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An Integrable Discrete Generalized Nonlinear Schrödinger Equation and Its Reductions*

LI Hong-Min (李红敏), LI Yu-Qi (李玉奇), and CHEN Yong (陈勇)†

Shanghai Key Laboratory of Trustworthy Computing, East China Normal University, Shanghai 200062, China

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Abstract *An integrable discrete system obtained by the algebraization of the difference operator is studied. The system is named discrete generalized nonlinear Schrödinger (GNLS) equation, which can be reduced to classical discrete nonlinear Schrödinger (NLS) equation. Furthermore, all of the linear reductions for the discrete GNLS equation are given through the theory of circulant matrices and the discrete NLS equation is obtained by one of the reductions. At the same time, the recursion operator and symmetries of continuous GNLS equation are successfully recovered by its corresponding discrete ones.*

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1 Introduction

The research of discrete nonlinear systems described by differential-difference equations (discrete in space and continuous in time) has received considerable attention.[1−5] Much effort has been spent by several research groups to derive analogous integrable nonlinear differential-difference equations for a given integrable partial differential equation (PDE). Especially, integrable discrete family of nonlinear Schrödinger (NLS) equations are involved in many physical applications.[6−10] In this paper, we construct an integrable discrete NLS systems which maybe have potential physical applications.

The construction of the discrete NLS equations for the given continuous NLS equations is not an easy task. Several works are devoted to this subject.[6,11−12] Among these works, Ablowitz and Ladik[6] presented the following integrable discrete NLS equation by introducing a new discrete eigenvalue problem

$$
\begin{aligned}
&\mathrm{i}u_{nt}+\frac{(u_{n+1}+u_{n-1}-2u_n)}{h^2}\\
&\qquad -u_nw_n(u_{n+1}+u_{n-1})=0\,,\\
&\mathrm{i}w_{nt}-\frac{(w_{n+1}+w_{n-1}-2w_n)}{h^2}\\
&\qquad +w_nu_n(w_{n+1}+w_{n-1})=0\,, \qquad (1)
\end{aligned}
$$

which is a discrete version of the generalized nonlinear Schrödinger (GNLS) equation

$$\mathrm{i}u_t+u_{xx}-2u^2w=0\,,\quad \mathrm{i}w_t-w_{xx}+2w^2u=0\,, \qquad (2)$$

and Eq. (2) belongs to the Ablowitz, Kaup, Newell and Segur or so-called AKNS hierarchy, so Eq. (2) is also called AKNS equation.

The systems (1) and (2) respectively reduce to the following integrable systems:

$$
\begin{aligned}
&\mathrm{i}u_{nt}+\frac{(u_{n+1}+u_{n-1}-2u_n)}{h^2}\\
&\qquad -|u_n|^2(u_{n+1}+u_{n-1})=0\,, \qquad (3)\\
&\mathrm{i}u_t+u_{xx}-2|u|^2u=0\,, \qquad (4)
\end{aligned}
$$

in the continuum limit ($h\to 0$) where in Eqs. (1) and (2) we take the linear reductions $w_n=-u_n^*$, $w=-u^*$, where $*$ denotes the complex conjugate.

Another discretization of the NLS equation is the diagonal discrete NLS

$$\mathrm{i}u_{nt}+\frac{(u_{n+1}+u_{n-1}-2u_n)}{h^2}-2|u_n|^2u_n=0\,. \qquad (5)$$

The systems (3) and (5) differ only in the discretization of the nonlinear term, yet they have very different properties. It is worthy to note that Eq. (3) is integrable via the inverse scattering transform, while Eq. (5) is not. Izergin and Korepin[11] also presented a discretization of NLS which is integrable but lengthy, see the book by Faddeev and Takhtajan[12] for details. As a contrast, our discrete NLS equations are neat and integrable.

As is known, the algebraization of the shift operator E is the usual way to generate integrable discrete equations. However, the algebraization of the difference operator Δ is more convenient to the algebraization of E as operator Δ^{-1} appears explicitly in the Lax pairs. Recently, Li, *et al.*,[13] gave a new residue formula to compute the conservation laws for the discrete Lax equations obtained by the algebraization of Δ and verified the validity of the formula by numerical experiments under the periodic bound-

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†E-mail: ychen@sei.ecnu.edu.cn

ary condition. More recently, we study the constraint of discrete Lax equations with some special forms of Lax pairs.[14] Furthermore, the work[15] extended the method for constructing recursion operators of continuous PDEs proposed in Ref. [16] to discrete Lax equations and recovered the recursion operators for the continuous equations by a limit process.

The arrangement of the paper is as follows: In Sec. 2, we recall a brief introduction of pseudodifference operators ring (see Refs. [13–15] for details). In Sec. 3, we study an integrable discrete GNLS equation's property such as it's recursion operator, symmetries and conservation quantities. In Sec. 4, we prove a theorem that clarifies all the linear reductions for the discrete GNLS equation, moreover from a reduction of discrete GNLS equation, we successfully obtain the discrete NLS equation. In Sec. 5, we summarize the conclusions and propose some problems.

For brevity, here we introduce some notes relevant to the contents of this paper: $u = u(n,t)$, $u_j = E^j \cdot u = u(n+j,t)$, $v = v(n,t)$, $v_j = E^j \cdot v = v(n+j,t)$, $w = w(n,t)$, $w_j = E^j \cdot w = w(n+j,t)$. All the other functions' subscript is understood as the usual way.

2 Basic Definitions and Propositions of Pseudodifference Operators Ring

Denote the forward difference, i.e.,

$$(\Delta \cdot g)(n) = \frac{g(n+1) - g(n)}{h}\,, \tag{6}$$

where h is a real constant and $g(n)$ is the n-th component of vector g.

The operator Δ acts on a vector space V of arbitrary but definite dimension N. In the continuous case, any vector v of dimension N can be identified as an operator by

$$(v \circ f)(n) = v(n)f(n)\,, \quad n = 1, 2, \ldots, N\,.$$

The basic formula for the ring is derived by rewriting the modified Leibnize rule

$$\Delta \cdot (fg) = (\Delta \cdot f)g + (f + h\Delta \cdot f)\Delta \cdot g \tag{7}$$

into an operator form

$$\Delta \circ f \circ g = (\Delta \cdot f) \circ g + (f + h\Delta \cdot f) \circ \Delta \circ g\,. \tag{8}$$

In Eq. (7), the multiplication among vectors f, $\Delta \cdot f$, and g is component multiplication. For example, $(fg)(n) = f(n)g(n)$. In Eq. (8), f and $\Delta \cdot f$ are operators acting on V and g is an arbitrary vector in V. So we have the operator equation

$$\Delta \circ f = (\Delta \cdot f) + (f + h\Delta \cdot f) \circ \Delta\,. \tag{9}$$

For the convenience, in the following paper, we will omit the composition symbol $\circ$ in the operator expressions. From Eq. (9), we can quickly derive the formula

$$\Delta^n f = \sum_{k=0}^{\infty} \binom{n}{k} (\Delta^k (1 + h\Delta)^{n-k}) \cdot f \Delta^{n-k}, \quad n \in Z. \tag{10}$$

The above formula (9) is the basic formula for the difference operators' ring and please pay attention to the definition of $\Delta^n \cdot f$ is $\Delta \cdot (\Delta^{n-1} \cdot f)$.

Suppose $A = \sum_{-\infty}^{n} a_j \Delta^j$, $a_n \neq 0$, we introduce the following definitions and propositions.

Definition 1 The residue of A is defined as follows:

$$\mathrm{Res}(A) = a_{-1}\,. \tag{11}$$

Definition 2 The Gateaux derivative for $f \in V_\Delta$ in the direction $g \in V_\Delta$ is defined by

$$f'[g] = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} f(u + \varepsilon g)|_{\varepsilon=0}\,. \tag{12}$$

The f' is called linearization operator of f and also can be expressed as:

$$f' = \sum_j \frac{Df}{D(E^j u)} E^j\,, \tag{13}$$

where $V_\Delta = \{\sum_{j=-\infty}^{\infty} f_j \Delta^j, f_j$ are operators corresponding to vectors of dimension $N\}$ with dimension infinity.

Definition 3 For a given discrete evolution equation

$$u_t = K(u(t,n))\,,$$

$\sigma(u(t,n)) \in V_\Delta$ is called its symmetry if $\sigma_t = K'[\sigma]$, where $K(u(t,n))$ is a function of t, n, u, $\Delta \cdot u$, $\ldots$, $\Delta^\alpha \cdot u$ and $\Delta^{-1} \cdot u$, $\ldots$, $\Delta^{-\beta} \cdot u$.

Proposition 1

$$\rho_{\frac{k}{n}} = \sum_{j=0}^{N-1} E^j \cdot \mathrm{Res}(L^{k/n} E^{-1})\,, \quad k = 1,\ 2,\ \ldots\,, \tag{14}$$

are all constants of motion. We call the above formula residue formula, which is the base for our calculations of the conserved quantities of the discrete Lax equation ($L_t = [P, L]$).

Proposition 2

$$\mathrm{Res}(AE^{-1}) = \frac{1}{h} \sum_{j=0}^{n} \Big(\frac{-1}{h}\Big)^j a_j\,. \tag{15}$$

Remark 1 The periodic boundary condition is very interesting in computation and application. In this paper, we suppose our Lax equations satisfy the periodic boundary condition.

3 Integrability of Discrete GNLS Equation

In this section, we construct the discrete GNLS equation and show the integrability for it by constructing it's recursion operator, symmetries and conserved quantities.

3.1 *Recursion Operator and Symmetries of Discrete GNLS Equation*

Let us begin with the following Lax pair

$$\begin{aligned} L &= \mathrm{i}(\Delta + v + u\Delta^{-1} w)\,, \\ P &= \mathrm{i}(L^2)_+ = -\mathrm{i}[\Delta^2 + (v + v_1)\Delta + \Delta \cdot v + v^2 \\ &\quad + uw_{-1} + u_1 w]\,, \end{aligned} \tag{16}$$

where u, v, w are N-dimensional vectors.

Substituting the above L, P into the Lax representation,

$$L_t = [P, L]\,, \tag{17}$$

by equating the coefficients of different powers of Δ in Eq. (17), we have the following differential equations:

$$\begin{aligned}\mathrm{i}v_t =& \frac{1}{h}(u_2 w - 2u_1 w + 2uw_{-1} - uw_{-2})\\ &+ u_1(v+v_1)w - u(v_{-1}+v)w_{-1}\,,\\ \mathrm{i}u_t =& \frac{1}{h^2}(u_2 - 2u_1 + u) + \frac{1}{h}(u_1v_1 + u_1 v - 2uv)\\ &+ u(v^2 + uw_{-1} + u_1 w)\,,\\ \mathrm{i}w_t =& -\frac{1}{h^2}(w - 2w_{-1} + w_{-2}) + \frac{1}{h}(2wv - w_{-1}v_{-1}\\ &- w_{-1}v) - w(v^2 + uw_{-1} + u_1 w)\,. \end{aligned} \tag{18}$$

From Eq. (18), we obtain

$$v_t = h(uw)_t\,.$$

Here for convenience, let the constant vector be 0, then $v = huw$. Therefore we have the discrete GNLS equation:

$$\begin{aligned}\mathrm{i}u_t =& \frac{1}{h^2}(u_2 - 2u_1 + u) + u_1^2 w_1 + u^2 w_{-1}\\ &+ 2uu_1 w - 2u^2 w + h^2u^3w^2\,,\\ \mathrm{i}w_t =& -\frac{1}{h^2}(w - 2w_{-1} + w_{-2}) - u_1 w^2 - u_{-1}w_{-1}^2\\ &+ 2uw^2 - 2uw_{-1}w - h^2u^2w^3\,. \end{aligned} \tag{19}$$

Remark 2 When $h \to 0$, the above equation is just the continuous GNLS equation (2).

It is well known that a recursion operator for a system of PDEs is extremely important, so several works are devoted to this subject. Among these works, Gürses *et al.*[16] proposed a powerful approach to construct the recursion operators for nonlinear integrable equations admitting Lax representation. Next, we use the same idea to investigate the recursion operator of discrete GNLS equation.

Theorem 1 For any n,

$$L_{t_{n+1}} = L_{t_n}L + [R_n, L]\,, \tag{20}$$

the above equation is just the recursion equation

$$\begin{pmatrix} u_{t_{n+1}} \\ w_{t_{n+1}} \end{pmatrix} = R \begin{pmatrix} u_{t_n} \\ w_{t_n} \end{pmatrix},$$

$$R = \mathrm{i}\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \tag{21}$$

where

$$R_{11} = \Delta + 2u(\Delta^{-1} + h)w\,,\quad R_{12} = u(2\Delta^{-1} + h)u\,,$$
$$R_{21} = -w(2\Delta^{-1} + h)w\,,\quad R_{22} = -\Delta E^{-1} - 2w\Delta^{-1}u\,.$$

Proof Obviously, from Eq. (16), we get

$$L_{t_{n+1}} = \mathrm{i}(hu_{t_{n+1}}w + huw_{t_{n+1}} + u_{t_{n+1}}\Delta^{-1}w + u\Delta^{-1}w_{t_{n+1}})\,,\quad L_{t_n} = \mathrm{i}(hu_{t_n}w + huw_{t_n} + u_{t_n}\Delta^{-1}w + u\Delta^{-1}w_{t_n})\,,$$

and $R_n = \mathrm{i}(a_n + b_n\Delta^{-1}w)$. Therefore, equating the coefficients of Δ, Δ^0, and Δ^{-1} of Eq. (20), we obtain

$$hu_{t_n}w + huw_{t_n} = E\cdot a_n - a_n\,, \tag{22}$$

$$\mathrm{i}(hu_{t_{n+1}}w + huw_{t_{n+1}}) = -[h^2(u_{t_n}w + uw_{t_n})uw + u_{t_n}w_{-1} + uw_{-1t_n} + b_n w_{-1} - \Delta\cdot a_n - wE\cdot b_n]\,, \tag{23}$$

$$\begin{aligned}\mathrm{i}(u_{t_{n+1}}\Delta^{-1}w + u\Delta^{-1}w_{t_{n+1}}) =& -[(hu_{t_n}w + huw_{t_n})u + ua_n - \Delta\cdot b_n - huwb_n\\ &+ (u_{t_n} + b_n)\Delta^{-1}uw]\Delta^{-1}w + (u_{t_n} + b_n)\Delta^{-1}(\Delta\cdot w_{-1}) - h(u_{t_n} + b_n)\Delta^{-1}(uw^2)\\ &- u\Delta^{-1}(uw_{t_n} - wb_n)\Delta^{-1}w - u\Delta^{-1}(-\Delta\cdot w_{-1t_n} - wa_n + huww_{t_n})\,. \end{aligned} \tag{24}$$

From Eq. (22), we get $\Delta\cdot a_n = u_{t_n}w + uw_{t_n}$. Observing Eq. (24), we use the anstaze $b_n = -u_{t_n}$, then substituting the above a_n, b_n to Eqs. (23) and (24), we find the two equations are compatible, furthermore the recursion operator R can be obtained easily which is just Eq. (21). When setting h approaches to 0 in the discrete R, we get

$$R = \mathrm{i}\begin{pmatrix} D + 2uD^{-1}w & 2uD^{-1}u \\ -2wD^{-1}w & -D - 2wD^{-1}u \end{pmatrix},$$

which is just the recursion operator for continuous GNLS equation. Furthermore, the compact recursion operator (21) is indeed more applicable than the recursion operator in Refs. [17–18] in determining the whole integrable family starting from one given equation and the families of Hamiltonian structures for the other integrable equations.

As in the differential case, the recursion operator acting on a seed symmetry of the discrete equation also generates an infinite hierarchy of symmetries of that discrete equation. So in the following, we investigate the symmetries of the discrete GNLS equation.

Proposition 3 $\sigma_1 = \mathrm{i}u$, $\sigma_2 = -\mathrm{i}w$ is a symmetry of discrete GNLS equation.

Proof For brevity, suppose

$$\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} \mathrm{i}u \\ -\mathrm{i}w \end{pmatrix}$$

and the discrete GNLS equation is

$$\begin{pmatrix} u_t \\ w_t \end{pmatrix} = K\begin{pmatrix} u \\ w \end{pmatrix}.$$

Set the linearization operator of K as follows:

$$K' = -\mathrm{i}\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix},$$

through Eq. (13), we get

$$K_{11} = \Delta^2 + 2(E+1)\cdot(uwE + uw_{-1}) - 4uw + 3h^2u^2w^2\,,\quad K_{12} = u_1^2E + u^2E^{-1} + 2u(u_1 - u) + 2h^2u^3w\,,$$
$$K_{21} = -w^2E - w_{-1}^2E^{-1} + 2w(w - w_{-1}) - 2h^2uw^3\,,$$
$$K_{22} = -\Delta^2E^{-2} - 2(E+1)\cdot(u_{-1}w_{-1}E^{-1} + uw_{-1}) + 4uw - 3h^2u^2w^2\,.$$

By direct calculation,

$$K'[\sigma] = -\mathrm{i}\begin{pmatrix} K_{11}\cdot\sigma_1 + K_{12}\cdot\sigma_2 \\ K_{21}\cdot\sigma_1 + K_{22}\cdot\sigma_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{h^2}(u_2 - 2u_1 + u) + u_1^2 w_1 + 2uu_1 w - 2u^2 w + h^2 u^3 w^2 + u^2 w_{-1} \\ \frac{1}{h^2}(w - 2w_{-1} + w_{-2}) - 2uw^2 + u_{-1}w_{-1}^2 + 2uw_{-1}w + h^2u^2w^3 + u_1 w^2 \end{pmatrix},$$

since u, w satisfy the discrete GNLS equation, therefore $\sigma_t = K'[\sigma]$, i.e., σ is a symmetry of discrete GNLS equation. Moreover, σ is just the seed symmetry for the continuous GNLS equation when h approaches to 0.

From the above recursion operator R and seed symmetry σ, we can get infinite symmetries of the discrete GNLS equation. For example

$$R\cdot\sigma = \begin{pmatrix} -\Delta\cdot u - hu^2 w \\ -\Delta\cdot w_{-1} + huw^2 \end{pmatrix},$$

$$R^2\cdot\sigma = \begin{pmatrix} -\frac{1}{h^2}(u_2 - 2u_1 + u) - u_1^2 w_1 - 2uu_1 w + 2u^2 w - h^2u^3w^2 - u^2 w_{-1} \\ \frac{1}{h^2}(w - 2w_{-1} + w_{-2}) - 2uw^2 + u_{-1}w_{-1}^2 + 2uw_{-1}w + h^2u^2w^3 + u_1w^2 \end{pmatrix}$$

are symmetries of the discrete GNLS equation.

3.2 *Conservation Laws of Discrete GNLS Equation*

The conservation laws play important roles in discussing the integrability for soliton equations. Many methods have been developed to find them. In this section, we will use the method proposed in Ref. [13] to compute the conservation laws for the discrete GNLS equation.

From Eqs. (14) and (15), we can get the infinitely conserved quantities and the first three are

$$\rho_1 = \mathrm{i}\sum_{j=0}^{N-1} u_j w_j\,,$$

$$\rho_2 = -\sum_{j=0}^{N-1} hu_j^2 w_j^2 + \frac{1}{h}(u_j w_{j-1} + u_{j+1}w_j)\,,$$

$$\begin{aligned}\rho_3 = -\mathrm{i}\sum_{j=0}^{N-1} & h^2 u_j^3 w_j^3 + u_j u_{j-1} w_{j-1}^2 \\ & + u_{j+1}^2 w_j w_{j+1} + 2u_j^2 w_j w_{j-1} + 2u_j u_{j+1} w_j^2 \\ & + \frac{1}{h^2}(u_{j+2}w_j + u_{j+1}w_{j-1} + u_j w_{j-2})\,, \end{aligned} \tag{25}$$

and it can be verified numerically that the ρ_1, ρ_2, and ρ_3 are indeed conserved quantities.

4 Reductions of Discrete GNLS Equation

As is well known, the continuous GNLS equation can be reduced to the NLS equation. Therefore an important problem is whether the discrete GNLS equation in the paper can also be reduced to the corresponding discrete NLS equation. The answer is affirmative. In the following, as the reduction for Eq. (1) given by Ablowitz *et al.* is linear, we will study all linear reductions of the discrete GNLS equation. The reduction to the NLS equation is just among the rest.

Case 1 From Eq. (19), we know $w = 0$ is a reasonable reduction. Then the discrete GNLS can be reduced as follows:

$$\mathrm{i}u_t = \frac{1}{h^2}(u_2 - 2u_1 + u)\,,$$

the corresponding continuous equation is $\mathrm{i}u_t = u_{xx}$.

Case 2 When $w \neq 0$, we have the following theorem:

Theorem 2 The discrete GNLS equation has the following linear reductions: $u^* = w_k$, $u_1^* = w_{k-1}, \ldots, u_{k-1}^* = w_1$, $u_k^* = w$, $u_{k+1}^* = w_{N-1}$, $u_{k+2}^* = w_{N-2}, \ldots$, $u_{N-2}^* = w_{k+2}$, $u_{N-1}^* = w_{k+1}$, where $0 \le k \le N-1$ and N is the grid number of the periodic lattice. Moreover, the discrete GNLS equation only has the above linear reductions up to a non-zero constant.

Proof For the convenience of understanding, from Eq. (19), the discrete GNLS equation is expressed as below:

$$\mathrm{i}u_{kt} = \frac{1}{h^2}(u_{k+2} - 2u_{k+1} + u_k) + u_{k+1}^2 w_{k+1} + 2u_k u_{k+1} w_k - 2u_k^2 w_k + h^2 u_k^3 w_k^2 + u_k^2 w_{k-1}\,, \quad (0 \le k \le N-1)\,, \tag{26}$$

$$\mathrm{i}w_{kt} = -\frac{1}{h^2}(w_k - 2w_{k-1} + w_{k-2}) - u_{k-1}w_{k-1}^2 + 2u_k w_k^2 - 2u_k w_{k-1} w_k - h^2 u_k^2 w_k^3 - u_{k+1} w_k^2\,, \quad (0 \le k \le N-1)\,. \tag{27}$$

Then Eq. (26)'s conjugate is as follows:

$$-\mathrm{i}u_{kt}^* = \frac{1}{h^2}(u_{k+2}^* - 2u_{k+1}^* + u_k^*) + u_{k+1}^{*2}w_{k+1}^* + 2u_k^* u_{k+1}^* w_k^* - 2u_k^{*2}w_k^* + h^2 u_k^{*3} w_k^{*2} + u_k^{*2} w_{k-1}^*\,, \quad (0 \le k \le N-1)\,.$$

Substituting $u^* = w_k, u_1^* = w_{k-1}, \ldots, u_{k-1}^* = w_1$, $u_k^* = w$, $u_{k+1}^* = w_{N-1}$, $u_{k+2}^* = w_{N-2}$, $u_{N-2}^* = w_{k+2}$, $u_{N-1}^* = w_{k+1}$ to the above equations, we find these equations are compatible with corresponding equalities in Eq. (27).

In order to prove the discrete GNLS equation only has the reductions in theorem 2, we have to make the following transformation for the discrete GNLS equation.

Obviously, the coefficient matrices for $1/h^2$ in u and w are the $N\times N$ circulant matrices

$$A=\begin{pmatrix}1 & -2 & 1 & \cdots & 0\\ 0 & 1 & -2 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -2 & 1 & 0 & \cdots & 1\end{pmatrix}=\mathrm{cir}(1,-2,1,0,\ldots,0)\,,\quad B=-A^{\mathrm{T}}\,.$$

From the theory of circulant matrices,[19] there must exist an invertible matrix C makes A, B be diagonalized, i.e.,

$$C^{-1}AC=\mathrm{diag}(f(1),f(\varepsilon_1),\ldots,f(\varepsilon_{N-1}))\,,\quad C^{-1}BC=-\mathrm{diag}(f(1),f(\varepsilon_{N-1}),\ldots,f(\varepsilon_1))\,,$$

where C is the Vandermonde matrix

$$C=\begin{pmatrix}1 & 1 & 1 & \cdots & 1\\ 1 & \varepsilon_1 & \varepsilon_1^2 & \cdots & \varepsilon_1^{N-1}\\ 1 & \varepsilon_2 & \varepsilon_2^2 & \cdots & \varepsilon_2^{N-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & \varepsilon_{N-1} & \varepsilon_{N-1}^2 & \cdots & \varepsilon_{N-1}^{N-1}\end{pmatrix}\quad\text{and}\quad f(\varepsilon_i)=1-2\varepsilon_i+\varepsilon_i^2\,,\quad \varepsilon_0=1\,,\quad \varepsilon_i\quad (i=1,\ldots,N-1)$$

are the N-th roots of unity.

Then we make the following transformation:

$$(\,u,\ u_1,\ldots,u_{N-2},\ u_{N-1}\,)^{\mathrm{T}}=C\,(\,U,\ U_1,\ldots,U_{N-2},\ U_{N-1}\,)^{\mathrm{T}}\,,$$
$$(\,w,\ w_1,\ldots,w_{N-2},\ w_{N-1}\,)^{\mathrm{T}}=C\,(\,W,\ W_1,\ldots,W_{N-2},\ W_{N-1}\,)^{\mathrm{T}}\,. \tag{28}$$

Obviously, after the transformation, the coefficient matrices for $1/h^2$ in U and W are $C^{-1}AC,\ C^{-1}BC$ respectively. These coefficient matrices mean the linear reductions

$$(\,U^*,\ U_1^*,\ldots,U_{N-2}^*,\ U_{N-1}^*\,)^{\mathrm{T}}=\Upsilon\,(\,W,\ W_1,\ldots,W_{N-2},\ W_{N-1}\,)^{\mathrm{T}}\,, \tag{29}$$

where $\Upsilon=\mathrm{diag}(\gamma,\gamma_1,\ldots,\gamma_{N-1})$ and γ,γ_i $(i=1,\ldots,N-1)$ are non-zero complex numbers.

From Eqs. (28) and (29), we know that

$$(\,u^*,\ u_1^*,\ldots,u_{N-2}^*,\ u_{N-1}^*\,)^{\mathrm{T}}=C^*\Upsilon C^{-1}\,(\,w,\ w_1,\ldots,w_{N-2},\ w_{N-1}\,)^{\mathrm{T}}\,, \tag{30}$$

where C^* is the complex adjoint of C.

Substituting Eq. (29) to the transformed discrete GNLS equation and comparing the coefficients of h, we get equalities about $\gamma,\gamma_1,\ldots,\gamma_{N-1}$. Solving the equalities, we obtain N non-zero solutions Υ_i $(i=0,1,\ldots,N-1)$. Furthermore, through calculating some examples, we find $\Upsilon_i=\mathrm{diag}\ (1,\varepsilon_i,\varepsilon_i^2,\ldots,\varepsilon_i^{N-1})$ $(i=0,1,\ldots,N-1)$ up to a non-zero constant, which we have verified for small values of N $(N=3,4,5,6,7,8)$. Substituting the N solutions respectively to the Eq. (30), we have the discrete GNLS equation's reductions

$$C^*\Upsilon_iC^{-1}=\begin{pmatrix}\begin{matrix}0 & \ldots & 1\\ \vdots & \iddots & \vdots\\ 1 & \ldots & 0\end{matrix} & \begin{matrix}0 & \ldots & 0\\ \vdots & \iddots & \vdots\\ 0 & \ldots & 0\end{matrix}\\ \underbrace{\hphantom{1\ \ldots\ 0}}_{(i+1)\times(i+1)} & \\ \begin{matrix}0 & \ldots & 0\\ \vdots & \iddots & \vdots\\ 0 & \ldots & 0\end{matrix} & \begin{matrix}0 & \ldots & 1\\ \vdots & \iddots & \vdots\\ 1 & \ldots & 0\end{matrix}\end{pmatrix},\quad (i=0,1,\ldots,N-1)\,,$$

which are just the reductions in the theorem 2.

Thus, we complete the proof of theorem 2.

In the following, we will give two specific examples $(N=3,4)$ to illustrate the theorem.

When $N=3$, the corresponding coefficient matrices A and B are as follows:

$$A=\mathrm{cir}(1,-2,1)\,,\quad B=-A^{\mathrm{T}}=\mathrm{cir}(-1,-1,2)\,.$$

The corresponding eigenmatrices

$$C=\begin{pmatrix}1 & 1 & 1\\ 1 & -\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}\mathrm{i} & -\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}\mathrm{i}\\ 1 & -\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}\mathrm{i} & -\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}\mathrm{i}\end{pmatrix}.$$

Then we have the transformation

$$(u,\ u_1,\ u_2)^{\mathrm{T}} = C\,(U,\ U_1,\ U_2)^{\mathrm{T}}\,, \quad (w,\ w_1,\ w_2)^{\mathrm{T}} = C\,(W,\ W_1,\ W_2)^{\mathrm{T}}\,.$$

After the transformation, the $N = 3$ discrete GNLS equation is transformed to

$$\begin{aligned}
\mathrm{i}U_t = {} & [U^3W^2 + U_1^3W^2 + U_2^3W^2 + 2(U_2^3W_1W_2 + U^3W_1W_2 + U_1^3W_1W_2) + 3(U^2U_2W_2^2 + U^2U_1W_1^2 \\
& + UU_1^2W_2^2 + UU_2^2W_1^2 + U_1^2U_2W_1^2 + U_1U_2^2W_2^2 + 2UU_1U_2W^2 + 2U^2U_1WW_2 + 2U^2U_2WW_1 + 2UU_1^2WW_1 \\
& + 2UU_2^2WW_2 + 2U_1^2U_2WW_2 + 2U_1U_2^2WW_1 + 4UU_1U_2W_1W_2)]h^2 - \frac{5}{2}(U_1^2W_1 + U_2^2W_2) \\
& + 2(U^2W - UU_1W_2 - UU_2W_1 - U_1U_2W) + \frac{\sqrt{3}}{2}\mathrm{i}(U_1^2W_1 - U_2^2W_2 + 4UU_1W_2 - 4UU_2W_1)\,,
\end{aligned} \tag{31}$$

$$\begin{aligned}
\mathrm{i}U_{1t} = {} & [U^3W_2^2 + U_1^3W_2^2 + U_2^3W_2^2 + 2(U_2^3WW_1 + U^3WW_1 + U_1^3WW_1) \\
& + 3(UU_2^2W^2 + U_1^2U_2W^2 + U_1U_2^2W_1^2 + U^2U_2W_1^2 + UU_1^2W_1^2 + U^2U_1W^2 \\
& + 2U_1^2U_2W_1W_2 + 2U_1U_2^2WW_2 + 2UU_1U_2W_2^2 + 2U^2U_1W_1W_2 + 2U^2U_2WW_2 \\
& + 2UU_1^2WW_2 + 2UU_2^2W_1W_2 + 4UU_1U_2WW_1)]h^2 - \frac{5}{2}(U_2^2W + 2UU_2W_2) \\
& - U^2W_1 + \frac{\sqrt{3}}{2}\mathrm{i}(-U_2^2W + 2UU_2W_2 + 4UU_1W + 4U_1^2W_2) \\
& - 2(UU_1W + 2U_1^2W_2 + 4U_1U_2W_1) + \frac{(-(3\sqrt{3}/2)\mathrm{i}U_1 + (3/2)U_1)}{h^2}\,,
\end{aligned} \tag{32}$$

$$\begin{aligned}
\mathrm{i}U_{2t} = {} & [U^3W_1^2 + U_1^3W_1^2 + U_2^3W_1^2 + 2(U^3WW_2 + U_1^3WW_2 + U_2^3WW_2) \\
& + 3(U^2U_1W_2^2 + U^2U_2W^2 + UU_1^2W^2 + UU_2^2W_2^2 + U_1^2U_2W_2^2 + U_1U_2^2W^2 \\
& + 2UU_1U_2W_1^2 + 2U^2U_1WW_1 + 2U^2U_2W_1W_2 + 2UU_1^2W_1W_2 + 2UU_2^2WW_1 \\
& + 2U_1^2U_2WW_1 + 2U_1U_2^2W_1W_2 + 4UU_1U_2WW_2)]h^2 - \frac{5}{2}(U_1^2W + 2UU_1W_1) \\
& - U^2W_2 + \frac{\sqrt{3}}{2}\mathrm{i}(U_1^2W - 2UU_1W_1 - 4UU_2W - 4U_2^2W_1) \\
& - 2(UU_2W + 2U_2^2W_1 + 4U_1U_2W_2) + \frac{((3\sqrt{3}/2)\mathrm{i}U_2 + (3/2)U_2)}{h^2}\,,
\end{aligned} \tag{33}$$

$$\begin{aligned}
\mathrm{i}W_t = {} & [-U^2W^3 - U^2W_1^3 - U^2W_2^3 - 2(U_1U_2W^3 + U_1U_2W_1^3 + U_1U_2W_2^3) \\
& - 3(U_1^2W^2W_1 + U_1^2WW_2^2 + U_1^2W_1^2W_2 + U_2^2W^2W_2 + U_2^2WW_1^2 + U_2^2W_1W_2^2 \\
& + 2UU_1W^2W_2 + 2UU_1WW_1^2 + 2UU_2W_1^2W_2 + 2UU_1W_1W_2^2 + 2UU_2W^2W_1 \\
& + 2UU_2WW_2^2 + 2U^2WW_1W_2 + 4U_1U_2WW_1W_2)]h^2 \\
& + \frac{5}{2}(U_1W_1^2 + U_2W_2^2) + 2(UW_1W_2 + U_1WW_2 + U_2WW_1 - UW^2) \\
& + \frac{\sqrt{3}}{2}\mathrm{i}(U_1W_1^2 - U_2W_2^2 + 4U_2WW_1 - 4U_1WW_2)\,,
\end{aligned} \tag{34}$$

$$\begin{aligned}
\mathrm{i}W_{1t} = {} & [-U_2^2W^3 - U_2^2W_1^3 - U_2^2W_2^3 - 2(UU_1W^3 + UU_1W_1^3 + UU_1W_2^3) \\
& - 3(U^2W^2W_1 + U^2WW_2^2 + U^2W_1^2W_2 + U_1^2W^2W_2 + U_1^2WW_1^2 + U_1^2W_1W_2^2 \\
& + 2UU_2W^2W_2 + 2UU_2WW_1^2 + 2UU_2W_1W_2^2 + 2U_2^2WW_1W_2 + 2U_1U_2W^2W_1 \\
& + 2U_1U_2WW_2^2 + 2U_1U_2W_1^2W_2 + 4UU_1WW_1W_2)]h^2 + \frac{5}{2}(UW_2^2 + 2U_2WW_2) \\
& + U_1W^2 + \frac{\sqrt{3}}{2}\mathrm{i}(-UW_2^2 + 4UWW_1 + 2U_2WW_2 + 4U_2W_1^2) \\
& + 2(UWW_1 + 2U_2W_1^2 + 4U_1W_1W_2) + \frac{(-(3\sqrt{3}/2)\mathrm{i}W_1 - (3/2)W_1)}{h^2}\,,
\end{aligned} \tag{35}$$

$$\mathrm{i}W_{2t} = [-U_1^2W^3 - U_1^2W_1^3 - U_1^2W_2^3 - 2(UU_2W^3 + UU_2W_1^3 + UU_2W_2^3)$$

$$- 3(U^2W^2W_2 + U^2WW_1^2 + U^2W_1W_2^2 + U_2^2W^2W_1 + U_2^2WW_2^2 + U_2^2W_1^2W_2 + 2UU_1WW_2^2 + 2UU_1W_1^2W_2 + 2UU_1W^2W_1 + 2U_1U_2W^2W_2 + 2U_1U_2WW_1^2 + 2U_1U_2W_1W_2^2 + 2U_1^2WW_1W_2 + 4UU_2WW_1W_2)]h^2 + \frac{5}{2}(UW_1^2 + 2U_1WW_1) + U_2W^2 + \frac{\sqrt{3}}{2}\mathrm{i}(UW_1^2 - 2U_1WW_1 - 4UWW_2 - 4U_1W_2^2) + 2(UWW_2 + 2U_1W_2^2 + 4U_2W_1W_2) + \frac{((3\sqrt{3}/2)\mathrm{i}W_2 - (3/2)W_2)}{h^2}\,. \tag{36}$$

From the coefficients of $1/h^2$ in Eqs. (31)–(36), we could only assume that

$$\begin{pmatrix} U^* \\ U_1^* \\ U_2^* \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \gamma_1 & 0 \\ 0 & 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} W \\ W_1 \\ W_2 \end{pmatrix} = \Upsilon \begin{pmatrix} W \\ W_1 \\ W_2 \end{pmatrix}. \tag{37}$$

Besides the above equalities, we can also get $W^* = (1/\gamma^*)U$, $W_1^* = (1/\gamma_1^*)U_1$, $W_2^* = (1/\gamma_2^*)U_2$. Substituting the above equalities to Eqs. (31)–(36), making the Eqs. (31) and (34), (32) and (35), (33) and (36) be compatible respectively. We have the following equalities:

$$\gamma = \gamma^*\,,\quad \gamma_1^* = \gamma_2\,,\quad \gamma_1\gamma_2 = \gamma\gamma^*\,,\quad \gamma_1^2 = \gamma\gamma_1^*\,.$$

Solving the above equations, we have the three solutions:

$$\gamma = \gamma_1 = \gamma_2 = c\,,$$
$$\gamma = c_1\,,\quad \gamma_1 = c_1\Big(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\Big)\,,$$
$$\gamma_2 = c_1\Big(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\Big)\,,$$
$$\gamma = c_2\,,\quad \gamma_1 = c_2\Big(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\Big)\,,$$
$$\gamma_2 = c_2\Big(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\Big)\,,$$

where c, c_1, c_2 are non-zero real numbers.

Substituting the above solutions to Eq. (30) respectively, we get three relations:

$$\begin{pmatrix} u^* \\ u_1^* \\ u_2^* \end{pmatrix} = \begin{pmatrix} c & 0 & 0 \\ 0 & 0 & c \\ 0 & c & 0 \end{pmatrix} \begin{pmatrix} w \\ w_1 \\ w_2 \end{pmatrix},$$
$$\begin{pmatrix} u^* \\ u_1^* \\ u_2^* \end{pmatrix} = \begin{pmatrix} 0 & c_1 & 0 \\ c_1 & 0 & 0 \\ 0 & 0 & c_1 \end{pmatrix} \begin{pmatrix} w \\ w_1 \\ w_2 \end{pmatrix},$$
$$\begin{pmatrix} u^* \\ u_1^* \\ u_2^* \end{pmatrix} = \begin{pmatrix} 0 & 0 & c_2 \\ 0 & c_2 & 0 \\ c_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} w \\ w_1 \\ w_2 \end{pmatrix},$$

which are just the reductions stated in the theorem 2 for GNLS equation as $c = 1$, $c_1 = 1$, $c_2 = 1$.

When $N = 4$, in the similar way, we get the following equations for γ, γ_1, γ_2

$$\gamma = \gamma^*\,,\quad \gamma_2 = \gamma_2^*\,,\quad \gamma_1\gamma_2 = \gamma\gamma_1^*\,,$$
$$\gamma_1^2 = \gamma\gamma_2^*\,,\quad \gamma_2^2 = \gamma\gamma^*\,,\quad \gamma_3 = \gamma_1^*\,.$$

Solving the above equalities, we obtain the following four solutions:

$$\gamma = \gamma_1 = \gamma_2 = \gamma_3 = c\,,$$
$$\gamma = c_1,\quad \gamma_1 = \mathrm{i}c_1,\quad \gamma_2 = -c_1,\quad \gamma_3 = -\mathrm{i}c_1\,,$$
$$\gamma = c_2,\quad \gamma_1 = -c_2,\quad \gamma_2 = c_2,\quad \gamma_3 = -c_2\,,$$
$$\gamma = c_3,\quad \gamma_1 = -\mathrm{i}c_3,\quad \gamma_2 = -c_3,\quad \gamma_3 = \mathrm{i}c_3\,,$$

where c, c_1, c_2, c_3 are non-zero real numbers. Substituting the above solutions to Eq. (30), we obtain the reductions

$$\begin{pmatrix} u^* \\ u_1^* \\ u_2^* \\ u_3^* \end{pmatrix} = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & c & 0 \\ 0 & c & 0 & 0 \end{pmatrix} \begin{pmatrix} w \\ w_1 \\ w_2 \\ w_3 \end{pmatrix},$$
$$\begin{pmatrix} u^* \\ u_1^* \\ u_2^* \\ u_3^* \end{pmatrix} = \begin{pmatrix} 0 & c_1 & 0 & 0 \\ c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 \\ 0 & 0 & c_1 & 0 \end{pmatrix} \begin{pmatrix} w \\ w_1 \\ w_2 \\ w_3 \end{pmatrix},$$
$$\begin{pmatrix} u^* \\ u_1^* \\ u_2^* \\ u_3^* \end{pmatrix} = \begin{pmatrix} 0 & 0 & c_2 & 0 \\ 0 & c_2 & 0 & 0 \\ c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 \end{pmatrix} \begin{pmatrix} w \\ w_1 \\ w_2 \\ w_3 \end{pmatrix},$$
$$\begin{pmatrix} u^* \\ u_1^* \\ u_2^* \\ u_3^* \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & c_3 \\ 0 & 0 & c_3 & 0 \\ 0 & c_3 & 0 & 0 \\ c_3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w \\ w_1 \\ w_2 \\ w_3 \end{pmatrix},$$

which are just the reductions in the theorem 2 when $c = 1$, $c_1 = 1$, $c_2 = 1$, $c_3 = 1$.

Because the transformed discrete GNLS equation is a bit complicated, here we omit the proof since we have not found a neat way to present it in a reasonable length even for a given N ($N \geq 5$).

Since $0 \leq k \leq N - 1$, the discrete GNLS equation has N reductions. Specially, when $k = N - 1$, the reduction $u^* = w_{N-1}$, $u_1^* = w_{N-2}, \ldots, u_{k-1}^* = w_{N-k}$, $u_k^* = w_{N-k-1}$, $u_{k+1}^* = w_{N-k-2}$, $u_{k+2}^* = w_{N-k-3}, \ldots, u_{N-2}^* = w_1$, $u_{N-1}^* = w$ makes the discrete GNLS equation reduce to the discrete NLS equation

$$\mathrm{i}u_{kt} = \frac{1}{h^2}(u_{k+2} - 2u_{k+1} + u_k) + u_{k+1}^2u_{N-k-2}^* + 2u_ku_{k+1}u_{N-k-1}^* - 2u_k^2u_{N-k-1}^* + h^2u_k^3u_{N-k-1}^{*2} + u_k^2u_{N-k}^*,\quad (0 \leq k \leq N-1)\,. \tag{38}$$

Alternatively, the discrete NLS equation can be written as

$$\begin{aligned} \mathrm{i}u_t = & \frac{1}{h^2}(u_2 - 2u_1 + u) + u_1^2 u_{-2}^* + 2uu_1u_{-1}^* \\ & - 2u^2u_{-1}^* + h^2u^3u_{-1}^{*2} + u^2u^* \,. \end{aligned} \tag{39}$$

Remark 3 As $h \to 0$, the above equation is just the continuous periodic NLS equation (4) symmetric at $x = 0$, i.e., $u(x) = u(-x)$.

5 Conclusions and Discussions

An integrable discrete GNLS equation is given explicitly by the algebraization of the difference operator. Then we compute its recursion operator, infinitely symmetries and conservation laws. Moreover, all the linear reductions for the discrete GNLS equation are presented and we can get the corresponding discrete nonlinear Schrödinger (NLS) equation by one of the reductions. From the results in the paper, it is worth noting that the discrete GNLS equation and discrete NLS equation should have soliton solutions, Hamiltonian structure, τ-function and so on. These problems deserve to study further.

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