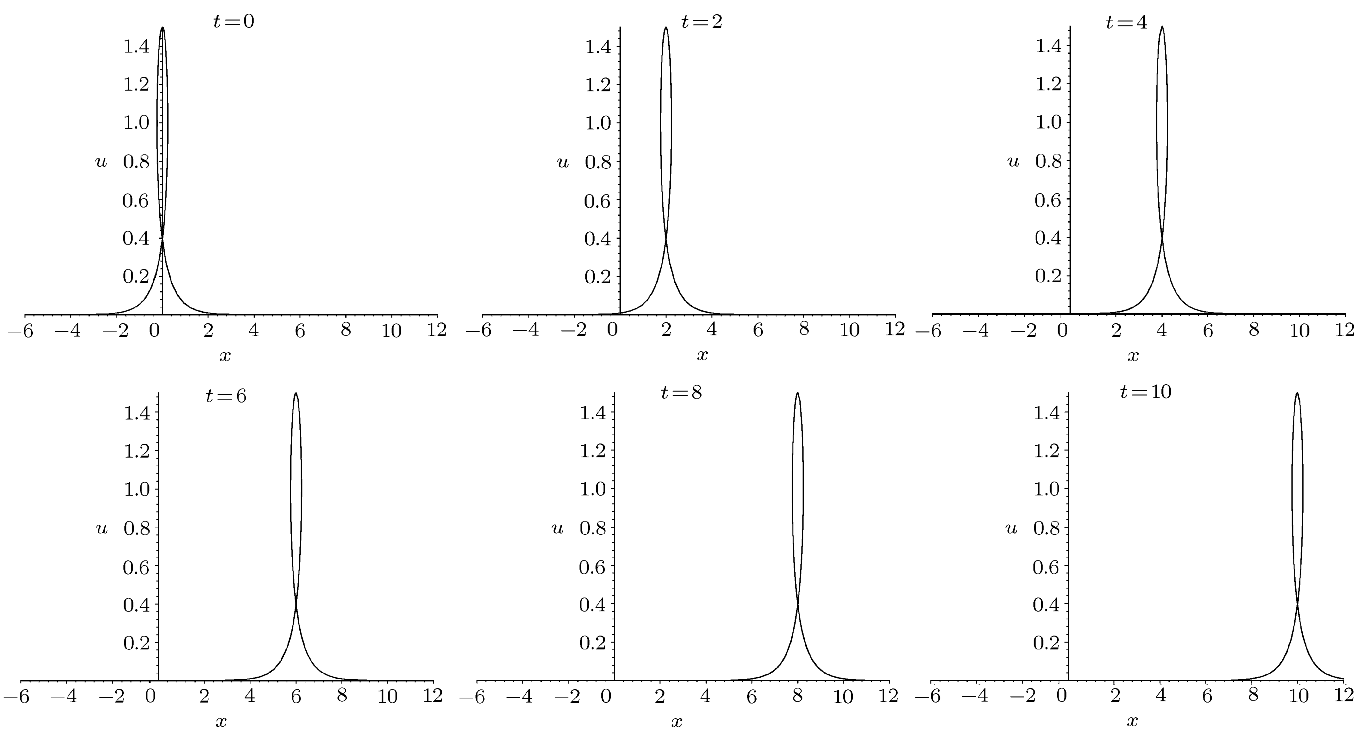

Fig. 10 The HAM 15-th order approximate analytic solution of Eq. (2) for $-6 < X < 12$ at different time levels when $\kappa = 1$ and $c = 1$.

4 Conclusion

In this paper, the short-wave models for CH and DP equations are investigated by means of HAM. The two types of solutions of them are obtained in parametric form, one is the one-cusp soliton for the CH equation while the other one is one-loop soliton for DP equation. The numerical results agree well with the exact solutions. To our knowledge, the numerical results presented here have never been reported. It demonstrates the validity and great potential of homotopy analysis method for complicated nonlinear solitary wave problems.

Acknowledgements

Thanks to the anonymous reviewers for their valuable comments.

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