A Laplace Decomposition Method for Nonlinear Partial Differential Equations with Nonlinear Term of Any Order*

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Abstract *A Laplace decomposition algorithm is adopted to investigate numerical solutions of a class of nonlinear partial differential equations with nonlinear term of any order,* $u_{tt} + au_{xx} + bu + cu^p + du^{2p-1} = 0$, *which contains some important equations of mathematical physics. Three distinct initial conditions are constructed and generalized numerical solutions are thereby obtained, including numerical hyperbolic function solutions and doubly periodic ones. Illustrative figures and comparisons between the numerical and exact solutions with different values of p are used to test the efficiency of the proposed method, which shows good results are achieved.*

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1 Introduction

In mathematical physics, it is usually very important to seek and construct explicit solutions of nonlinear partial differential equations (PDEs). Such solutions can help us understand the physical phenomena they describe in nature. There are many powerful methods developed, such as inverse scattering theory,[1] Bäcklund and Darboux transformation,[2–3] Hirota's bilinear technique,[4] variable separation approach,[5–6] Wronskian and Casoratian techniques,[7–8] various tanh methods,[9–10] rational expansion method[11] etc. However, it is very difficult to find explicit solutions of nonlinear partial differential equations generally. With the rapid development of nonlinear science, the development of numerical techniques for solving nonlinear equations is a subject of considerable interest. Recently many scientists and engineers have done excellent work, such as Laplace decomposition algorithm,[12–14] homotopy perturbational method (HPM),[15–17] Adomian decomposition method (ADM),[18–22] variational iteration method (VIM)[23–24] etc. It is known that there exist many useful differential equations wherein the functions are expressed by complex, especially in the fields of engineering and electronic circuits.[25–26] Usually, it is not very easy to do some calculations via the ADM directly. However, the Laplace decomposition algorithm (LDA) based on the ADM can effectively turn the calculations from the complex differential procedure to a purely algebraic procedure. With this approach, one can always obtain a kind of more realistic series solutions that generally converge fast to real physics models. However, no special discretization, linearization techniques or small parameter ansätz are required. Particularly, the LDA can accelerate the rapid convergence of series solutions when compared with the ADM and therefore provide major progress.[26] Recently, such an algorithm has been extensively applied to investigate approximate solutions of a class of nonlinear differential equations and even fractional differential equations (see e.g. [27–29]).

Here, our main concern will be with investigation on numerical solutions of a class of nonlinear partial differential equations with nonlinear term of any order:[30]

$$u_{tt} + au_{xx} + bu + cu^p + du^{2p-1} = 0\,, \tag{1}$$

via the Laplace decomposition algorithm. In the above, a, b, c, d and $p \neq 1$ are arbitrary constants. It is noted that distinct equations will be constructed when setting different values to those parameters. In fact, the study on Eq. (1) is very necessary and significant since it contains many important nonlinear equations of mathematical physics, such as Duffing equation,[1] Klein–Gordon equation,[31] Landau–Ginburg–Higgs equation,[32] Sin–Gordon equation,[31–32] ϕ^4 equation[1] as well as the nonlinear evolution equation considered in [33]. It is shown in Ref. [30] that via introduction of a particular transformation and the improved tanh method, Chen *et*

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al. derived some explicit exact solutions of (1), which included the kink-profile, bell-profile solitary-wave solutions and periodic wave solutions. Subsequently, such solutions of numerical type were investigated by An and Chen via the ADM.[22] Therefore, it is natural to inquire whether an alternative method can be devised to construct the explicit exact solutions or numerical solutions that can rapidly converge to the known exact solutions but without of the special transformation? This will be the subject of our present paper.

The paper is organized as follows. In Sec. 2, we give some necessary descriptions on the Laplace decomposition algorithm of (1). In Sec. 3, three distinct initial conditions are constructed and thereby the generalized numerical series solutions are derived. Meanwhile, the efficiency and accuracy of the proposed method are verified by the illustrative figures together with the comparisons between the numerical and exact solutions. Finally, a short conclusion is attached.

2 Laplace Decomposition Algorithm of the Nonlinear Partial Differential Equations

In this section, the Laplace decomposition algorithm will be extended to the nonlinear partial differential equations with the following initial conditions:

$$u_{tt} + au_{xx} + bu + cu^p + du^{2p-1} = 0\,, \tag{2}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)\,, \tag{3}$$

where $f(x)$ and $g(x)$ are analytical functions to be determined later.

In order to construct the numerical solution, it proves convenient to apply the Laplace transform $\mathcal{L}$ to both sides of (2) firstly and then on use of properties of Laplace transform, yields:

$$s^2\mathcal{L}[u] - su(x,0) - u_t(x,0) + a\mathcal{L}[u_{xx}] + b\mathcal{L}[u] + c\mathcal{L}[u^p] + d\mathcal{L}[u^{2p-1}] = 0\,. \tag{4}$$

Insertion of the given initial conditions (3) into it, produces

$$s^2\mathcal{L}[u] = sf(x) + g(x) - a\mathcal{L}[u_{xx}] - b\mathcal{L}[u] - c\mathcal{L}[u^p] - d\mathcal{L}[u^{2p-1}]\,, \tag{5}$$

or

$$\mathcal{L}[u] = \frac{1}{s}f(x) + \frac{1}{s^2}g(x) - \frac{1}{s^2}\{a\mathcal{L}[u_{xx}] + b\mathcal{L}[u]\} - \frac{1}{s^2}\{c\mathcal{L}[u^p] + d\mathcal{L}[u^{2p-1}]\}\,. \tag{6}$$

Secondly, according to LDA,[12–14] the unknown solution of (2)–(3) can be expressed in an infinite series form

$$u = \sum_{n=0}^{\infty} u_n(x,t)\,, \tag{7}$$

where the terms $u_n(x,t)$ may be determined recursively. While, the nonlinear terms u^p and u^{2p-1} are decomposed as follows:

$$u^p = \sum_{n=0}^{\infty} A_n, \quad u^{2p-1} = \sum_{n=0}^{\infty} B_n\,, \tag{8}$$

where A_n and B_n are the so-called Adomian polynomials[18–22,34] defined as

$$A_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big(\sum_{k=0}^{\infty}\lambda^k u_k\Big)^p_{\lambda=0}\,,$$

$$B_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\Big(\sum_{k=0}^{\infty}\lambda^k u_k\Big)^{2p-1}_{\lambda=0}, \quad n = 0,1,2,\ldots \tag{9}$$

In general, these tedious Adomian polynomials can be easily computed with the aid of symbolic computational software *Maple*. For convenience, we list the first few terms of the polynomials A_n:

$$\begin{aligned}
A_0 &= u_0^p\,,\\
A_1 &= pu_1u_0^{p-1}\,,\\
A_2 &= pu_2u_0^{p-1} + C_p^2u_1^2u_0^{p-2}\,,\\
A_3 &= pu_3u_0^{p-1} + 2C_p^2u_2u_1u_0^{p-2} + C_p^3u_1^3u_0^{p-3}\,,\\
A_4 &= pu_4u_0^{p-1} + 2C_p^2\Big(u_3u_1 + \frac{1}{2}u_2^2\Big)u_0^{p-2}\\
&\quad + 3C_p^3u_2u_1^2u_0^{p-3} + C_p^4u_1^4u_0^{p-4}\,,\\
&\cdots
\end{aligned} \tag{10}$$

It is remarked here that the Adomian polynomials A_n only depend on u_i $(i = 0,1,\ldots,n)$ and the summation of the subscripts in each term of A_n is n while the summation of the superscripts is p. In a similar way, one may readily write out the first few terms of B_n so the related details are omitted. More details on the Adomian polynomials can be seen in Refs. [18–22,34].

Substitution of (7)–(8) into (6) and on use of the linearity of Laplace transformation, one can obtain the following recursive relations:

$$\mathcal{L}[u_0] = \frac{1}{s}f(x) + \frac{1}{s^2}g(x)\,, \tag{11a}$$

$$\mathcal{L}[u_{n+1}] = -\frac{a}{s^2}\mathcal{L}[u_{nxx}] - \frac{b}{s^2}\mathcal{L}[u_n] - \frac{c}{s^2}\mathcal{L}[A_n] - \frac{d}{s^2}\mathcal{L}[B_n], \quad n \geq 0\,. \tag{11b}$$

Implementation the inverse Laplace transform to (11a) yields

$$u_0 = f(x) + tg(x)\,. \tag{12}$$

Therefore, the Adomian polynomial A_0 and B_0 can be obtained according to the expression (10). Insertion of the known u_0, A_0 and B_0 into the iteration (11b) with $n = 1$ and then applying the inverse Laplace transform delivers the value of u_1. Repeating the analogous process described above will enable us to obtain the values of u_2, u_3, … iteratively. Therefore, the series solution of (7) will be generated. In general, the series solution obtained can rapidly converge to an exact solution if such a solution exists (otherwise, the series solution can be used for numerical purposes). As for the proof of the convergence, readers may refer to Refs. [35-37].

3 Three Distinct Numerical Solutions of the Nonlinear Partial Differential Equations

In the preceding section, some necessary preparations were made on sought of approximate solutions of (2). It may be easily seen there that the class of solutions is intimately related to initial conditions chosen. In this section, we shall construct three distinct initial values which, remarkably, lead to three different types of numerical solutions. When appropriate p is chosen, the numerical solutions derived can rapidly converge to the exact solutions obtained by other authors.

3.1 *The Doubly Periodic Numerical Solution*

With the construction of doubly periodic solution in mind, we introduce the initial condition in the form of

$$u(x,0)=\big[\sqrt{-2m^2k^2(\lambda^2+a)/d}\,\mathrm{sn}(kx,m)\big]^{2/p},\quad u_t(x,0)=0\,. \tag{13}$$

According to LDA analyzed in the above together with the relation (12), we obtain the first component of the series solution

$$u_0=u(x,0)+tu_t(x,0)=\big[\sqrt{-2m^2k^2(\lambda^2+a)/d}\,\mathrm{sn}(kx,m)\big]^{2/p}. \tag{14}$$

Accordingly, the Adomian polynomials A_0 and B_0 are :

$$A_0=u_0^p=\big[\sqrt{-2m^2k^2(\lambda^2+a)/d}\,\mathrm{sn}(kx,m)\big]^2,$$
$$B_0=u_0^{2p-1}=\big[\sqrt{-2m^2k^2(\lambda^2+a)/d}\,\mathrm{sn}(kx,m)\big]^{(4p-2)/p}. \tag{15}$$

Substitution of (14) and (15) into the iterative relation (11b) with $n=0$, yields

$$\mathcal{L}[u_1]=-\frac{a}{s^2}\mathcal{L}[u_{0xx}]-\frac{b}{s^2}\mathcal{L}[u_0]-\frac{c}{s^2}\mathcal{L}[A_0]-\frac{d}{s^2}\mathcal{L}[B_0]$$
$$=\frac{1}{s^3p^2\mathrm{sn}^2(kx,m)}[a_0+a_1\mathrm{sn}^2(kx,m)+a_2\mathrm{sn}^4(kx,m)]\,, \tag{16}$$

whence, on use of the inverse Laplace transform, we obtain

$$u_1=\frac{t^2}{2p^2\mathrm{sn}^2(kx,m)}[a_0+a_1\mathrm{sn}^2(kx,m)+a_2\mathrm{sn}^4(kx,m)]\,, \tag{17}$$

where

$$A=\big[\sqrt{-2m^2k^2(\lambda^2+a)/d}\,\mathrm{sn}(kx,m)\big]^{2/p},\quad a_0=2ak^2A(p-2)\,,$$
$$a_1=4ak^2A(m^2+1)-bp^2A-cp^2A^p-dp^2A^{2p-1},\quad a_2=-2ak^2m^2A(p+2)\,. \tag{18}$$

By using the relation of (10), one may readily calculate the Adomain polynomials A_1 and B_1 in the form of

$$A_1=\frac{t^2A^{p-1}}{2p\mathrm{sn}^2(kx,m)}[a_0+a_1\mathrm{sn}^2(kx,m)+a_2\mathrm{sn}^4(kx,m)]\,,$$
$$B_1=\frac{(2p-1)t^2A^{2p-2}}{2p^2\,\mathrm{sn}^2(kx,m)}[a_0+a_1\mathrm{sn}^2(kx,m)+a_2\mathrm{sn}^4(kx,m)]\,. \tag{19}$$

Substitution of (17) and (19) into the iterative relation (11b) with $n=1$, produces

$$\mathcal{L}[u_2]=-\frac{a}{s^2}\mathcal{L}[u_{1,xx}]-\frac{b}{s^2}\mathcal{L}[u_1]-\frac{c}{s^2}\mathcal{L}[A_1]-\frac{d}{s^2}\mathcal{L}[B_1]$$
$$=\frac{1}{s^5p^2}\Big[\frac{a_3}{\mathrm{sn}^4(kx,m)}+\frac{a_4}{\mathrm{sn}^2(kx,m)}+a_5+a_6\mathrm{sn}(kx,m)+a_7\mathrm{sn}^2(kx,m)+a_8\mathrm{sn}^4(kx,m)\Big]\,, \tag{20}$$

therefore, we have

$$u_2=\frac{t^4}{4!p^2}\Big[\frac{a_3}{\mathrm{sn}^4(kx,m)}+\frac{a_4}{\mathrm{sn}^2(kx,m)}+a_5+a_6\mathrm{sn}(kx,m)+a_7\mathrm{sn}^2(kx,m)+a_8\mathrm{sn}^4(kx,m)\Big]\,, \tag{21}$$

where

$$a_3=-18k^2aa_0\,,$$
$$a_4=3a_0dA^{2p-2}(1-2p)-12k^2aa_0(1+m^2)-3aa_{0xx}-3a_0b-3a_0cpA^{p-1}\,,$$
$$a_5=3a_1dA^{2p-2}(1-2p)-3a_1b-3a_1cpA^{p-1}-3a(a_{1xx}+2a_2k^2+2a_0m^2k^2)\,,$$
$$a_6=-12kdaa_{2x}\,\mathrm{cn}(kx,m)\mathrm{dn}(kx,m)\,,$$
$$a_7=12k^2aa_2(1+m^2)+3a_2d(1-2p)A^{2p-2}-3a_2b-3aa_{2xx}-3a_2cpA^{p-1}\,,$$

$$a_8 = -18k^2m^2aa_2\,. \tag{22}$$

In a similar manner, we can iteratively calculate the other components $u_3,\ u_4,\ldots$ Therefore, the numerical solution subject to the initial condition of the Jacobian elliptic function is now generated, namely

$$u = u_0 + u_1 + u_2 + \cdots = A + \frac{t^2}{2p^2}\Big[\frac{a_0}{\mathrm{sn}^2(kx,m)} + a_1 + a_2\mathrm{sn}^2(kx,m)\Big] + \frac{t^4}{4!p^2}\Big[\frac{a_3}{\mathrm{sn}^4(kx,m)} + \frac{a_4}{\mathrm{sn}^2(kx,m)} + a_5 + \cdots + a_8\mathrm{sn}^4(kx,m)\Big] + \cdots \tag{23}$$

It is noted that the arbitrariness of p involved in (2) and (23) enables the numerical solution to be a general one. That is to say, a particular approximate solution will occur when we take p a certain constant. For example, if setting $p = 2$ and $c = 0$, then (2) becomes the famous Klein–Gordon equation:

$$u + au_{xx} + bu + du^3 = 0\,, \tag{24}$$

and (23) stands for the approximate solution of this special equation. It is known that in Ref. [38], Liu *et al.* showed when $a < 0$, such Klein–Gordon equation (24) admitted the doubly periodic exact solution:

$$u(x,t) = \sqrt{-\frac{2m^2k^2(\lambda^2+a)}{d}}\ \mathrm{sn}\Big(\sqrt{\frac{b}{(\lambda^2+a)(1+m^2)}}\ (x-\lambda t), m\Big)\,. \tag{25}$$

Interestingly, the numerical solution given in (23) not only holds for the special case of $p = 2$, $c = 0$, $a < 0$ but also for the other cases.

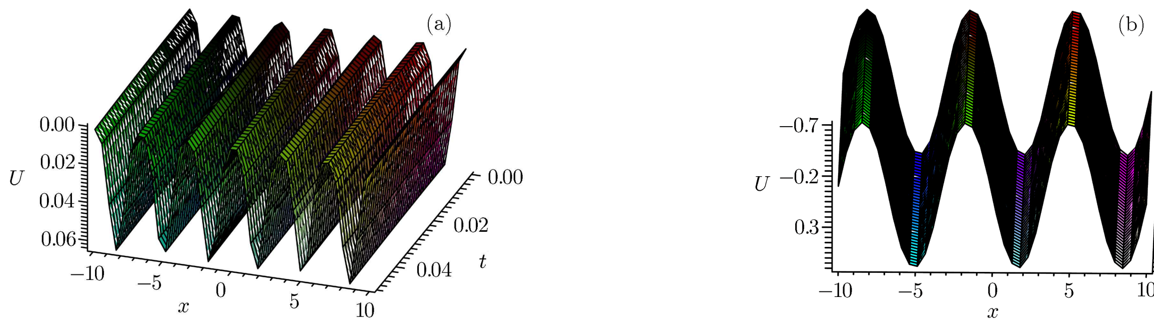

Fig. 1 Numerical figures of the generalized solution (23): (a) is for $p = 1/2$ and (b) for $p = 3$. The other parameters are given by $(a,b,c,d) = (-1, 1/3, 1, -2)$ and $(m,\lambda,k) = (1/2, -\sqrt{2}, 1)$.

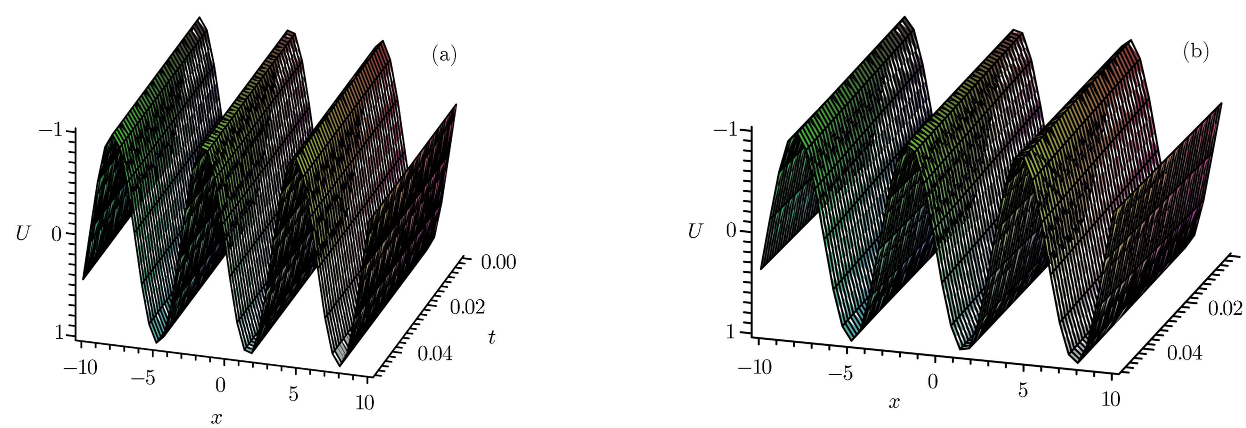

Fig. 2 Figures of the solutions of the Klein–Gordon equation (24), namely $p = 2$: (a) is for approximate solution (23) and (b) for the exact (25). The parameters are chosen as $(a,b,c,d) = (-2, 26/25, 0, -2/25)$, $(k,m,\lambda) = (1, 1/5, -\sqrt{3})$.

Here, we present some numerical simulations to show the efficiency and accuracy of the generalized approximate solutions obtained by the proposed method. It is known that for the LDA, low-order approximate solution can lead to high accuracy if suitable initial conditions are chosen. Without loss of generality, a three-order numerical solution of (23) is adopted here. Figure 1 shows the generalized numerical solutions (23) when $p = 1/2$ and $p = 3$, respectively. Figure 2 depicts the numerical solution (23) and exact solution (25) for the Klein–Gordon equation, namely $p = 2$. The comparison of them at $t = 0.05$ is exhibited in Fig. 3.

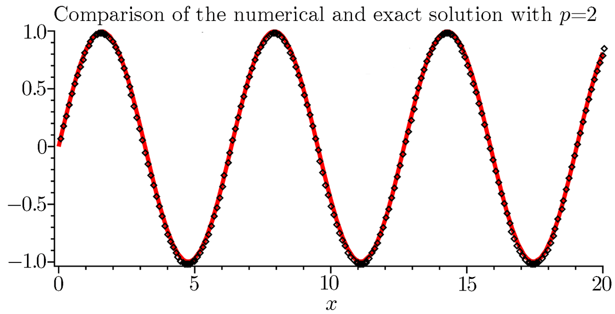

Fig. 3 The comparison of the exact and numerical solution for the Klein–Gordon equation at $t = 0.05$. Line depicts the numerical solution and points the exact.

It is evident from Fig. 3 that for a special value of p, the numerical solution obtained by us can rapidly converge to the known exact solution, which shows that good results are achieved.

3.2 *The Kink-Profile Numerical Solution*

In order to seek the kink-profile solution, we take the initial condition in the form of

$$u(x,0) = [m - k\tanh(\alpha x)]^{1/(p-1)}, \quad u_t(x,0) = 0\,. \tag{26}$$

Therefore, according to the expression (12), we have

$$u_0 = [m - k\tanh(\alpha x)]^{1/(p-1)}, \tag{27}$$

whence the Adomian polynomials A_0 and B_0 are:

$$A_0 = u_0^p = [m - k\tanh(\alpha x)]^{p/(p-1)}, \quad B_0 = u_0^{2p-1} = [m - k\tanh(\alpha x)]^{(2p-1)/(p-1)}. \tag{28}$$

Insertion of (27) and (28) into the iterative relation (11b) with $n = 0$, produces

$$\begin{aligned}\mathcal{L}[u_1] &= -\frac{a}{s^2}\mathcal{L}[u_{0,xx}] - \frac{b}{s^2}\mathcal{L}[u_0] - \frac{c}{s^2}\mathcal{L}[A_0] - \frac{d}{s^2}\mathcal{L}[B_0] \\ &= \frac{A}{s^3(p-1)^2}[a_0 + a_1\tanh(\alpha x) + a_2\tanh(\alpha x)^2 + a_3\tanh(\alpha x)^3 + a_4\tanh(\alpha x)^4]\,,\end{aligned} \tag{29}$$

which, in turn, yields

$$u_1 = \frac{At^2}{2(p-1)^2}[a_0 + a_1\tanh(\alpha x) + a_2\tanh(\alpha x)^2 + a_3\tanh(\alpha x)^3 + a_4\tanh(\alpha x)^4]\,. \tag{30}$$

In the above, the parameters are given by

$$\begin{aligned}&A = [m - k\tanh(\alpha x)]^{1/(p-1)}, \\ &a_0 = ak^2\alpha^2(p-2)A^{2-2p} - (p-1)^2(b + cA^{p-1} + dA^{2p-2})\,, \\ &a_1 = 2ak\alpha^2(1-p)A^{1-p}, \quad a_2 = 2ak^2\alpha^2(2-p)A^{2-2p}\,, \\ &a_3 = 2ak\alpha^2(p-1)A^{1-p}, \quad a_4 = ak^2\alpha^2(p-2)A^{2-2p}\,.\end{aligned} \tag{31}$$

Substitution of (30) into the iterative representation (11b) with $n = 1$, we obtain

$$\begin{aligned}\mathcal{L}[u_2] &= -\frac{a}{s^2}\mathcal{L}[u_{1,xx}] - \frac{b}{s^2}\mathcal{L}[u_1] - \frac{c}{s^2}\mathcal{L}[A_1] - \frac{d}{s^2}\mathcal{L}[B_1] \\ &= \frac{1}{s^5(p-1)^2}[a_5 + a_6\tanh(\alpha x) + a_7\tanh(\alpha x)^2 + \cdots + a_{11}\tanh(\alpha x)^6]\,,\end{aligned} \tag{32}$$

whence

$$u_2 = \frac{t^4}{4!(p-1)^2}[a_5 + a_6\tanh(\alpha x) + a_7\tanh(\alpha x)^2 + \cdots + a_{11}\tanh(\alpha x)^6]\,, \tag{33}$$

with

$$\begin{aligned}a_5 &= a_0d(1-2p)A^{2p-1} - aA(a_{0xx} + 2\alpha a_{1x} + 2\alpha^2 a_2) - aa_0A_{xx}\,, \\ a_6 &= a_1d(1-2p)A^{2p-1} - aA(a_{1xx} + 4\alpha a_{2x} - 2\alpha^2 a_1 + 6\alpha^2 a_3) \\ &\quad - 2aA_x(a_{1x} + 2\alpha a_2) - a_1cpA^p - a_1bA - aa_1A_{xx}\,, \\ a_7 &= a_2d(1-2p)A^{2p-1} - aA(a_{2xx} - 2\alpha a_{1x} - 8\alpha^2 a_2 + 6\alpha a_{3x} + 12\alpha^2 a_4) \\ &\quad - 2aA_x(a_{2x} - \alpha a_1 + 3\alpha a_3) - a_2cpA^p - a_2bA - aa_2A_{xx}\,, \\ a_8 &= a_3d(1-2p)A^{2p-1} - aA(a_{3xx} + 2\alpha^2 a_1 - 4\alpha a_{2x} - 18\alpha^2 a_3 + 8\alpha a_{4x}) \\ &\quad - 2aA_x(a_{3x} - 2\alpha a_2 + 4\alpha a_4) - a_3cpA^p - a_3bA - aa_3A_{xx}\,, \\ a_9 &= a_4d(1-2p)A^{2p-1} - aA(a_{4xx} + 6\alpha^2 a_2 - 6\alpha a_{3x} - 32\alpha^2 a_4) \\ &\quad - 2aA_x(a_{4x} - 3\alpha a_3) - a_4cpA^p - a_4bA - aa_4A_{xx}\,,\end{aligned}$$

$$a_{10} = 4a\alpha A(2a_{4x} - 3\alpha a_3), \quad a_{11} = -2aa_4\alpha^2 A\,. \tag{34}$$

Likewise, the other components of u_3, u_4, u_5, ... can be obtained iteratively. Therefore, the numerical solution associated with the initial condition (26) is derived:

$$u = u_0 + u_1 + u_2 + \cdots = A + \frac{At^2}{2(p-1)^2}[a_0 + a_1\tanh(\alpha x) + \cdots + a_4\tanh(\alpha x)^4] + \frac{t^4}{4!(p-1)^2}[a_5 + a_6\tanh(\alpha x) + a_7\tanh(\alpha x)^2 + \cdots + a_{11}\tanh(\alpha x)^6] + \cdots \tag{35}$$

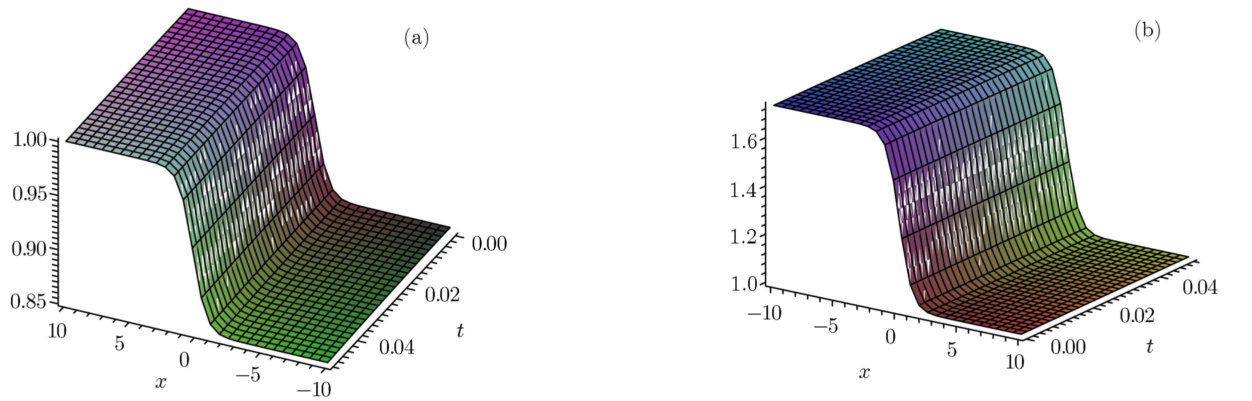

Fig. 4 Figures of the generalized numerical kink-profile solution (35): (a) is for $p = -6$ and (b) for $p = 2$.

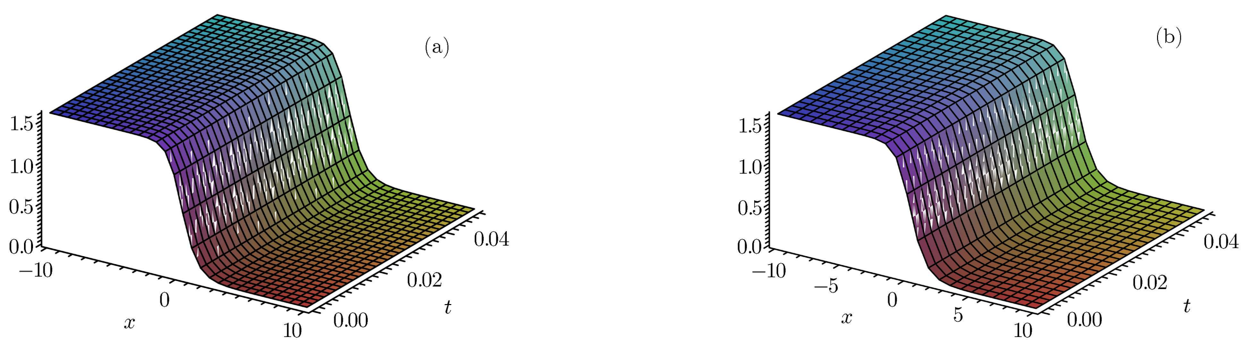

Fig. 5 Figures of the numerical and exact solution when $p = 5/2$: (a) depicts the approximate solution (35) and (b) shows the exact solution (36) derived by Chen *et al.* The parameters are taken as $(a, b, c, d) = (-29/80, 1/5, 7/20, 1/8)$ and $(k, m, \alpha, \lambda, R) = (1, 1, 1, 1, -1)$.

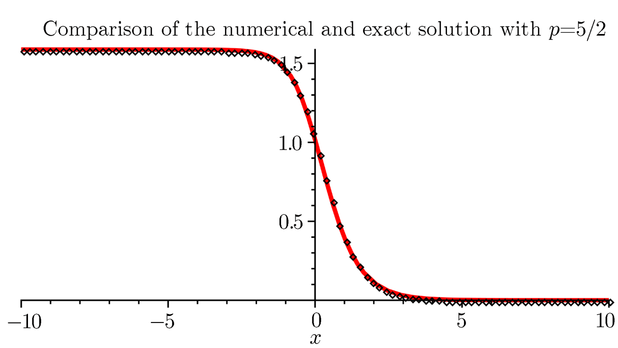

Fig. 6 The comparison of numerical and exact solution described in Fig. 5 with $p = 5/2$ and $t = 0.05$. Line stands for numerical solution and points the exact.

The arbitrariness of p allows one to take $p = 5/2$, in which case the original system (1) has the exact solution

$$u = \Big[-\frac{5c}{14d} \mp \sqrt{\frac{5b}{8d}}\tanh\Big(\sqrt{-R}\Big(x \mp t\sqrt{-a + \frac{9b}{16R}} + \xi_0\Big)\Big)\Big]^{2/3}, \tag{36}$$

that was derived by Chen *et al.* via the improved method.[30]

Below, we perform some numerical simulations to show the efficiency of this type approximate solution. Figure 4 exhibits the generalized numerical solutions (35) when $p = -6$ and $p = 2$, respectively. When $p = 5/2$, the corresponding numerical solution (35) and exact solution (36) derived by Chen *et al.* are illustrated in Fig. 5. While, the comparison of them at $t = 0.05$ are presented in Fig. 6. The comparison result enables us to believe that another hight accuracy solution is achieved.

3.3 *The Bell-Profile Numerical Solution*

For construction of the bell-profile numerical solution, we shall proceed with the following type of initial values

$$u(x,0) = \operatorname{sech}(kx)^{1/(p-1)}, \quad u_t(x,0) = 0\,. \tag{37}$$

As the steps are analogous to the above, we omit the tedious calculations and just give the final result, namely

$$\begin{aligned} u = u_0 + u_1 + u_2 + \cdots = A + \frac{t^2}{2(p-1)^2}[a_0 A + a_1 A^p + a_2 A^{2p-1}] \\ + \frac{t^4}{4!(p-1)^4}[a_3 A + a_4 A^p + a_5 A^{2p-1} + a_6 A^{3p-2} + a_7 A^{4p-3}] + \cdots\,, \end{aligned} \tag{38}$$

with

$$\begin{aligned} &A = \operatorname{sech}(kx)^{1/(p-1)}, \quad a_0 = -ak^2 - b(p-1)^2, \quad a_1 = -c(p-1)^2, \quad a_2 = apk^2 - d(p-1)^2\,, \\ &a_3 = 2aa_{0x}k(1-p)\tanh(kx) - (p-1)^2(aa_{0xx} + a_0 b) - aa_0 k^2, \\ &a_4 = 2aa_{1x}k(p-1)\tanh(kx) - (p-1)^2(aa_{1xx} + a_1 b + a_0 cp) - aa_1 p^2 k^2, \\ &a_5 = aa_0 pk^2 - aa_2 k^2(p-2)^2 + 2kaa_{2x}(p-1)(2p-1)\tanh(kx) \\ &\qquad - (p-1)^2(aa_{2xx} + a_2 b + a_1 cp) - a_0 d(2p-1)(p-1)^2, \\ &a_6 = aa_1 pk^2(2p-1) - a_3 cp(p-1)^2 - a_1 d(2p-1)(p-1)^2, \\ &a_7 = aa_2 k^2(2p-1)(3p-2) - a_2 d(2p-1)(p-1)^2. \end{aligned} \tag{39}$$

It is known that when $c = 0$, Chen *et al.* obtained the bell-profile exact solution of the original equation (1):

$$u = \Big[\pm\sqrt{\frac{bp}{d}}\operatorname{sech}\Big(-\sqrt{R}\Big(x \mp t\sqrt{\frac{b(p-1)^2 - aR}{R}} + \xi_0\Big)\Big)\Big]^{1/(p-1)}\,, \tag{40}$$

which can be seen in the case 4 of Ref. [30]. However, it is remarkable that the bell-profile numerical solution (38) derived here not only holds for $c = 0$ but also for $c \neq 0$. Therefore, we conclude that this type of approximate solutions is also a generalized one.

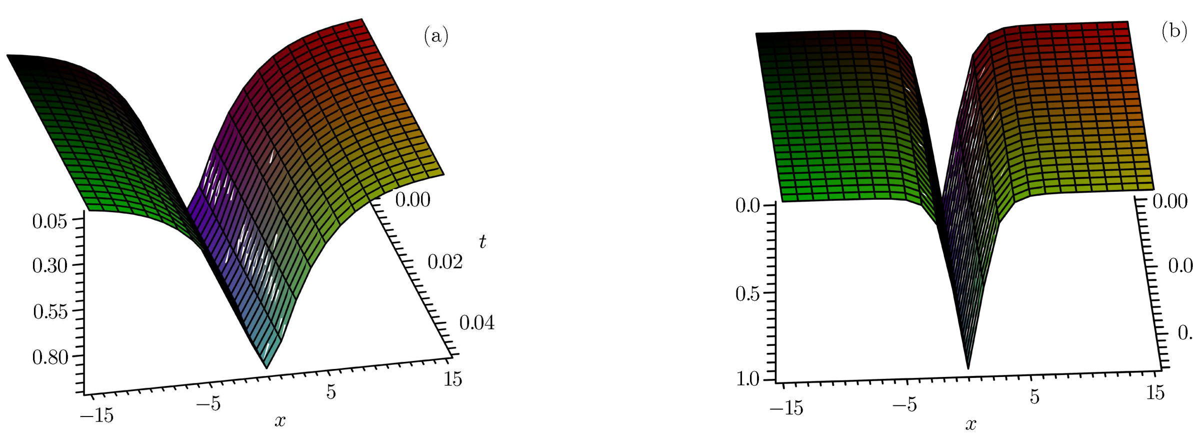

Fig. 7 The figures of the generalized numerical solution (38): (a) is for $p = 5$ and (b) for $p = 1$.

In Fig. 7, we show the time evolutions of the generalized numerical solutions (38) with $p = 5$ and $p = 1$, respectively. It is seen that such solution is indeed of bell-profile type for the arbitrary p. In Fig. 8, we exhibit the particular numerical solution (38) and exact solution (40) with $p = 3/2$. The comparison of these two kinds of solutions described in Fig. 8 at $t = 0.05$ is depicted in Fig. 9.

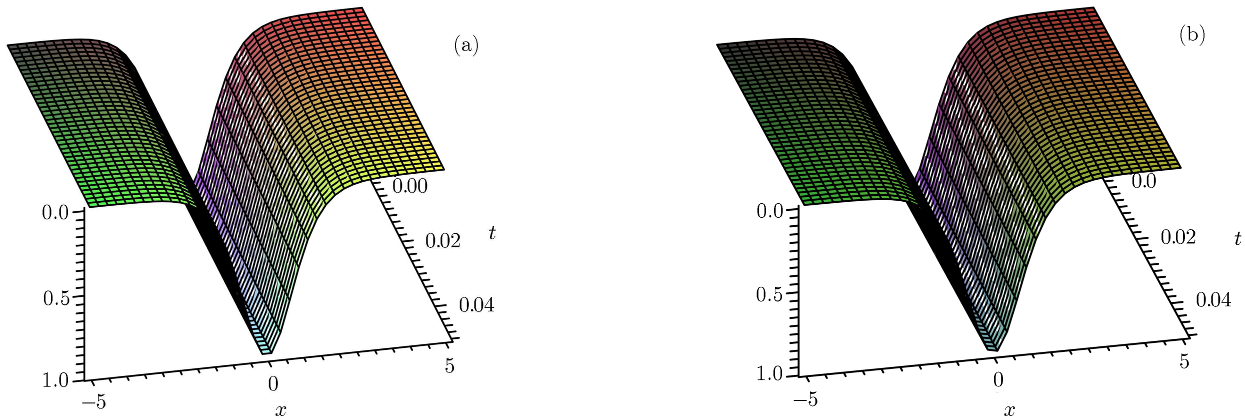

Fig. 8 Figures of the numerical solution (38) and the exact solution (40) with $p = 3/2$. The parameters are given by $(a, b, c, d) = (-3/4, 2, 0, 3)$ and $(k, R) = (1, -1)$.

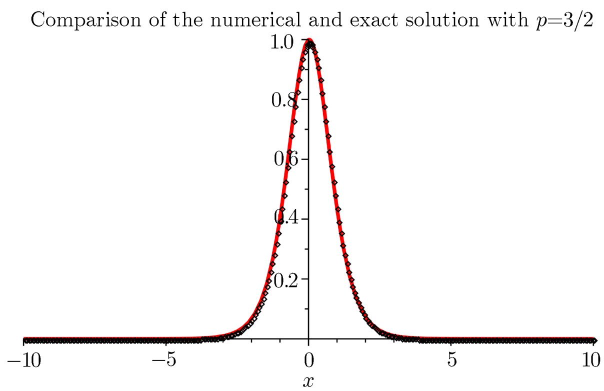

Fig. 9 The comparison of numerical and exact solution described in Fig. 8 at $t = 0.05$. Line is for the numerical solution and points the exact.

Remark Here, we want to point out that except the three types of approximate solutions given above, other classes of numerical solutions such as soliton, rational, antikink-profile solution etc. can also be constructed if suitable forms of initial conditions are chosen.

4 Conclusion

A class of partial differential equations with nonlinear term of any order has been revisited by the LDA. It is very interesting and surprising that only by constructing special forms of initial values and using the LDA, three distinct generalized numerical solutions are obtained successfully, namely the doubly periodic numerical solution, kink-profile and bell-profile solution. Numerical results show that such approximate solutions are very realistic series solutions, which generally converge rapidly. In particular, when p is chosen by the same values as that given in [30] and [38], the numerical solutions obtained by us can converge to the exact ones derived by Liu and Chen *et al.* Therefore, we predict that the LDA is an effective technique to investigate numerical solutions of nonlinear problems, especially for the partial differential equations with nonlinear term of any order. However, to our knowledge, it is a very difficult problem to investigate multi-soliton solutions of integrable nonlinear equations by the known numerical methods and few papers are reported. Whether the LDA or other new algorithms can be effectively used to solve multi-soliton solution of nonlinear partial differential equations? It is worthy of deep study in our future work.

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