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Bilinear Bäcklund transformation, Lax pair and multi-soliton solution for a vector Ramani equation

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In this paper, a vector Ramani equation is proposed by using the bilinear approach. With the help of the bilinear exchange formulae, bilinear Bäcklund transformation and the corresponding Lax pair for the vector Ramani equation are derived. Besides, multi-soliton solution expressed by pfaffian is given and proved by pfaffian techniques.

Keywords: Vector Ramani equation; bilinear Bäcklund transformation; Lax pair; multi-soliton solution; pfaffian.

1. Introduction

In recent years, several approaches have been developed to search for various integrable coupled versions of soliton equations,[1–8] since there are much richer mathematical structures behind integrable coupled systems than scalar ones. One of them is to use bilinear approach to construct the vector extension from the bilinear form of the original nonlinear equation.[1,6–8] For example, the celebrated Korteweg–de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

can be transformed into the bilinear form

$$D_x(D_t + D_x^3)f \cdot f = 0\,, \tag{2}$$

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through the dependent variable transformation $u = 2(\ln f)_{xx}$, where the Hirota's bilinear differential operators[1] are defined by

$$D_x^n D_t^m (a \cdot b) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m a(x,t)b(x',t')|_{x=x',t=t'} .$$

In Ref. 6, it is shown that the extension of the bilinear equation (2) into a coupled bilinear form

$$D_x(D_t + D_x^3) f \cdot f = g^2 , \tag{3}$$

$$(D_t - 2D_x^3) f \cdot g = 0 , \tag{4}$$

results in the following Hirota–Satauma coupled KdV equation:

$$u_t + 6uu_x + u_{xxx} = 2vv_x , \tag{5}$$

$$v_t - 2v_{xxx} - 6uv_x = 0 , \tag{6}$$

where $u = 2(\ln f)_{xx}$ and $v = g/f$.

In what follows, we list several extensions of the KdV equation by using the bilinear approach.

(i) Taking into account the fact that the KdV equation (1) can be transformed into another bilinear form

$$(D_t + D_x^3) g \cdot f = 0 , \tag{7}$$

$$D_x^2 f \cdot f = 2gf , \tag{8}$$

through the rational transformation $u = g/f$, the following bilinear equations

$$(D_t + D_x^3) g_j \cdot f = 0 , \tag{9}$$

$$D_x^2 f \cdot f = 2\left(\sum_{j=1}^{N} g_j\right) f, \quad j = 1, 2, \ldots, N , \tag{10}$$

give rise to a coupled KdV equation of the form

$$\frac{\partial u_j}{\partial t} + 6\left(\sum_{k=1}^{N} u_k\right)\frac{\partial u_j}{\partial x} + \frac{\partial^3 u_j}{\partial x^3} = 0, \quad j = 1, 2, \ldots, N , \tag{11}$$

through the dependent variable transformation $u_j = g_j/f$. Equation (11) has been studied by Yoneyama in Ref. 7.

(ii) Based on the fact that the bilinear form of the KdV equation (1) can be rewritten as

$$(D_t + D_x^3) f_x \cdot f = 0 , \tag{12}$$

Hirota *et al.*[8] noticed that Eq. (12) can be cast into an alternative form

$$(D_t + D_x^3) g \cdot f = 0 , \tag{13}$$

$$D_x^2 f \cdot f = 2D_x g \cdot f , \tag{14}$$

by letting $g = f_x$, and extending Eqs. (13) and (14) to the following coupled form

$$(D_t + D_x^3) g_j \cdot f = 0\,, \tag{15}$$

$$D_x^2 f \cdot f = 2D_x \left(\sum_{j=1}^{N} g_j \right) f, \quad j = 1, 2, \ldots, N\,. \tag{16}$$

By using the dependent variable transformation $v_j = 2g_j/f$, Eqs. (15) and (16) are converted to the N-component potential KdV equation

$$\frac{\partial v_j}{\partial t} + 3 \left(\sum_{k=1}^{N} v_k \right) \frac{\partial v_j}{\partial x} + \frac{\partial^3 v_j}{\partial x^3} = 0, \quad j = 1, 2, \ldots, N\,. \tag{17}$$

(iii) Starting from the bilinear form of the Hirota–Satauma coupled KdV equations (5) and (6), we considered a generalized vector Hirota–Satauma coupled KdV equation[9]

$$\left(D_x D_t + \tfrac{1}{2} D_x^4 \right) f \cdot f = 3 \sum_{1 \le j < k \le N} c_{jk} g_j g_k\,, \tag{18}$$

$$(D_t - D_x^3) g_i \cdot f = 0, \quad i = 1, 2, \ldots, N\,, \tag{19}$$

which has the nonlinear form

$$u_t + \tfrac{1}{2} u_{xxx} + 3uu_x = 3 \sum_{1 \le j < k \le N} c_{jk} (\phi_j \phi_k)_x\,, \tag{20}$$

$$\phi_{i,t} - \phi_{i,xxx} - 3u\phi_{i,x} = 0, \quad i = 1, 2, \ldots, N\,, \tag{21}$$

by the dependent variable transformation $u = 2(\ln f)_{xx}$ and $\phi_i = g_i/f$. In particular, when $N = 2$, (20) and (21) are reduced to the generalized Hirota–Satauma coupled KdV equation[10] under some adequate scaling transformations.

The Ramani equation:

$$\begin{aligned} &u_{xxxxxx} + 15u_{xx}u_{xxx} + 15u_x u_{xxxx} + 45u_x^2 u_{xx} \\ &\quad - 5(u_{xxxt} + 3u_{xx}u_t + 3u_x u_{xt}) - 5u_{tt} = 0\,, \end{aligned} \tag{22}$$

was first proposed in Ref. 11, which can be obtained as a 5-reduction of the BKP equation.[12] It has been shown that the Ramani equation possesses bilinear Bäcklund transformation and conservation laws.[13] In Ref. 14, a coupled Ramani equation of the following form has been proposed as follows:

$$\begin{aligned} &u_{xxxxxx} + 15u_{xx}u_{xxx} + 15u_x u_{xxxx} + 45u_x^2 u_{xx} \\ &\quad - 5(u_{xxxt} + 3u_{xx}u_t + 3u_x u_{xt}) - 5u_{tt} + 18w_x = 0\,, \end{aligned} \tag{23}$$

$$w_t - w_{xxx} - 3w_x u_x - 3wu_{xx} = 0\,. \tag{24}$$

Moreover, several extensions of the coupled Ramani equation were studied in Refs. 15–17.

The purpose of this paper is to extend the coupled Ramani equations (23) and (24) to a vector form and to study its integrability using the bilinear approach. We will give bilinear Bäcklund transformation, Lax pair and soliton solution expressed by pfaffian for the vector Ramani equation.

The rest of the paper is organized as follows. In Sec. 2, we give the vector Ramani equation by extending the bilinear form of Eqs. (23) and (24). In Sec. 3, with the help of the exchange formulae, we derive a bilinear Bäcklund transformation and the corresponding Lax pair for the vector Ramani equation. In Sec. 4, multi-soliton solution expressed by pfaffian is given and proved by pfaffian techniques. Finally, this paper is concluded by Sec. 5.

2. The Vector Ramani Equation and Its Bilinear Form

By the dependent variable transformations

$$u = 2(\ln f)_x, \quad w = \left(\frac{g}{f}\right)_x. \tag{25}$$

Equations (23) and (24) are cast into the following bilinear form:

$$(D_x^6 - 5D_x^3 D_t - 5D_t^2) f \cdot f + 18 D_x g \cdot f = 0, \tag{26}$$

$$(D_t - D_x^3) g \cdot f = 0. \tag{27}$$

Furthermore, by introducing an auxiliary variable z and letting $g = f_z$, (26) and (27) become the following bilinear equations for a single field f:

$$(D_x^6 - 5D_x^3 D_t - 5D_t^2 + 9D_x D_z) f \cdot f = 0, \tag{28}$$

$$D_z(D_t - D_x^3) f \cdot f = 0. \tag{29}$$

Based on the bilinear equations (28) and (29), the Lax pair and Bäcklund transformation for (23) and (24) were discussed in Ref. 14. Multi-soliton solution for Eqs. (23) and (24) was derived and expressed using pfaffians in a compact form.[18]

We note that the bilinear equations (28) and (29) are similar to those of the KdV equation (13). Thus, we consider the following bilinear equations as an extension of the coupled Ramani equations (23) and (24):

$$(D_x^6 - 5D_x^3 D_t - 5D_t^2) f \cdot f + 18 \sum_{\mu=1}^{M} D_x g_\mu \cdot f = 0, \tag{30}$$

$$(D_t - D_x^3) g_\mu \cdot f = 0, \tag{31}$$

$$\sum_{\mu=1}^{M} g_\mu = f_z, \quad \mu = 1, 2, \ldots, M, \tag{32}$$

which is transformed into the following nonlinear form:

$$\begin{aligned} &u_{xxxxxx} + 15u_{xx}u_{xxx} + 15u_x u_{xxxx} + 45u_x^2 u_{xx} \\ &\quad - 5(u_{xxxt} + 3u_{xx}u_t + 3u_x u_{xt}) - 5u_{tt} + 18 \sum_{\mu=1}^{M} w_{\mu,x} = 0, \end{aligned} \tag{33}$$

$$w_{\mu,t} - w_{\mu,xxx} - 3w_{\mu,x}u_x - 3w_\mu u_{xx} = 0\,, \tag{34}$$

$$u_z = 2\sum_{\mu=1}^{M} w_\mu, \quad \mu = 1, 2, \ldots, M\,, \tag{35}$$

or the equivalent vector form

$$\begin{aligned} &u_{xxxxxx} + 15u_{xx}u_{xxx} + 15u_x u_{xxxx} + 45u_x^2 u_{xx} \\ &\quad - 5(u_{xxxt} + 3u_{xx}u_t + 3u_x u_{xt}) - 5u_{tt} + 18(\mathbf{C}\cdot\mathbf{W})_x = 0\,, \end{aligned} \tag{36}$$

$$\mathbf{W}_t - \mathbf{W}_{xxx} - 3u_x\mathbf{W}_x - 3u_{xx}\mathbf{W} = 0\,, \tag{37}$$

$$u_z = 2\mathbf{C}\cdot\mathbf{W}\,, \tag{38}$$

through the dependent variable transformations

$$u = 2(\ln f)_x, \quad w_\mu = \left(\frac{g_\mu}{f}\right)_x\,, \tag{39}$$

where $\mathbf{W} = (w_1, w_2, \ldots, w_M)$, $\mathbf{C} = (1, 1, \ldots, 1)$ and the inner product $\mathbf{C}\cdot\mathbf{W}$ is defined by $\mathbf{C}\cdot\mathbf{W} = \sum_{\mu=1}^{M} w_\mu$.

Similar to the vector potential KdV equation and vector Ito equation,[8] the vector asymmetrical Nizhnik–Novikov–Veselov equation[19] and the multi-component higher-order Ito equation,[20] this is a natural extension from the bilinear equations (28) and (29) of the coupled Ramani equations (23) and (24).

3. Bilinear Bäcklund Transformation and Lax Pair

In this section, we first derive a bilinear Bäcklund transformation for the vector Ramani equations (33) and (35). To this end, we consider

$$\begin{aligned} P_1 = &\left[(D_x^6 - 5D_x^3D_t - 5D_t^2)f\cdot f + 18\sum_{\mu=1}^{M} D_x g_\mu\cdot f\right] f'^2 \\ &- \left[(D_x^6 - 5D_x^3D_t - 5D_t^2)f'\cdot f' + 18\sum_{\mu=1}^{M} D_x g'_\mu\cdot f'\right] f^2\,, \end{aligned} \tag{40}$$

$$P_2 = [(D_t - D_x^3)g_\mu\cdot f]f'^2 - [(D_t - D_x^3)g'_\mu\cdot f']f^2\,, \tag{41}$$

where g'_μ and f' satisfy

$$(D_x^6 - 5D_x^3D_t - 5D_t^2)f'\cdot f' + 18\textstyle\sum_{\mu=1}^{M} D_x g'_\mu\cdot f' = 0\,, \tag{42}$$

$$(D_t - D_x^3)g'_\mu\cdot f' = 0\,, \tag{43}$$

$$\sum_{\mu=1}^{M} g'_\mu = f'_z, \quad \mu = 1, 2, \ldots, M\,. \tag{44}$$

With the help of the bilinear operator identities:

$$\begin{aligned}
b^2D_x^6 a\cdot a - a^2 D_x^6 b\cdot b &= 3D_x[ab\cdot(D_x^5 a\cdot b) + 5(D_x^3 a\cdot b)\cdot(D_x^2 a\cdot b)]\\
&\quad + 5D_x^3(D_x^3 a\cdot b)\cdot ab\,,\\
b^2D_x^3D_t a\cdot a - a^2D_x^3D_t b\cdot b &= 2D_t(D_x^3 a\cdot b)\cdot ab + 6D_x(D_x a\cdot b)\cdot(D_xD_t a\cdot b)\,,\\
b^2D_t^2 a\cdot a - a^2 D_t^2 b\cdot b &= 2D_t(D_t a\cdot b)\cdot ab\,,\\
c^2D_t a\cdot b - b^2 D_t d\cdot c &= bcD_t(a\cdot c - d\cdot b) - (ac+bd)D_t b\cdot c\,,\\
c^2D_x^3 a\cdot b - b^2D_x^3 d\cdot c &= bcD_x^3(a\cdot c - d\cdot b) - (ac+bd)D_x^3 b\cdot c\\
&\quad - 3D_x[D_x(a\cdot c - b\cdot d)]\cdot[D_x b\cdot c]\,,\\
c^2D_x a\cdot b - b^2D_x d\cdot c &= D_x(ac - db)\cdot bc\,,\\
D_x^3(D_t a\cdot b)\cdot ab &= D_t(D_x^3 a\cdot b)\cdot ab + 3D_t(D_x a\cdot b)\cdot(D_x^2 a\cdot b)\,,
\end{aligned}$$

$$\begin{aligned}
&D_t(D_x a\cdot b)\cdot(D_x^2 a\cdot b) + D_x[(D_t a\cdot b)\cdot(D_x^2 a\cdot b) - 2(D_x a\cdot b)\cdot(D_xD_t a\cdot b)]\\
&\quad = D_t(D_x^3 a\cdot b)\cdot ab - D_x(D_x^2D_t a\cdot b)\cdot ab\,,
\end{aligned}$$

P_1 and P_2 can be rewritten as

$$\begin{aligned}
P_1 &= 3D_x[ff'\cdot(D_x^5 f\cdot f') + 5(D_x^3 f\cdot f')\cdot(D_x^2 f\cdot f')] + 5D_x^3(D_x^3 f\cdot f')\cdot ff'\\
&\quad - 10D_t(D_x^3 f\cdot f')\cdot ff' - 30D_x(D_x f\cdot f')\cdot(D_xD_t f\cdot f') - 10D_t(D_t f\cdot f')\cdot ff'\\
&\quad + 18\sum_{\mu=1}^{M} D_x(g_\mu f' - g'_\mu f)\cdot ff'\\
&= -3D_x(D_x^5 f\cdot f')\cdot ff' + 15D_x[(D_x^3 - D_t)f\cdot f']\cdot(D_x^2 f\cdot f')\\
&\quad + 15D_x(D_t f\cdot f')\cdot(D_x^2 f\cdot f') + 5D_x^3[(D_x^3 - D_t)f\cdot f']\cdot ff' + 5D_x^3(D_t f\cdot f')\cdot ff'\\
&\quad - 10D_t(D_x^3 f\cdot f')\cdot ff' - 30D_x(D_x f\cdot f')\cdot(D_xD_t f\cdot f') - 10D_t(D_t f\cdot f')\cdot ff'\\
&\quad + 18\sum_{\mu=1}^{M} D_x(g_\mu f' - g'_\mu f)\cdot ff'\\
&= -3D_x(D_x^5 f\cdot f')\cdot ff' + 15D_x(D_t f\cdot f')\cdot(D_x^2 f\cdot f') + 5D_x^3(D_t f\cdot f')\cdot ff'\\
&\quad -10D_t(D_x^3 f\cdot f')\cdot ff' - 30D_x(D_x f\cdot f')\cdot(D_xD_t f\cdot f') - 10D_t(D_t f\cdot f')\cdot ff'\\
&\quad + 15D_x[(D_x^3 - D_t)f\cdot f']\cdot(D_x^2 f\cdot f') + 5D_x^3[(D_x^3 - D_t)f\cdot f']\cdot ff'\\
&\quad + 18\sum_{\mu=1}^{M} D_x(g_\mu f' - g'_\mu f)\cdot ff'\\
&= -3D_x(D_x^5 f\cdot f')\cdot ff' + 15D_x(D_t f\cdot f')\cdot(D_x^2 f\cdot f') + 5[D_t(D_x^3 f\cdot f')\cdot ff'\\
&\quad + 3D_t(D_x f\cdot f')\cdot(D_x^2 f\cdot f')] - 10D_t(D_x^3 f\cdot f')\cdot ff'
\end{aligned}$$

$$
\begin{aligned}
& -30D_x(D_xf\cdot f')\cdot(D_xD_tf\cdot f') - 10D_t(D_tf\cdot f')\cdot ff' \\
& + 5D_x^3[(D_x^3 - D_t)f\cdot f']\cdot ff' + 18\sum_{\mu=1}^{M} D_x(g_\mu f' - g'_\mu f)\cdot ff' \\
= & -3D_x[(D_x^5 f\cdot f') + 5D_x^2D_tf\cdot f' - 6\sum_{\mu=1}^{M}(g_\mu f' - g'_\mu f)]\cdot ff' \\
& + 15D_x[(D_x^3 - D_t)f\cdot f']\cdot(D_x^2 f\cdot f') + 5D_x^3[(D_x^3 - D_t)f\cdot f']\cdot ff' \\
& + 10D_t[(D_x^3 - D_t)f\cdot f']\cdot ff',
\end{aligned}
$$

$$
\begin{aligned}
P_2 = & ff'(D_t - D_x^3)(g_\mu\cdot f' - g'_\mu\cdot f) - (g_\mu f' + g'_\mu f)(D_t - D_x^3)(f\cdot f') \\
& + 3D_x[D_x(g_\mu\cdot f' - f\cdot g'_\mu)]\cdot[D_xf\cdot f'].
\end{aligned}
$$

Thus, one can have the following bilinear Bäcklund transformation

$$(D_x^5 f\cdot f') + 5D_x^2D_tf\cdot f' - 6\sum_{\mu=1}^{M}(g_\mu f' - g'_\mu f) = 0\,, \tag{45}$$

$$(D_t - D_x^3)(g_\mu\cdot f' - g'_\mu\cdot f) = 0\,, \tag{46}$$

$$(D_t - D_x^3)(f\cdot f') = 0\,, \tag{47}$$

$$D_x(g_\mu\cdot f' - f\cdot g'_\mu) = \lambda_\mu D_xf\cdot f'\,. \tag{48}$$

In order to find Lax pair for the vector Ramani equations (33)–(35), we let

$$
\begin{gathered}
f = \phi f', \quad g_\mu = \psi_\mu f' + \phi g'_\mu, \quad u = 2(\ln f')_x, \quad w_\mu = \left(\frac{g'_\mu}{f'}\right)_x, \\
\mathbf{C}\cdot\mathbf{W} = \frac{D_xD_zf'\cdot f'}{2f'^2}\,,
\end{gathered}
\tag{49}
$$

and thus deduce the Lax pair

$$\psi_{\mu,x} = -2w_\mu\phi + \lambda_\mu\phi_x\,, \tag{50}$$

$$\psi_{\mu,t} = \psi_{\mu,xxx} + 3u_x\psi_{\mu,x} + 6w_{\mu,x}\phi_x\,, \tag{51}$$

$$\phi_{xxxxx} = -5u_x\phi_{xxx} - 5u_{xx}\phi_{xx} - \left(\frac{10}{3}u_{xxx} - 5u_x^2 - \frac{5}{3}u_t\right)\phi_x + \sum_{\mu=1}^{M}\psi_\mu\,, \tag{52}$$

$$\phi_t = \phi_{xxx} + 3u_x\phi_x, \quad \phi_z = \sum_{\mu=1}^{M}\psi_\mu\,, \tag{53}$$

for $\mu = 1, 2, \ldots M$. One can check that the compatibility condition of (50)–(53) gives Eqs. (33)–(35).

4. Multi-Soliton Solution Expressed by Pfaffian

Using a perturbational method, we obtain a few soliton solutions to the vector Ramani equations (33)–(35) for $N = 3$. These solutions contain one-soliton solution

$$f = 1 + \exp(\eta_1), \quad g_\mu = c_\mu(1)\exp(\eta_1)\,, \tag{54}$$

two-soliton solution

$$f = 1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_3) + c_0(1,2)\exp(\eta_1+\eta_2)\,, \tag{55}$$

$$g_\mu = c_\mu(1)\exp(\eta_1) + c_\mu(2)\exp(\eta_2) + c_\mu(3)\exp(\eta_3) + c_\mu(1,2)\exp(\eta_1+\eta_2)\,, \tag{56}$$

and three-soliton solution

$$\begin{aligned} f &= 1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_3) + c_0(1,2)\exp(\eta_1+\eta_2) + c_0(1,3)\exp(\eta_1+\eta_3) \\ &\quad + c_0(2,3)\exp(\eta_2+\eta_3) + c_0(1,2,3)\exp(\eta_1+\eta_2+\eta_3), \end{aligned} \tag{57}$$

$$\begin{aligned} g_\mu &= c_\mu(1)\exp(\eta_1) + c_\mu(2)\exp(\eta_2) + c_\mu(3)\exp(\eta_3) + c_\mu(1,2)\exp(\eta_1+\eta_2) \\ &\quad + c_\mu(1,3)\exp(\eta_1+\eta_3) + c_\mu(2,3)\exp(\eta_2+\eta_3) \\ &\quad + c_\mu(1,2,3)\exp(\eta_1+\eta_2+\eta_3). \end{aligned} \tag{58}$$

Here,

$$\eta_j = p_j x + p_j^3 t + p_j^5 z + \eta_{j,0}, \quad c_0(j,k) = \frac{(p_j-p_k)(p_j^5-p_k^5)}{(p_j+p_k)(p_j^5+p_k^5)},$$

$$c_\mu(j,k) = \frac{(p_j-p_k)}{(p_j+p_k)}[c_\mu(j)-c_\mu(k)], \quad c_0(1,2,3) = c_0(1,2)c_0(1,3)c_0(2,3)\,,$$

$$c_\mu(1,2,3) = \frac{(p_1-p_2)(p_1-p_3)(p_2-p_3)}{(p_1+p_2)(p_1+p_3)(p_2+p_3)(p_1^5+p_2^5)(p_1^5+p_3^5)(p_2^5+p_3^5)}\Delta_\mu(1,2,3),$$

$$\begin{aligned} \Delta_\mu(1,2,3) &= c_\mu(1)(p_1^5+p_2^5)(p_1^5+p_3^5)(p_2^5-p_3^5) + c_\mu(2)(p_1^5+p_2^5)(p_3^5-p_1^5)(p_2^5+p_3^5) \\ &\quad + c_\mu(3)(p_1^5-p_2^5)(p_1^5+p_3^5)(p_2^5+p_3^5)\,, \end{aligned}$$

$$\sum_{\mu=1}^{3} c_\mu(j) = p_j^5\,,$$

where $j, \mu, k = 1,2,3$ and $p_j, c_1(j), c_2(j), c_3(j)$ for $j = 1,2,3$ are free parameters. Plots of u and w_μ ($\mu = 1,2,3$) defined by (39), (55) and (56) are shown in Fig. 1, respectively, to illustrate the two solitons interaction.

In the following, we give N-soliton solution to Eqs. (30)–(32) by pfaffians. In fact, we find that

$$f = \mathrm{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_0) \triangleq \mathrm{pf}(d_0, \bullet\beta_0), \tag{59}$$

$$g_\mu = \mathrm{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu) \triangleq \mathrm{pf}(d_0, \bullet\beta_\mu) \tag{60}$$

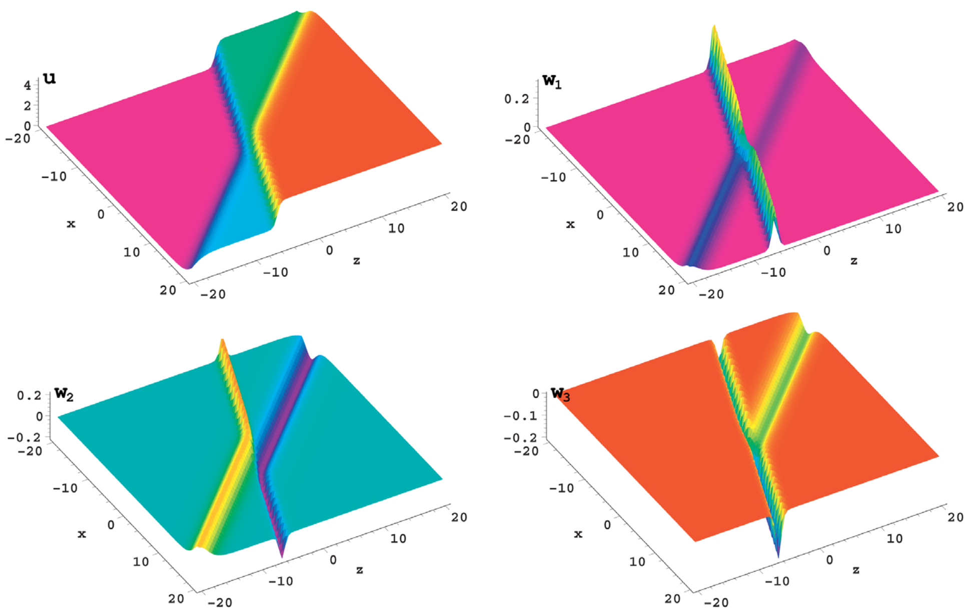

Fig. 1. (Color online) Two-soliton solution given by (39), (55) and (56) with the parameters $p_1 = 1$, $p_2 = \frac{13}{10}$, $c_1(1) = \frac{1}{3}$, $c_2(1) = c_2(2) = \frac{2}{3}$, $c_1(2) = 1$, $c_3(2) = -\frac{11}{30}$ and $c_3(1) = \eta_{1,0} = \eta_{2,0} = 0$ at $t = 0$.

for $\mu = 1, 2, \ldots, M$, where the elements of the pfaffians are defined as follows:

$$\mathrm{pf}(d_0, a_j) = \exp(\eta_j), \quad \mathrm{pf}(d_0, b_j) = -1, \quad \mathrm{pf}(d_0, \beta_0) = 1\,,$$
$$\mathrm{pf}(a_j, a_k) = -a_{j,k}\exp(\eta_j + \eta_k), \quad \mathrm{pf}(a_j, b_k) = \delta_{j,k}, \quad \mathrm{pf}(a_j, \beta_0) = 0\,,$$
$$\mathrm{pf}(b_j, b_k) = b_{j,k}, \quad \mathrm{pf}(d_0, \beta_\mu) = 0, \quad \mathrm{pf}(b_j, \beta_0) = 1\,,$$
$$\mathrm{pf}(a_j, \beta_\mu) = 0, \quad \mathrm{pf}(b_j, \beta_\mu) = c_\mu(j)$$

with

$$\eta_j = p_j x + p_j^3 t + p_j^5 z + \eta_{j,0}\,,$$
$$a_{j,k} = \frac{p_j - p_k}{p_j + p_k}, \quad b_{j,k} = \frac{p_j^5 - p_k^5}{p_j^5 + p_k^5}\,,$$

where p_j, $\eta_{j,0}$ and $c_\mu(j)$ are constant parameters, and $c_\mu(j)$ needs to satisfy the constraint conditions

$$\sum_{\mu}^{M} c_\mu(j) = p_j^5\,. \tag{61}$$

From the definition of the functions f and g_μ, we can derive the following pfaffian's rules:

$$f_x = -\mathrm{pf}(d_0, d_1, \bullet)\,, \tag{62}$$

$$f_{xx} = -\mathrm{pf}(d_0, d_2, \bullet)\,, \tag{63}$$

$$f_{xxx} = -\mathrm{pf}(d_0, d_3, \bullet) - \mathrm{pf}(d_0, d_1, d_2, \bullet, \beta_0)\,, \tag{64}$$

$$f_{xxxx} = -\mathrm{pf}(d_0, d_4, \bullet) - 2\mathrm{pf}(d_0, d_1, d_3, \bullet, \beta_0)\,, \tag{65}$$

$$f_{xxxxx} = -\mathrm{pf}(d_0, d_5, \bullet) - 3\mathrm{pf}(d_0, d_1, d_4, \bullet, \beta_0) - 2\mathrm{pf}(d_0, d_2, d_3, \bullet, \beta_0), \tag{66}$$

$$\begin{aligned} f_{xxxxxx} &= -\mathrm{pf}(d_0, d_6, \bullet) - 4\mathrm{pf}(d_0, d_1, d_5, \bullet, \beta_0) \\ &\quad - 5\mathrm{pf}(d_0, d_2, d_4, \bullet, \beta_0) + 2\mathrm{pf}(d_0, d_1, d_2, d_3, \bullet)\,, \end{aligned} \tag{67}$$

$$f_t = -\mathrm{pf}(d_0, d_3, \bullet) + 2\mathrm{pf}(d_0, d_1, d_2, \bullet, \beta_0)\,, \tag{68}$$

$$f_{xt} = -\mathrm{pf}(d_0, d_4, \bullet) + \mathrm{pf}(d_0, d_1, d_3, \bullet, \beta_0)\,, \tag{69}$$

$$f_{xxt} = -\mathrm{pf}(d_0, d_5, \bullet) + \mathrm{pf}(d_0, d_2, d_3, \bullet, \beta_0)\,, \tag{70}$$

$$\begin{aligned} f_{xxxt} &= -\mathrm{pf}(d_0, d_6, \bullet) - \mathrm{pf}(d_0, d_1, d_5, \bullet, \beta_0) + \mathrm{pf}(d_0, d_2, d_4, \bullet, \beta_0) \\ &\quad - \mathrm{pf}(d_0, d_1, d_2, d_3, \bullet)\,, \end{aligned} \tag{71}$$

$$\begin{aligned} f_{tt} &= -\mathrm{pf}(d_0, d_6, \bullet) + 2\mathrm{pf}(d_0, d_1, d_5, \bullet, \beta_0) - 2\mathrm{pf}(d_0, d_2, d_4, \bullet, \beta_0) \\ &\quad - 4\mathrm{pf}(d_0, d_1, d_2, d_3, \bullet)\,, \end{aligned} \tag{72}$$

$$f_z = -\mathrm{pf}(d_0, d_5, \bullet) + 2\mathrm{pf}(d_0, d_1, d_4, \bullet, \beta_0) - 2\mathrm{pf}(d_0, d_2, d_3, \bullet, \beta_0)\,, \tag{73}$$

$$f_{xz} = -\mathrm{pf}(d_0, d_6, \bullet) + \mathrm{pf}(d_0, d_1, d_5, \bullet, \beta_0) + 2\mathrm{pf}(d_0, d_1, d_2, d_3, \bullet)\,, \tag{74}$$

and

$$g_{\mu,x} = \mathrm{pf}(d_0, d_1, \bullet, \beta_\mu, \beta_0)\,, \tag{75}$$

$$g_{\mu,xx} = \mathrm{pf}(d_0, d_2, \bullet, \beta_\mu, \beta_0)\,, \tag{76}$$

$$g_{\mu,xxx} = \mathrm{pf}(d_0, d_3, \bullet, \beta_\mu, \beta_0) - \mathrm{pf}(d_0, d_1, d_2, \bullet, \beta_\mu)\,, \tag{77}$$

$$g_{\mu,t} = \mathrm{pf}(d_0, d_3, \bullet, \beta_\mu, \beta_0) + 2\mathrm{pf}(d_0, d_1, d_2, \bullet, \beta_\mu)\,, \tag{78}$$

where

$$\mathrm{pf}(d_0, d_k) = 0, \quad \mathrm{pf}(d_k, d_l) = 0, \quad \mathrm{pf}(d_k, b_j) = 0\,,$$
$$\mathrm{pf}(d_k, a_j) = p_j^k \exp(\eta_j), \quad \mathrm{pf}(d_k, \beta_0) = 0\,, \quad \mathrm{pf}(d_k, \beta_\mu) = 0\,,$$

for $k, l = 1, 2, 3, 4, 5, 6$ and $j = 1, 2, \ldots, N$.

Substituting Eqs. (32), (62)–(78) into the bilinear equations (30) and (31) yields

$$\begin{aligned}
&(D_x^6 - 5D_x^3D_t - 5D_t^2)f\cdot f + 18\sum_{\mu=1}^{M} D_x g_\mu\cdot f\\
&\quad = (D_x^6 - 5D_x^3D_t - 5D_t^2 + 9D_xD_z)f\cdot f\\
&\quad = 90\mathrm{pf}(d_0,d_2,\bullet)\mathrm{pf}(d_0,d_1,d_3,\bullet,\beta_0) - 90\mathrm{pf}(d_0,d_3,\bullet)\mathrm{pf}(d_0,d_1,d_2,\bullet,\beta_0)\\
&\qquad + 90\mathrm{pf}(d_0,\bullet,\beta_0)\mathrm{pf}(d_0,d_1,d_2,d_3,\bullet) - 90\mathrm{pf}(d_0,d_1,\bullet)\mathrm{pf}(d_0,d_2,d_3,\bullet,\beta_0)\,,
\end{aligned}$$

and

$$\begin{aligned}
&(D_t - D_x^3)g_\mu\cdot f\\
&\quad = 3\mathrm{pf}(d_0,\bullet,\beta_0)\mathrm{pf}(d_0,d_1,d_2,\bullet,\beta_\mu) - 3\mathrm{pf}(d_0,\bullet,\beta_\mu)\mathrm{pf}(d_0,d_1,d_2,\bullet,\beta_0)\\
&\quad - 3\mathrm{pf}(d_0,d_2,\bullet,\beta_\mu,\beta_0)\mathrm{pf}(d_0,d_1,\bullet) + 3\mathrm{pf}(d_0,d_1,\bullet,\beta_\mu,\beta_0)\mathrm{pf}(d_0,d_2,\bullet)
\end{aligned}$$

which vanish by the pfaffian identities.

Next, we prove that f and g_μ satisfy the linear equation (32).

In order to utilize the expansion formulae in Ref. 1 as follows:

$$\begin{aligned}
&\mathrm{pf}(\alpha_1,\alpha_2,c_1,c_2,\ldots,c_{2n})\\
&\quad = \sum_{1\le j<k\le 2n}(-1)^{j+k-1}\mathrm{pf}(\alpha_1,\alpha_2,c_j,c_k)\mathrm{pf}(c_1,c_2,\ldots,\hat{c}_j,\ldots,\hat{c}_k,\ldots,c_{2n}),\\
&\qquad \text{if}\quad \mathrm{pf}(\alpha_1,\alpha_2)=0,
\end{aligned}\tag{79}$$

$$\begin{aligned}
&\mathrm{pf}(\alpha_1,\alpha_2,c_1,c_2,\ldots,c_{2n})\\
&\quad = \sum_{j=2}^{2n}(-1)^j[\mathrm{pf}(\alpha_1,\alpha_2,c_1,c_j)\mathrm{pf}(c_2,\ldots,\hat{c}_j,\ldots,c_{2n})\\
&\qquad + \mathrm{pf}(c_1,c_j)\mathrm{pf}(\alpha_1,\alpha_2,c_2,\ldots,\hat{c}_j,\ldots,c_{2n})],\quad \text{if}\quad \mathrm{pf}(\alpha_1,\alpha_2)=0,
\end{aligned}\tag{80}$$

we introduce a new character β_0' defined by

$$\mathrm{pf}(d_0,\beta_0')=0,\quad (\mathrm{pf}(d_0,\beta_0)=1)\,,\tag{81}$$

$$\mathrm{pf}(a_j,\beta_0')=0,\quad \mathrm{pf}(b_j,\beta_0')=1,\quad (\mathrm{pf}(a_j,\beta_0)=0,\quad \mathrm{pf}(b_j,\beta_0)=1)\tag{82}$$

for $j=1,2,\ldots,N$. Then

$$\begin{aligned}
f &= \mathrm{pf}(d_0,\beta_0,a_1,a_2,\ldots,a_N,b_1,b_2,\ldots,b_N)\\
&= \mathrm{pf}(d_0,\beta_0,c_1,c_2,\ldots,c_{2N})\\
&= \mathrm{pf}(d_0,\beta_0)\mathrm{pf}(c_1,c_2,\ldots,c_{2N})\\
&\quad + \sum_{j=1}^{2N}\sum_{k=1}^{2N}(-1)^{j+k}\mathrm{pf}(d_0,c_j)\mathrm{pf}(\beta_0,c_k)\mathrm{pf}(c_1,c_2,\ldots,\hat{c}_j,\ldots,\hat{c}_k,\ldots,c_{2N})
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{pf}(c_1, c_2, \ldots, c_{2N}) \\
&\quad + \sum_{1\le j<k\le 2N} (-1)^{j+k}[\mathrm{pf}(d_0, c_j)\mathrm{pf}(\beta_0, c_k) - \mathrm{pf}(d_0, c_k)\mathrm{pf}(\beta_0, c_j)] \\
&\quad \times \mathrm{pf}(c_1, c_2, \ldots, \hat{c}_j, \ldots, \hat{c}_k, \ldots, c_{2N}) \\
&= \mathrm{pf}(c_1, c_2, \ldots, c_{2N}) \\
&\quad + \sum_{1\le j<k\le 2N} (-1)^{j+k}[\mathrm{pf}(d_0, c_j)\mathrm{pf}(\beta_0', c_k) - \mathrm{pf}(d_0, c_k)\mathrm{pf}(\beta_0', c_j)] \\
&\quad \times \mathrm{pf}(c_1, c_2, \ldots, \hat{c}_j, \ldots, \hat{c}_k, \ldots, c_{2N}) \\
&= \mathrm{pf}(c_1, c_2, \ldots, c_{2N}) \\
&\quad + \sum_{1\le j<k\le 2N} (-1)^{j+k-1}\mathrm{pf}(d_0, \beta_0', c_j, c_k)\mathrm{pf}(c_1, c_2, \ldots, \hat{c}_j, \ldots, \hat{c}_k, \ldots, c_{2N}) \\
&= \mathrm{pf}(c_1, c_2, \ldots, c_{2N}) + \mathrm{pf}(d_0, \beta_0', c_1, c_2, \ldots, c_{2N})\,.
\end{aligned}
$$

Thus, f can be expressed as

$$f = f_0 + f', \tag{83}$$

$$f_0 = \mathrm{pf}(a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N), \tag{84}$$

$$f' = \mathrm{pf}(d_0, \beta_0', a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N)\,. \tag{85}$$

We note that f_0 and f' are invariant under the transformation

$$
\begin{aligned}
a_j &\to a_j' \quad (= a_j \exp(-\eta_j))\,, \\
b_j &\to b_j' \quad (= b_j \exp(\eta_j))\,,
\end{aligned}
$$

so that

$$
\begin{aligned}
f_0 &= \mathrm{pf}(a_1', a_2', \ldots, a_N', b_1', b_2', \ldots, b_N')\,, \\
f' &= \mathrm{pf}(d_0, \beta_0', a_1', a_2', \ldots, a_N', b_1', b_2', \ldots, b_N')
\end{aligned}
$$

with the entries

$$
\begin{gathered}
\mathrm{pf}(a_j', a_k') = -a_{j,k}, \quad \mathrm{pf}(a_j', b_k') = \delta_{j,k}, \quad \mathrm{pf}(b_j', b_k') = b_{j,k}\exp(\eta_j + \eta_k)\,, \\
\mathrm{pf}(d_0, \beta_0') = 0, \quad \mathrm{pf}(d_0, a_j') = 1, \quad \mathrm{pf}(d_0, b_j') = -\exp(\eta_j)\,, \\
\mathrm{pf}(\beta_0', a_j') = 0, \quad \mathrm{pf}(\beta_0', b_j') = -\exp(\eta_j)\,.
\end{gathered}
$$

Furthermore, we introduce another character $d_0'(= d_0 - \beta_0')$, so that

$$
\begin{aligned}
\mathrm{pf}(d_0', a_j') &= \mathrm{pf}(d_0, a_j') - \mathrm{pf}(\beta_0', a_j') = 1\,, \\
\mathrm{pf}(d_0', b_j') &= \mathrm{pf}(d_0, b_j') - \mathrm{pf}(\beta_0', b_j') = 0\,.
\end{aligned}
$$

Therefore, we can find the following differential formulae:

$$\frac{\partial}{\partial z}\mathrm{pf}(a'_j, a'_k) = 0 = \mathrm{pf}(\alpha_5, \beta'_0, a'_j, a'_k)\,,$$

$$\frac{\partial}{\partial z}\mathrm{pf}(a'_j, b'_k) = 0 = \mathrm{pf}(\alpha_5, \beta'_0, a'_j, b'_k)\,,$$

$$\frac{\partial}{\partial z}\mathrm{pf}(b'_j, b'_k) = (p_j^5 - p_k^5)\exp(\eta_j + \eta_k) = \mathrm{pf}(\alpha_5, \beta'_0, b'_j, b'_k)\,,$$

and

$$\frac{\partial}{\partial z}\mathrm{pf}(d_0, \beta'_0, a'_j, a'_k) = 0 = \mathrm{pf}(\alpha_5, d'_0, a'_j, a'_k)\,,$$

$$\frac{\partial}{\partial z}\mathrm{pf}(d_0, \beta'_0, a'_j, b'_k) = p_j^5\exp(\eta_j) = \mathrm{pf}(\alpha_5, d'_0, a'_j, b'_k)\,,$$

$$\frac{\partial}{\partial z}\mathrm{pf}(d_0, \beta'_0, b'_j, b'_k) = 0 = \mathrm{pf}(\alpha_5, d'_0, b'_j, b'_k)\,,$$

where the new entries are defined by

$$\mathrm{pf}(\alpha_5, \beta'_0) = 0, \quad \mathrm{pf}(\alpha_5, b'_j) = p_j^5\exp(\eta_j)\,,$$

$$\mathrm{pf}(\alpha_5, d_0) = 0, \quad \mathrm{pf}(\alpha_5, a'_j) = 0\,.$$

By defining $c'_j = a'_j$ and $c'_{N+j} = b'_j$ for $j = 1, 2, \ldots, N$, f_0, f' and the differential formulae are written as

$$f_0 = \mathrm{pf}(c'_1, c'_2, \ldots, c'_{2N})\,, \tag{86}$$

$$f' = \mathrm{pf}(d_0, \beta'_0, c'_1, c'_2, \ldots, c'_{2N})\,, \tag{87}$$

$$\frac{\partial}{\partial z}\mathrm{pf}(c'_j, c'_k) = \mathrm{pf}(\alpha_5, \beta'_0, c'_j, c'_k)\,, \tag{88}$$

$$\frac{\partial}{\partial z}\mathrm{pf}(d_0, \beta'_0, c'_j, c'_k) = \mathrm{pf}(\alpha_5, d'_0, c'_j, c'_k)\,. \tag{89}$$

Following the procedures described in Ref. 8, we can derive

$$\frac{\partial}{\partial z}f_0 = \mathrm{pf}(\alpha_5, \beta'_0, c'_1, c'_2, \ldots, c'_{2N})\,, \tag{90}$$

$$\frac{\partial}{\partial z}f' = \mathrm{pf}(\alpha_5, d'_0, c'_1, c'_2, \ldots, c'_{2N})\,. \tag{91}$$

In fact, by expanding f_0 with respect to c'_1, we can obtain

$$\begin{aligned}\frac{\partial}{\partial z}f_0 &= \frac{\partial}{\partial z}\mathrm{pf}(c'_1, c'_2, \ldots, c'_{2N})\\ &= \sum_{j=2}^{2N}\frac{\partial}{\partial z}[\mathrm{pf}(c'_1, c'_j)(-1)^j\mathrm{pf}(c'_2, \ldots, \hat{c}'_j, \ldots, c'_{2N})]\end{aligned}$$

$$= \sum_{j=2}^{2N} \left\{ \frac{\partial}{\partial z}[\mathrm{pf}(c_1', c_j')](-1)^j \mathrm{pf}(c_2', \ldots, \hat{c}_j', \ldots, c_{2N}') \right.$$
$$\left. + \mathrm{pf}(c_1', c_j')(-1)^j \frac{\partial}{\partial z}[\mathrm{pf}(c_2', \ldots, \hat{c}_j', \ldots, c_{2N}')] \right\} . \tag{92}$$

Next, we consider an induction. If $N = 1$, the formula (90) is nothing but the differential formula (88). Assuming the differential formula (90) holds for $n = n-1$ and utilizing Eq. (88), one has

$$\frac{\partial}{\partial z} f_0 = \sum_{j=2}^{2N} \{ [\mathrm{pf}(\alpha_5, \beta_0', c_1', c_j')](-1)^j \mathrm{pf}(c_2', \ldots, \hat{c}_j', \ldots, c_{2N}')$$
$$+ \mathrm{pf}(c_1', c_j')(-1)^j \mathrm{pf}(\alpha_5, \beta_0', c_2', \ldots, \hat{c}_j', \ldots, c_{2N}') \} . \tag{93}$$

With the help of the expansion formula (80), the differential formula (90) is obtained.

In order to derive the differential formula (91), we expand f' and use the the expansion formula (79),

$$\frac{\partial}{\partial z} f' = \frac{\partial}{\partial z} \mathrm{pf}(d_0, \beta_0', c_1', c_2', \ldots, c_{2N}') ,$$
$$= \frac{\partial}{\partial z} \sum_{1 \le j < k \le 2N} (-1)^{j+k-1} \mathrm{pf}(d_0, \beta_0', c_j', c_k') \mathrm{pf}(c_1', c_2', \ldots, \hat{c}_j', \ldots, \hat{c}_k', \ldots, c_{2N}')$$
$$= \sum_{1 \le j < k \le 2N} (-1)^{j+k-1} \left\{ \frac{\partial}{\partial z} [\mathrm{pf}(d_0, \beta_0', c_j', c_k')] \mathrm{pf}(c_1', c_2', \ldots, \hat{c}_j', \ldots, \hat{c}_k', \ldots, c_{2N}') \right.$$
$$\left. + \mathrm{pf}(d_0, \beta_0', c_j', c_k') \frac{\partial}{\partial z} [\mathrm{pf}(c_1', c_2', \ldots, \hat{c}_j', \ldots, \hat{c}_k', \ldots, c_{2N}')] \right\} . \tag{94}$$

Considering the differential formulae (89) and (90) and the expansion formula (79), we have

$$\frac{\partial}{\partial z} f' = \sum_{1 \le j < k \le 2N} (-1)^{j+k-1} \{ \mathrm{pf}(\alpha_5, d_0', c_j', c_k') \mathrm{pf}(c_1', c_2', \ldots, \hat{c}_j', \ldots, \hat{c}_k', \ldots, c_{2N}')$$
$$+ \mathrm{pf}(d_0, \beta_0', c_j', c_k') \mathrm{pf}(\alpha_5, \beta_0', c_1', c_2', \ldots, \hat{c}_j', \ldots, \hat{c}_k', \ldots, c_{2N}') \}$$
$$= \sum_{1 \le j < k \le 2N} (-1)^{j+k-1} \{ \mathrm{pf}(\alpha_5, d_0', c_j', c_k') \mathrm{pf}(c_1', c_2', \ldots, \hat{c}_j', \ldots, \hat{c}_k', \ldots, c_{2N}') \}$$
$$= \mathrm{pf}(\alpha_5, d_0', c_1', c_2', \ldots, c_{2N}') . \tag{95}$$

Subsequently,

$$\frac{\partial}{\partial z} f = \frac{\partial}{\partial z} (f_0 + f') = \mathrm{pf}(\alpha_5, d_0, c_1', c_2', \ldots, c_{2N}') . \tag{96}$$

Expanding (96) with respect to α_5 leads to

$$\begin{aligned}\frac{\partial}{\partial z}f &= \sum_{j=1}^{N} p_j^5(-1)^{N+j}\exp(\eta_j)\mathrm{pf}(d_0, c_1', c_2', \ldots, c_N', c_{N+1}', \ldots, \hat{c}_{N+j}', \ldots, c_{2N}') \\ &= \sum_{j=1}^{N} p_j^5(-1)^{N+j}\mathrm{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, \hat{b}_j, \ldots, b_N)\,. \end{aligned} \tag{97}$$

On the other hand,

$$\begin{aligned} g_\mu &= \mathrm{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, \beta_\mu) \\ &= \sum_{j=1}^{N} c_\mu(j)(-1)^{N+j}\mathrm{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, \hat{b}_j, \ldots, b_N)\,, \end{aligned} \tag{98}$$

then the sum of g_μ over μ gives

$$\sum_{\mu=1}^{M} g_\mu = \sum_{j=1}^{N} p_j^5(-1)^{N+j}\mathrm{pf}(d_0, a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, \hat{b}_j, \ldots, b_N) \tag{99}$$

which is equal to Eq. (97). Accordingly, we have shown that f and g_μ satisfy the linear equation (32).

5. Conclusion and Summary

In this paper, we study a vector Ramani equation based on its bilinear form. By means of the bilinear exchange formulae, bilinear Bäcklund transformation for the vector Ramani equation is given. It is also shown that the bilinear Bäcklund transformation can be linearized into the corresponding Lax pair. Moreover, multi-soliton solution expressed by pfaffian can be obtained and proved by pfaffian techniques.

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