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Constructing two-dimensional optimal system of the group invariant solutions

Xiaorui Hu,[1,a)] Yuqi Li,[2] and Yong Chen[2,b)]
[1]*Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou 310023, People's Republic of China*
[2]*Shanghai Key Laboratory of Trustworthy Computing, East China Normal University, Shanghai 200062, People's Republic of China*

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To search for inequivalent group invariant solutions of two-dimensional optimal system, a direct and systematic approach is established, which is based on commutator relations, adjoint matrix, and the invariants. The details of computing all the invariants for two-dimensional algebra are presented, which is shown more complex than that of one-dimensional algebra. The optimality of two-dimensional optimal systems is shown clearly for each step of the algorithm, with no further proof. To leave the algorithm clear, each stage is illustrated with a couple of examples: the heat equation and the Novikov equation. Finally, two-dimensional optimal system of the (2+1)-dimensional Navier-Stokes (NS) equation is found and used to generate intrinsically different reduced ordinary differential equations. Some interesting explicit solutions of the NS equation are provided. © *2016 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4941990]

I. INTRODUCTION

The study of group-invariant solutions of differential equations plays an important role in mathematics and physics. The machinery of Lie group theory provides a systematic method to search for these special group invariant solutions. For a system of differential equations with n variables, any of its m-dimensional ($m < n$) symmetry subgroup can transform it into a system of differential equations with $n - m$ variables, which is generally easier to solve than the original system. By solving these reduced equations, rich group-invariant solutions are found. For two group-invariant solutions, one may connect them with some group transformation and in this case, one calls them equivalent. Naturally, it is a significant job to find these inequivalent branches of group-invariant solutions, which leads to the concept of the optimal systems. For the classification of group-invariant solutions, it is more convenient to work in the space of Lie algebra and this problem reduces to the problem of finding an optimal system of subalgebras under the adjoint representation.

The adjoint representation of a Lie group on its Lie algebra was known to Lie. The construction of one-dimensional optimal systems of Lie algebra was demonstrated by Ovsiannikov,[1] using a global matrix for the adjoint transformation. This is also the technique used by Galas[2] and Ibragimov.[3] Then Olver[4] used a slightly different and elegant technique for one-dimensional optimal system, which is based on commutator table and adjoint table, and presented detailed instructions on the KdV equation and the heat equation. For two-dimensional optimal systems, Ovsiannikov sketched the construction by showing a simple example. Galas refined Ovsiannikov's method by removing equivalent subalgebras for the solvable algebra, and he also discussed the problem of a nonsolvable algebra, which is generally harder. In Ref. 9, the details for constructing two-dimensional optimal systems were shown for the three-dimensional, one-temperature hydrodynamic equations. In a fundamental series of papers, Patera *et al.*[5–8] developed a different and

a)Electronic mail: lansexiaoer@163.com
b)Electronic mail: ychen@sei.ecnu.edu.cn

powerful method to classify subalgebras and many optimal systems of important Lie algebras arising in mathematical physics are obtained.

In this paper, we devote ourselves to investigating two-dimensional optimal system of invariant solutions, which simultaneously imply two kinds of invariance of the differential systems. Using two-dimensional optimal system of a Lie algebra, systems of differential equations with n variables would be reduced to some inequivalent systems with $n-2$ variables. Especially, for a given system of differential equations with three variables, one-dimensional optimal system can only reduce it to several systems of differential equations with two variables while two-dimensional optimal system would directly lead it to ordinary differential equations which are more easily solved in principle than those partial differential equations. Hence it is a meaningful job to consider two-dimensional optimal systems independently. In almost all of the existing literatures, one-dimensional optimal system is required for the calculation of two-dimensional optimal system, which takes too much work, and one needs master rich algebraic knowledge before setting about the operation. The purpose of this paper is to introduce a direct and systematic method for constructing two-dimensional optimal system, which starts from the Lie algebra itself and only depends on fragments of the theory of Lie algebras, without the prior one-dimensional optimal system.

For the case of one-dimensional optimal system, Olver pointed out that the Killing form of the Lie algebra as an "invariant" for the adjoint representation is very important since it places restrictions on how far one can expect to simplify the Lie algebra. Chou *et al.*[10–12] introduced more numerical invariants (which are different from common invariants such as the Casimir operator, harmonics, and rational invariants) to demonstrate the inequivalence among different elements of the optimal system. However, to the best of our knowledge, in spite of the importance of the common invariants for the Lie algebra, there are few literatures to use more invariants except the Killing form in the process of constructing optimal systems. For this, in our early paper,[13] we introduced a direct and valid method to compute all the general invariants for a given one-dimensional Lie algebra and then made the best of them to construct one-dimensional optimal system. On the basis of all the invariants, the new method can both guarantee the comprehensiveness and the inequivalence of the one-dimensional optimal system. Here, we develop the ideas to two-dimensional optimal systems of invariant solutions.

The layout of this paper is as follows. Section II provides a theoretical background on Lie algebras and all machinery needed to develop the algorithm. In Section III, a direct and systematic algorithm is proposed for constructing two-dimensional optimal system of a general symmetry algebra. Since the realization of our new algorithm builds on different invariants, a valid method for computing the invariants of two-dimensional subalgebras is also given in this section. To leave our algorithm clear, we would illustrate each stage with two examples, i.e., the heat equation and the Novikov equation. In Section IV, the two-dimensional optimal system of (2+1)-dimensional Navier-Stokes equation is presented and all the corresponding reduced ordinary differential equations with some interesting exact group invariant solutions are obtained. Finally, a brief conclusion is given in Section V.

II. THEORETICAL BACKGROUND ON LIE ALGEBRA

Consider an n-dimensional Lie algebra $\mathcal{G}$ of a differential system with p independent variables $\{x_1, x_2, \ldots, x_p\}$ and q dependent variables $\{u_1, u_2, \ldots, u_q\}$, which is generated by n vector fields $\{v_1, v_2, \ldots, v_n\}$. The corresponding n-parameter symmetry group of $\mathcal{G}$ is denoted as G, which is the collections of transformations

$$(\tilde{x}_1, \ldots, \tilde{x}_p, \tilde{u}_1, \ldots, \tilde{u}_q) = \exp\left(\sum_{i=1}^{n} a_i v_i\right)(x_1, \ldots, x_p, u_1, \ldots, u_q) \tag{1}$$

for all allowed values of the group parameters. The Lie bracket $[v_i, v_j] = v_i v_j - v_j v_i$ is the commutator of two of the differential operators. The complete information of the group structure is contained by

$$[v_i, v_j] = \sum_{k=1}^{n} C_{ij}^k v_k, \tag{2}$$

where the C_{ij}^k's are called structure constants.

The group invariant solutions are large classes of special explicit solutions which are characterized by their invariance under some symmetry group of the system of partial differentials equations. Let $H \subset G$ be an s-parameter subgroup. An H-invariant solution can be transformed into another one by the elements $g \in G$ not belonging the subgroup H. That is to say, two group invariant solutions are essentially different if it is impossible to connect them with any group transformation in (1). In fact, if ψ is an H-invariant solution, $\tilde{\psi} = g \cdot \psi$ is a $\tilde{H}$-invariant solution with $\tilde{H} = gHg^{-1} = \{ghg^{-1}, g \in G, h \in H\}$. This group $\tilde{H}$ is called the conjugate subgroup to H under G.

For each $g \in G$, group conjugation $K_g(h) \equiv ghg^{-1}$, $h \in G$, determines a global group action of G on itself. Then the corresponding differential dK_g determines a linear map on the Lie algebra $\mathcal{G}$ of G, called the adjoint representation,

$$Ad_g(w) \equiv dK_g(w) \tag{3}$$

for w being any vector field form $\mathcal{G}$. Furthermore, if w generates the one-parameter subgroup $H = \{\exp(\epsilon w) : \epsilon \in \mathcal{R}\}$, then $Ad_g(w)$ generates the conjugate one-parameter subgroup $Ad_g H = gHg^{-1}$. Let the group element g be generated by the vector field v, seen $g = \exp(\epsilon v)$. More simply, adjoint representation (3) can be expressed through commutators as

$$Ad_{\exp(\epsilon v)}(w) = w - \epsilon[v, w] + \frac{1}{2!}\epsilon^2[v, [v, w]] - \frac{1}{3!}\epsilon^3[v, [v, [v, w]]] + \cdots. \tag{4}$$

The infinitesimal adjoint action of (4) is

$$\mathrm{ad}_v(w) \equiv \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(Ad_{\exp(\epsilon v)}(w)\right)\bigg|_{\epsilon=0} = [w, v]. \tag{5}$$

The Killing form is a bilinear form defined on Lie algebra $\mathcal{G}$ by

$$K(v, w) = \mathrm{trace}(\mathrm{ad}_v \cdot \mathrm{ad}_w). \tag{6}$$

A real function ϕ defined on a Lie algebra $\mathcal{G}$ is called an invariant if $\phi(Ad_g(w)) = \phi(w)$ for all w in $\mathcal{G}$ and g in the Lie group G. By the definition of the Killing form, we have

$$K(Ad_g(w), Ad_g(w)) = K(w, w) \tag{7}$$

for all $w \in \mathcal{G}$ and $g \in G$. Therefore, the function

$$f(w) = K(w, w) \tag{8}$$

is invariant under the adjoint action. It was shown in our previous paper[13] that the invariants play a very important role in the construction of one-dimensional optimal system.

For a given Lie algebra $\mathcal{G}$, a family of r-dimensional subalgebras $\{\mathfrak{g}_\alpha\}_{\alpha \in \mathcal{A}}$ forms an r-parameter optimal system named as O_r if (1) any r-dimensional subalgebra is equivalent to some $\mathfrak{g}_\alpha$ and (2) $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are inequivalent for distinct α and β. Each member $\mathfrak{g}_\alpha \in O_r$ is a collection of r linear combinations of generators. In this paper, we focus on constructing two-dimensional optimal system O_2.

Let $\mathfrak{G}(v, w) \equiv \mathfrak{G}(\sum_{i=1}^{n} a_i v_i, \sum_{i=1}^{n} b_i v_i)$ be a general two-dimensional algebra, which remains closed under commutation. In $\mathfrak{G}(v, w)$, two subalgebras $\{w_1, w_2\}$ and $\{w_1', w_2'\}$ are called equivalent if one can find some transformation $g \in G$ and some constants $\{k_1, k_2, k_3, k_4\}$ so that

$$w_1' = k_1 Ad_g(w_1) + k_2 Ad_g(w_2), \quad w_2' = k_3 Ad_g(w_1) + k_4 Ad_g(w_2). \tag{9}$$

Since w_1' and w_2' are linearly independent, it requires $k_1 k_4 - k_2 k_3 \neq 0$ in (9) or else $w_1' = c w_2'$.

On the one hand, for the above equivalent two-dimensional subalgebras $\{w_1, w_2\}$ and $\{w_1', w_2'\}$, there is

$$\begin{aligned}[w_1', w_2'] &= [k_1 Ad_g(w_1) + k_2 Ad_g(w_2), k_3 Ad_g(w_1) + k_4 Ad_g(w_2)] \\ &= (k_1k_4 - k_2k_3)[Ad_g(w_1), Ad_g(w_2)] \\ &= (k_1k_4 - k_2k_3)[g(w_1)g^{-1}, g(w_2)g^{-1}] \\ &= (k_1k_4 - k_2k_3)g([w_1, w_2])g^{-1}.\end{aligned} \tag{10}$$

It is clear that $[w_1', w_2'] = 0$ if and only if $[w_1, w_2] = 0$; $[w_1', w_2'] \neq 0$ if and only if $[w_1, w_2] \neq 0$.

On the other hand, for any given two-dimensional subalgebra $\{w_1, w_2\}$ with $[w_1, w_2] = \lambda w_1 + \mu w_2$, one can easily find an equivalent one $\{\hat{w}_1, \hat{w}_2\}$ so that $[\hat{w}_1, \hat{w}_2] = 0$ or $[\hat{w}_1, \hat{w}_2] = \hat{w}_1$. Hence, to find all the inequivalent elements in the optimal system O_2, without loss of generality, we require each member $\{v, w\} \in O_2$ satisfy $[v, w] = 0$ or $[v, w] = v$. For the latter case, we give out the following remark.

Remark 1. If two subalgebras $\mathfrak{g}_\alpha = \{w_1, w_2\}$ and $\mathfrak{g}_{\alpha'} = \{w_1', w_2'\}$, with $[w_1, w_2] = w_1$ and $[w_1', w_2'] = w_1'$ are equivalent in the form of (9), there must be $k_2 = 0$ and $k_4 = 1$.

Proof. If we make $[w_1, w_2] = w_1$ and $[w_1', w_2'] = w_1'$, Eq. (10) become

$$[w_1', w_2'] = (k_1k_4 - k_2k_3)g(w_1)g^{-1} = (k_1k_4 - k_2k_3)Ad_g(w_1) = w_1' = k_1 Ad_g(w_1) + k_2 Ad_g(w_2). \tag{11}$$

For the independence of $Ad_g(w_1)$ and $Ad_g(w_2)$, there must be $k_2 = 0$ and $k_4 = 1$.

III. A GENERAL ALGORITHM FOR CONSTRUCTING TWO-DIMENSIONAL OPTIMAL SYSTEM

In this section, we will demonstrate how to construct the adjoint transformation matrix and invariants on the refined two-dimensional algebra and apply them to present an algorithm for two-dimensional optimal system stage by stage. Each step is illustrated by two examples, the heat and Novikov equations.

A. Construction of the refined two-dimensional algebra

To find out all the inequivalent elements in the two-dimensional optimal system O_2, which represent the respective equivalent classes, we first require each $\{v, w\} \in O_2$ satisfy

$$[v, w] = \delta v, \quad \text{where} \quad \delta \equiv 0, 1. \tag{12}$$

Let

$$v = \sum_{i=1}^{n} a_i v_i, \quad w = \sum_{i=1}^{n} b_i v_i. \tag{13}$$

Requirement (12) will provide a set of restrictive equations for a_i and b_i.

1. Refined two-dimensional algebra of the heat equation

The equation for the conduction of heat in a one-dimensional road is written as

$$u_t = u_{xx}. \tag{14}$$

The Lie algebra of infinitesimal symmetries for this equation is spanned by six vector fields

$$\begin{aligned} &v_1 = \partial_x, \quad v_2 = \partial_t, \quad v_3 = u\partial_u, \quad v_4 = x\partial_x + 2t\partial_t, \\ &v_5 = 2t\partial_x - xu\partial_u, \quad v_6 = 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u, \end{aligned} \tag{15}$$

and by the infinitesimal generator of an infinity dimensional subalgebra

$$v_h = h(x, t)\partial_u,$$

TABLE I. Commutator table of the heat equation.

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	0	0	v_1	$-v_3$	$2v_5$
v_2	0	0	0	$2v_2$	$2v_1$	$4v_4-2v_3$
v_3	0	0	0	0	0	0
v_4	$-v_1$	$-2v_2$	0	0	v_5	$2v_6$
v_5	v_3	$-2v_1$	0	$-v_5$	0	0
v_6	$-2v_5$	$2v_3-4v_4$	0	$-2v_6$	0	0

where $h(x,t)$ is an arbitrary solution of the heat equation. Since the infinite-dimensional subalgebra $\langle v_h \rangle$ does not lead to group invariant solutions, it will not be considered in the classification problem.

The commutator table and actions of the adjoint representation, which are taken from Ref. 13, are given in Tables I and II, respectively.

For six-dimensional Lie algebra (15), take

$$w_1 = \sum_{i=1}^{6} a_i v_i, \quad w_2 = \sum_{j=1}^{6} b_j v_j. \tag{16}$$

With the help of Table I, substituting (16) into (12) leads six restrictive equations:

$$\begin{aligned} &a_1 b_4 + 2a_2 b_5 - a_4 b_1 - 2a_5 b_2 = \delta a_1, \quad 2a_2 b_4 - 2a_4 b_2 = \delta a_2, \\ &- a_1 b_5 - 2a_2 b_6 + a_5 b_1 + 2a_6 b_2 = \delta a_3, \quad 2a_4 b_6 - 2a_6 b_4 = \delta a_6, \\ &4a_2 b_6 - 4a_6 b_2 = \delta a_4, \quad 2a_1 b_6 + a_4 b_5 - a_5 b_4 - 2a_6 b_1 = \delta a_5. \end{aligned} \tag{17}$$

Later on, two inequivalent cases of Eqs. (17) with $\delta = 0$ and $\delta = 1$ should be considered, respectively.

2. *Refined two-dimensional algebra of the Novikov equation*

The Novikov equation reads

$$u_t - u_{txx} + 4u^2 u_x - 3u u_x u_{xx} - u^2 u_{xxx} = 0, \tag{18}$$

which was discovered by Novikov in a recent communication[14] and can be considered as a type of generalization of the known Camassa-Holm equation. In Ref. 15, the authors gave out a five-dimensional Lie algebra of Eq. (18), which was spanned by the following basis:

$$\begin{aligned} &v_1 = \partial_t, \qquad v_2 = \partial_x, \qquad v_3 = e^{2x}\partial_x + e^{2x} u \partial_u, \\ &v_4 = e^{-2x}\partial_x - e^{-2x} u \partial_u, \qquad v_5 = -2t\partial_t + u\partial_u. \end{aligned} \tag{19}$$

In our early paper,[13] one-dimensional optimal system of this five-dimensional Lie algebra (19) was constructed and used to find rich group invariant solutions of the Novikov equation. The corresponding commutator and adjoint representation relations are shown by Tables III and IV.

TABLE II. Adjoint representation table of the heat equation.

Ad	v_1	v_2	v_3	v_4	v_5	v_6
v_1	v_1	v_2	v_3	$v_4-\epsilon v_1$	$v_5+\epsilon v_3$	$v_6-2\epsilon v_5-\epsilon^2 v_3$
v_2	v_1	v_2	v_3	$v_4-2\epsilon v_2$	$v_5-2\epsilon v_1$	$v_6-4\epsilon v_4+2\epsilon v_3+4\epsilon^2 v_2$
v_3	v_1	v_2	v_3	v_4	v_5	v_6
v_4	$e^{\epsilon} v_1$	$e^{2\epsilon} v_2$	v_3	v_4	$e^{-\epsilon} v_5$	$e^{-2\epsilon} v_6$
v_5	$v_1-\epsilon v_3$	$v_2+2\epsilon v_1-\epsilon^2 v_3$	v_3	$v_4+\epsilon v_5$	v_5	v_6
v_6	$v_1+2\epsilon v_5$	$v_2-2\epsilon v_3+4\epsilon v_4+4\epsilon^2 v_6$	v_3	$v_4+2\epsilon v_6$	v_5	v_6

TABLE III. Commutator table of the Novikov equation.

	v_1	v_2	v_3	v_4	v_5
v_1	0	0	0	0	$-2v_1$
v_2	0	0	$2v_3$	$-2v_4$	0
v_3	0	$-2v_3$	0	$-4v_2$	0
v_4	0	$2v_4$	$4v_2$	0	0
v_5	$2v_1$	0	0	0	0

In terms of the refined algebra for $[w_1, w_2] = \delta w_1$, we have the restrictions,

$$\begin{aligned}&\delta a_5 = 0, \quad 2(a_5 b_1 - a_1 b_5) = \delta a_1, \quad 2(a_2 b_3 - a_3 b_2) = \delta a_3,\\ &2(a_4 b_2 - a_2 b_4) = \delta a_4, \quad 4(a_4 b_3 - a_3 b_4) = \delta a_2.\end{aligned} \tag{20}$$

B. Calculation of the adjoint transformation matrix

For $w_1 = \sum_{i=1}^{n} a_i v_i$, its general adjoint transformation matrix A is the product of the matrices of the separate adjoint actions $A_1, A_2, \ldots, A_n$, each corresponding to $Ad_{\exp(\epsilon v_i)}(w_1), i = 1 \cdots n$.

For example, applying the adjoint action of v_1 to $w_1 = \sum_{i=1}^{n} a_i v_i$ and with the help of adjoint representation table, one has

$$\begin{aligned}&Ad_{\exp(\epsilon_1 v_1)}(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n)\\ &= a_1 Ad_{\exp(\epsilon_1 v_1)} v_1 + a_2 Ad_{\exp(\epsilon_1 v_1)} v_2 + \cdots + a_n Ad_{\exp(\epsilon_1 v_1)} v_n\\ &= R_1 v_1 + R_2 v_2 + \cdots + R_n v_n,\end{aligned} \tag{21}$$

with $R_i \equiv R_i(a_1, a_2, \ldots, a_n, \epsilon_1), i = 1 \ldots n$. To be intuitive, formula (21) can be rewritten into the following matrix form:

$$v \doteq (a_1, a_2, \ldots, a_n) \xrightarrow{Ad_{\exp(\epsilon_1 v_1)}} (R_1, R_2, \ldots, R_n) = (a_1, a_2, \ldots, a_n) A_1.$$

Similarly, the matrices $A_2, A_3, \ldots, A_n$ of the separate adjoint actions of $v_2, v_3, \ldots, v_n$ can be constructed, respectively. Then the general adjoint transformation matrix A is the product of $A_1, \ldots, A_n$ taken in any order

$$A \equiv A(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) = A_1 A_2 \cdots A_n. \tag{22}$$

Since only the existence of the element of the group is needed in our algorithm, the orders of the product shown in (22) can be arbitrary. Applying the most general adjoint action $Ad_g = Ad_{\exp(\epsilon_n v_n)} \cdots Ad_{\exp(\epsilon_2 v_2)} Ad_{\exp(\epsilon_1 v_1)}$ to w_1 and w_2, we have

$$\begin{aligned}&w_1 \doteq (a_1, a_2, \ldots, a_n) \xrightarrow{Ad} Ad_g(w_1) \doteq (a_1, a_2, \ldots, a_n) A,\\ &w_2 \doteq (b_1, b_2, \ldots, b_n) \xrightarrow{Ad} Ad_g(w_2) \doteq (b_1, b_2, \ldots, b_n) A.\end{aligned} \tag{23}$$

TABLE IV. Adjoint representation table of the Novikov equation.

Ad	v_1	v_2	v_3	v_4	v_5
v_1	v_1	v_2	v_3	v_4	$v_5+2\epsilon v_1$
v_2	v_1	v_2	$e^{-2\epsilon} v_3$	$e^{2\epsilon} v_4$	v_5
v_3	v_1	$v_2+2\epsilon v_3$	v_3	$v_4+4\epsilon v_2+4\epsilon^2 v_3$	v_5
v_4	v_1	$v_2-2\epsilon v_4$	$v_3-4\epsilon v_2+4\epsilon^2 v_4$	v_4	v_5
v_5	$e^{-2\epsilon} v_1$	v_2	v_3	v_4	v_5

Hence, the equivalence between $\{w_1', w_2'\}$ and $\{w_1, w_2\}$ shown in (9) can be rewritten as

$$\begin{cases} (a_1', a_2', \ldots, a_n') = k_1(a_1, a_2, \ldots, a_n)A + k_2(b_1, b_2, \ldots, b_n)A, \\ (b_1', b_2', \ldots, b_n') = k_3(a_1, a_2, \ldots, a_n)A + k_4(b_1, b_2, \ldots, b_n)A. \end{cases} \quad (k_1k_4 - k_2k_3 \neq 0). \tag{24}$$

Remark 2. Eqs. (24) can be regarded as $2n$ algebraic equations with respect to $\epsilon_1, \ldots, \epsilon_n$ and k_1, k_2, k_3, k_4, which will be taken to judge the equivalence of two given two-dimensional algebras $\{w_1, w_2\}$ and $\{w_1', w_2'\}$. If Eqs. (24) have the solution, it means that $\{\sum_{i=1}^{n} a_i v_i, \sum_{j=1}^{n} b_j v_j\}$ is equivalent to $\{\sum_{i=1}^{n} a_i' v_i, \sum_{j=1}^{n} b_j' v_j\}$.

1. Adjoint matrix of the heat equation

For the heat equation, its general adjoint transformation matrix A of (15) is the product of the matrices of the separate adjoint actions $A_1, A_2, \ldots, A_6$, each corresponding to $Ad_{\exp(\epsilon v_i)}(w_1), i = 1 \ldots 6$.

First, under the adjoint action of v_1 and with the help of Table II, w_1 can be transformed into

$$\begin{aligned} &Ad_{\exp(\epsilon_1 v_1)}(a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6) \\ &= (a_1 - a_4\epsilon_1)v_1 + a_2v_2 + (a_3 + a_5\epsilon_1 - a_6\epsilon_1^2)v_3 + a_4v_4 + (a_5 - 2\epsilon_1 a_6)v_5 + a_6v_6. \end{aligned} \tag{25}$$

One can rewrite above formula (25) into the following matrix form:

$$w_1 \doteq (a_1, a_2, \ldots, a_6) \xrightarrow{Ad_{\exp(\epsilon_1 v_1)}} (a_1, a_2, \ldots, a_6)A_1,$$

where

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 0 & 1 & 0 \\ 0 & 0 & \epsilon_1^2 & 0 & -2\epsilon_1 & 1 \end{pmatrix}. \tag{26}$$

Similarly, the rest matrices of the separate adjoint actions of $v_2, \ldots, v_6$ are found to be

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2\epsilon_2 & 0 & 1 & 0 & 0 \\ -2\epsilon_2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 4\epsilon_2^2 & 2\epsilon_2 & -4\epsilon_2 & 0 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} e^{\epsilon_4} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2\epsilon_4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\epsilon_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-2\epsilon_4} \end{pmatrix}, \tag{27}$$

$$A_5 = \begin{pmatrix} 1 & 0 & -\epsilon_5 & 0 & 0 & 0 \\ 2\epsilon_5 & 1 & -\epsilon_5^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \epsilon_5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 2\epsilon_6 & 0 \\ 0 & 1 & -2\epsilon_6 & 4\epsilon_6 & 0 & 4\epsilon_6^2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2\epsilon_6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{28}$$

with $A_3 = E$ being the identity matrix. Then the general adjoint transformation matrix A is the product of $A_1, \ldots, A_6$ which can be taken in any order,

$$A \equiv (a_{ij})_{6\times 6} = A_1A_2A_3A_4A_5A_6, \tag{29}$$

with

$$
\begin{aligned}
&a_{11} = e^{\epsilon_4}, \quad a_{12} = a_{16} = 0, \quad a_{13} = -\epsilon_5 e^{\epsilon_4}, \quad a_{14} = 0, \quad a_{15} = 2\epsilon_6 e^{\epsilon_4},\\
&a_{21} = 2\epsilon_5 e^{2\epsilon_4}, \quad a_{22} = e^{2\epsilon_4}, \quad a_{23} = -(\epsilon_5^2 + 2\epsilon_6)e^{2\epsilon_4}, \quad a_{24} = 4\epsilon_6 e^{2\epsilon_4},\\
&a_{25} = 4\epsilon_5\epsilon_6 e^{2\epsilon_4}, \quad a_{26} = 4\epsilon_6^2 e^{2\epsilon_4}, \quad a_{31} = a_{32} = a_{34} = a_{35} = a_{36} = 0,\\
&a_{33} = 1, \quad a_{41} = -e^{\epsilon_4}(\epsilon_1 + 4\epsilon_2\epsilon_5 e^{\epsilon_4}), \quad a_{42} = -2\epsilon_2 e^{2\epsilon_4},\\
&a_{43} = 4\epsilon_2\epsilon_6 e^{2\epsilon_4} + \epsilon_5 e^{\epsilon_4}(\epsilon_1 + 2\epsilon_2\epsilon_5 e^{\epsilon_4}), \quad a_{44} = 1 - 8\epsilon_2\epsilon_6 e^{2\epsilon_4},\\
&a_{45} = \epsilon_5 - 2\epsilon_6 e^{\epsilon_4}(\epsilon_1 + 4\epsilon_2\epsilon_5 e^{\epsilon_4}), \quad a_{46} = 2\epsilon_6(1 - 4\epsilon_2\epsilon_6 e^{2\epsilon_4}),\\
&a_{51} = -2\epsilon_2 e^{\epsilon_4}, \quad a_{52} = 0, \quad a_{53} = \epsilon_1 + 2\epsilon_2\epsilon_5 e^{\epsilon_4}, \quad a_{54} = 0,\\
&a_{55} = e^{-\epsilon_4}(1 - 4\epsilon_2\epsilon_6 e^{2\epsilon_4}), \quad a_{56} = 0, \quad a_{61} = 4\epsilon_2 e^{\epsilon_4}(\epsilon_1 + 2\epsilon_2\epsilon_5 e^{\epsilon_4}),\\
&a_{62} = 4\epsilon_2^2 e^{2\epsilon_4}, \quad a_{63} = -(\epsilon_1 + 2\epsilon_2\epsilon_5 e^{\epsilon_4})^2 + 2\epsilon_2(1 - 4\epsilon_2\epsilon_6 e^{2\epsilon_4}),\\
&a_{64} = -4\epsilon_2(1 - 4\epsilon_2\epsilon_6 e^{2\epsilon_4}), \quad a_{66} = e^{-2\epsilon_4}(1 - 4\epsilon_2\epsilon_6 e^{2\epsilon_4})^2,\\
&a_{65} = 8\epsilon_2\epsilon_6 e^{\epsilon_4}(\epsilon_1 + 2\epsilon_2\epsilon_5 e^{\epsilon_4}) - 4\epsilon_2\epsilon_5 - 2\epsilon_1 e^{-\epsilon_4}.
\end{aligned}
$$

2. Adjoint matrix of the Novikov equation

For the Novikov equation, its matrices of the separate adjoint actions $A_1, \ldots, A_5$ were found in Ref. 13, which are rewritten in the follows:

$$
A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 2\epsilon_1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad
A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & e^{-2\epsilon_2} & 0 & 0\\ 0 & 0 & 0 & e^{2\epsilon_2} & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad
A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 2\epsilon_3 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 4\epsilon_3 & 4\epsilon_3^2 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & -2\epsilon_4 & 0\\ 0 & -4\epsilon_4 & 1 & 4\epsilon_4^2 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad
A_5 = \begin{pmatrix} e^{-2\epsilon_5} & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
$$

The general matrix A is selected as

$$
A = A_1A_2A_3A_4A_5 = \begin{pmatrix} e^{-2\epsilon_5} & 0 & 0 & 0 & 0\\ 0 & 1-8\epsilon_3\epsilon_4 & 2\epsilon_3 & 2\epsilon_4(4\epsilon_3\epsilon_4 - 1) & 0\\ 0 & -4\epsilon_4 e^{-2\epsilon_2} & e^{-2\epsilon_2} & 4\epsilon_4^2 e^{-2\epsilon_2} & 0\\ 0 & 4\epsilon_3 e^{2\epsilon_2}(1-4\epsilon_3\epsilon_4) & 4\epsilon_3^2 e^{2\epsilon_2} & e^{2\epsilon_2}(1-4\epsilon_3\epsilon_4)^2 & 0\\ 2\epsilon_1 e^{-2\epsilon_5} & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{30}
$$

C. Calculation of the invariants for the refined two-dimensional algebra

For a two-dimensional Lie algebra $\mathfrak{G}(v,w)$, a real function ϕ is called an invariant if $\phi(a_{11}Ad_g(w_1) + a_{12}Ad_g(w_2), a_{21}Ad_g(w_1) + a_{22}Ad_g(w_2)) = \phi(w_1, w_2)$ for any $\{w_1, w_2\} \in \mathfrak{G}(v,w)$ and all $g \in G$ with $a_{11}, a_{12}, a_{21}, a_{22}$ being arbitrary constants. For a general two-dimensional subalgebra $\{\sum_{i=1}^{n} a_i v_i, \sum_{j=1}^{n} b_j v_j\} \in \mathfrak{G}(v,w)$, the corresponding invariant is a function of $a_1, \ldots, a_n, b_1, \ldots, b_n$.

Let $v = \sum_{k=1}^{n} c_k v_k$ be a general element of $\mathcal{G}$. In conjunction with the commutator table, we have

$$\begin{aligned} Ad_g(w_1) &= Ad_{\exp(\epsilon v)}(w_1) \\ &= w_1 - \epsilon[v, w_1] + \frac{1}{2!}\epsilon^2[v,[v,w_1]] - \cdots \\ &= (a_1 v_1 + \cdots + a_n v_n) - \epsilon[c_1 v_1 + \cdots + c_n v_n, a_1 v_1 + \cdots + a_n v_n] + O(\epsilon^2) \\ &= (a_1 v_1 + \cdots + a_n v_n) - \epsilon(\Theta_1^a v_1 + \cdots + \Theta_n^a v_n) + O(\epsilon^2) \\ &= (a_1 - \epsilon\Theta_1^a)v_1 + (a_2 - \epsilon\Theta_2^a)v_2 + \cdots + (a_n - \epsilon\Theta_n^a)v_n + O(\epsilon^2), \end{aligned} \tag{31}$$

where $\Theta_i^a \equiv \Theta_i^a(a_1,\ldots,a_n,c_1,\ldots,c_n)$ can be easily obtained from the commutator table. Similarly, applying the same adjoint action $v = \sum_{k=1}^{n} c_k v_k$ to w_2, we get

$$Ad_g(w_2) = Ad_{\exp(\epsilon v)}(w_2) = (b_1 - \epsilon\Theta_1^b)v_1 + (b_2 - \epsilon\Theta_2^b)v_2 + \cdots + (b_6 - \epsilon\Theta_n^b)v_n + O(\epsilon^2), \tag{32}$$

where $\Theta_i^b \equiv \Theta_i^b(b_1,\ldots,b_n,c_1,\ldots,c_n)$ is obtained directly by replacing a_i with b_i in $\Theta_i^a (i = 1\ldots n)$.

More intuitively, we denote

$$\begin{aligned} w_1 &\doteq (a_1, a_2, \ldots, a_n), \quad w_2 \doteq (b_1, b_2, \ldots, b_n), \\ Ad_g(w_1) &\doteq (a_1 - \epsilon\Theta_1^a, a_2 - \epsilon\Theta_2^a, \ldots, a_n - \epsilon\Theta_n^a) + O(\epsilon^2), \\ Ad_g(w_2) &\doteq (b_1 - \epsilon\Theta_1^b, b_2 - \epsilon\Theta_2^b, \ldots, b_n - \epsilon\Theta_n^b) + O(\epsilon^2). \end{aligned} \tag{33}$$

For the two-dimensional subalgebra $\{w_1, w_2\}$, according to the definition of the invariant, we have

$$\phi(a_{11}Ad_g(w_1) + a_{12}Ad_g(w_2), a_{21}Ad_g(w_1) + a_{22}Ad_g(w_2)) = \phi(w_1, w_2). \tag{34}$$

Further, to guarantee $a_{11}Ad_g(w_1) + a_{12}Ad_g(w_2) = w_1$ and $a_{21}Ad_g(w_1) + a_{22}Ad_g(w_2) = w_2$ after the substitution of $\epsilon = 0$, we require

$$a_{11} \equiv 1 + \epsilon a_{11}, \quad a_{12} \equiv \epsilon a_{12}, \quad a_{21} \equiv \epsilon a_{21}, \quad a_{22} \equiv 1 + \epsilon a_{22}. \tag{35}$$

Then Eq. (34) is modified as

$$\phi(w_1, w_2) = \phi((1 + \epsilon a_{11})Ad_g(w_1) + \epsilon a_{12}Ad_g(w_2), \epsilon a_{21}Ad_g(w_1) + (1 + \epsilon a_{22})Ad_g(w_2)). \tag{36}$$

Remark 3. Since we just need consider the refined two-dimensional algebra, two cases in (36) are discussed.

(a) When $[w_1, w_2] = 0$, taking the derivative of Eq. (36) with respect to ϵ and setting $\epsilon = 0$ after the substitution of (33), extracting all the coefficients of $c_i, a_{11}, a_{12}, a_{21}, a_{22}$, some linear differential equations of ϕ are obtained. By solving these equations, all the invariants ϕ on $[w_1, w_2] = 0$ can be found.

(b) When $[w_1, w_2] = w_1$, according to "Remark 2," first we should make $a_{12} = 0$ and $a_{22} = 0$ in Eq. (36). Then one does the same procedure just as case (a) to obtain linear differential equations of ϕ, which keep invariable for $[w_1, w_2] = w_1$.

1. Invariant equations of the heat equation

For a general two-dimensional subalgebra $\{w_1, w_2\}$ of the heat equation, the corresponding invariant ϕ is a real function with twelve independent variables. Let $v = \sum_{k=1}^{6} c_k v_k$ be a general element of $\mathcal{G}$, then in conjunction with the commutator Table I, we have

$$\begin{aligned} Ad_g(w_1) &= Ad_{\exp(\epsilon v)}(w_1) \\ &= w_1 - \epsilon[v, w_1] + \frac{1}{2!}\epsilon^2[v,[v,w_1]] - \cdots \\ &= (a_1 v_1 + \cdots + a_6 v_6) - \epsilon[c_1 v_1 + \cdots + c_6 v_6, a_1 v_1 + \cdots + a_6 v_6] + O(\epsilon^2) \\ &= (a_1 v_1 + \cdots + a_6 v_6) - \epsilon(\Theta_1^a v_1 + \cdots + \Theta_6^a v_6) + O(\epsilon^2) \\ &= (a_1 - \epsilon\Theta_1^a)v_1 + (a_2 - \epsilon\Theta_2^a)v_2 + \cdots + (a_6 - \epsilon\Theta_6^a)v_6 + O(\epsilon^2), \end{aligned} \tag{37}$$

with

$$\begin{aligned}
\Theta_1^a &= -c_4a_1 - 2c_5a_2 + c_1a_4 + 2c_2a_5, \quad \Theta_2^a = -2c_4a_2 + 2c_2a_4,\\
\Theta_3^a &= c_5a_1 + 2c_6a_2 - c_1a_5 - 2c_2a_6 \quad \Theta_4^a = -4c_6a_2 + 4c_2a_6,\\
\Theta_5^a &= -2c_6a_1 - c_5a_4 + c_4a_5 + 2c_1a_6, \quad \Theta_6^a = -2c_6a_4 + 2c_4a_6.
\end{aligned} \tag{38}$$

Similarly, applying the same adjoint action $v = \sum_{k=1}^{6} c_k v_k$ to w_2, we get

$$Ad_g(w_2) = Ad_{\exp(\epsilon v)}(w_2) = (b_1 - \epsilon\Theta_1^b)v_1 + (b_2 - \epsilon\Theta_2^b)v_2 + \cdots + (b_6 - \epsilon\Theta_6^b)v_6 + O(\epsilon^2), \tag{39}$$

with

$$\begin{aligned}
\Theta_1^b &= -c_4b_1 - 2c_5b_2 + c_1b_4 + 2c_2b_5, \quad \Theta_2^b = -2c_4b_2 + 2c_2b_4,\\
\Theta_3^b &= c_5b_1 + 2c_6b_2 - c_1b_5 - 2c_2b_6 \quad \Theta_4^b = -4c_6b_2 + 4c_2b_6,\\
\Theta_5^b &= -2c_6b_1 - c_5b_4 + c_4b_5 + 2c_1b_6, \quad \Theta_6^b = -2c_6b_4 + 2c_4b_6.
\end{aligned} \tag{40}$$

Following "Remark 3," Eq. (36) is separated into two cases.

(a) For $[w_1, w_2] = 0$, all the $c_i(i = 1 \ldots 6), a_{11}, a_{12}, a_{21}, a_{22}$ in Eq. (36) are arbitrary. Now taking the derivative of Eq. (36) with respect to ϵ and then setting $\epsilon = 0$, extracting the coefficients of all $c_i, a_{11}, a_{12}, a_{21}, a_{22}$, one can directly obtain nine differential equations about $\phi = \phi(a_1, \ldots, a_6, b_1, \ldots, b_6)$,

$$\begin{aligned}
a_1\phi_{a_1} + a_2\phi_{a_2} + a_3\phi_{a_3} + a_4\phi_{a_4} + a_5\phi_{a_5} + a_6\phi_{a_6} &= 0,\\
a_1\phi_{b_1} + a_2\phi_{b_2} + a_3\phi_{b_3} + a_4\phi_{b_4} + a_5\phi_{b_5} + a_6\phi_{b_6} &= 0,\\
b_1\phi_{b_1} + b_2\phi_{b_2} + b_3\phi_{b_3} + b_4\phi_{b_4} + b_5\phi_{b_5} + b_6\phi_{b_6} &= 0,\\
2a_2\phi_{a_1} + 2b_2\phi_{b_1} - a_1\phi_{a_3} - b_1\phi_{b_3} + a_4\phi_{a_5} + b_4\phi_{b_5} &= 0,\\
-a_4\phi_{a_1} - b_4\phi_{b_1} + a_5\phi_{a_3} + b_5\phi_{b_3} - 2a_6\phi_{a_5} - 2b_6\phi_{b_5} &= 0,\\
-a_5\phi_{a_1} - b_5\phi_{b_1} - a_4\phi_{a_2} - b_4\phi_{b_2} + a_6\phi_{a_3} + b_6\phi_{b_3} - 2a_6\phi_{a_4} - 2b_6\phi_{b_4} &= 0,\\
-a_2\phi_{a_3} - b_2\phi_{b_3} + 2a_2\phi_{a_4} + 2b_2\phi_{b_4} + a_1\phi_{a_5} + b_1\phi_{b_5} + a_4\phi_{a_6} + b_4\phi_{b_6} &= 0,
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
a_1\phi_{a_1} + b_1\phi_{b_1} + 2a_2\phi_{a_2} + 2b_2\phi_{b_2} - a_5\phi_{a_5} - b_5\phi_{b_5} - 2a_6\phi_{a_6} - 2b_6\phi_{b_6} &= 0,\\
b_1\phi_{a_1} + b_2\phi_{a_2} + b_3\phi_{a_3} + b_4\phi_{a_4} + b_5\phi_{a_5} + b_6\phi_{a_6} &= 0.
\end{aligned} \tag{42}$$

Here the subscripts indicate partial derivatives.

(b) For $[w_1, w_2] = w_1$, it requires $a_{12} = a_{22} = 0$ in Eq. (36) and seven equations about ϕ which are just Eqs. (41) are obtained.

2. *Invariant equations of the Novikov equation*

For the Novikov equation, using the commutator Table III, we have

$$Ad_g(w_1) = Ad_{\exp(\epsilon v)}\left(\sum_{i=1}^{5} a_i v_i\right) = (a_1 - \epsilon\Theta_1^a)v_1 + \cdots + a_5v_5 + O(\epsilon^2), \tag{43}$$

with

$$\Theta_1^a = 2(a_1c_5 - a_5c_1), \Theta_2^a = 4(a_3c_4 - a_4c_3), \Theta_3^a = 2(a_3c_2 - a_2c_3), \Theta_4^a = 2(a_2c_4 - a_4c_2). \tag{44}$$

Similarly, there is

$$Ad_g(w_2) = Ad_{\exp(\epsilon v)}\left(\sum_{i=1}^{5} b_i v_i\right) = (b_1 - \epsilon\Theta_1^b)v_1 + \cdots + b_5v_5 + O(\epsilon^2). \tag{45}$$

Substituting (43) and (45) into Eq. (36), two cases about the invariant ϕ are obtained.

(a) For $[w_1, w_2] = 0$, nine differential equations about ϕ are found,

$$\begin{aligned} &a_5\phi_{a_1} + b_5\phi_{b_1} = 0, \quad a_1\phi_{a_1} + b_1\phi_{b_1} = 0, \quad a_2\phi_{a_3} + b_2\phi_{b_3} + 2a_4\phi_{a_2} + 2b_4\phi_{b_2} = 0, \\ &a_2\phi_{a_4} + b_2\phi_{b_4} + 2a_3\phi_{a_2} + 2b_3\phi_{b_2} = 0, \quad a_3\phi_{a_3} + b_3\phi_{b_3} - a_4\phi_{a_4} - b_4\phi_{b_4} = 0, \\ &a_1\phi_{a_1} + a_2\phi_{a_2} + a_3\phi_{a_3} + a_4\phi_{a_4} + a_5\phi_{a_5} = 0, \quad a_1\phi_{b_1} + a_2\phi_{b_2} + a_3\phi_{b_3} + a_4\phi_{b_4} + a_5\phi_{b_5} = 0, \end{aligned} \tag{46}$$

and

$$\begin{aligned} b_1\phi_{b_1} + b_2\phi_{b_2} + b_3\phi_{b_3} + b_4\phi_{b_4} + b_5\phi_{b_5} = 0, \\ b_1\phi_{a_1} + b_2\phi_{a_2} + b_3\phi_{a_3} + b_4\phi_{a_4} + b_5\phi_{a_5} = 0. \end{aligned} \tag{47}$$

(b) For $[w_1, w_2] = w_1$, one just needs consider seven equations (46).

D. Construction of two-dimensional optimal system

(1) *First step*: Present the commutator table and adjoint representation table of the generators $\{v_i\}_{i=1}^n$ for a given algebra. Then in terms of $[w_1, w_2] = \delta w_1$ with $\delta \equiv 0, 1$, give out the restrictions about $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$.

(2) *Second step*: Following sections **B** and **C**, compute the adjoint transformation matrix A and determine the general equations about the invariants ϕ.

(3) *Third step*: For two different cases $[w_1, w_2] = 0$ and $[w_1, w_2] = w_1$, in terms of every restricted condition given by step 1, compute their respective invariants and select the corresponding eligible representative elements $\{w_1', w_2'\}$. For ease of calculations, we rewrite Eq. (9) as

$$\begin{cases} Ad_g(w_1) = k_1'w_1' + k_2'w_2', \\ Ad_g(w_2) = k_3'w_1' + k_4'w_2', \end{cases} \qquad (k_1'k_4' \neq k_2'k_3'), \tag{48}$$

which is usually expressed by

$$\begin{cases} (a_1, a_2, \ldots, a_n)A = k_1'(a_1', a_2', \ldots, a_n') + k_2'(b_1', b_2', \ldots, b_n'), \\ (b_1, b_2, \ldots, b_n)A = k_3'(a_1', a_2', \ldots, a_n') + k_4'(b_1', b_2', \ldots, b_n'), \end{cases} (k_1'k_4' \neq k_2'k_3'). \tag{49}$$

Following "Remark 2," if Eqs. (49) have the solution with respect to $\epsilon_1, \ldots \epsilon_n, k_1', k_2', k_3', k_4'$, it signifies that the selected representative element $\{w_1', w_2'\}$ is right; if Eqs. (49) have no solution, another new representative element $\{w_1'', w_2''\}$ should be reselected. Repeat the process until all the cases are finished in the restrictions of step 1.

1. Two-dimensional optimal system of the heat equation

(a) The case of $\delta = 0$ in restrictive equations (17)

Case 1: Not all a_2, a_4, a_6, b_2, b_4 and b_6 are zeroes.

Without loss of generality, we adopt $a_6 \neq 0$. In fact, when only one of a_2, a_4, a_6, b_2, b_4, and b_6 is not zero, one can choose appropriate adjoint transformation to transform it into the case $\tilde{a}_6 \neq 0$. By solving Eqs. (17) with $\delta = 0$ and $a_6 \neq 0$, we have three kinds of solutions.

(i) a_3, a_4, a_5, a_6, b_3, and b_6 are independent with

$$a_1 = \frac{1}{2}\frac{a_4a_5}{a_6}, a_2 = \frac{1}{4}\frac{a_4^2}{a_6}, b_1 = \frac{1}{2}\frac{a_4a_5b_6}{a_6^2}, b_2 = \frac{1}{4}\frac{a_4^2b_6}{a_6^2}, b_4 = \frac{a_4b_6}{a_6}, b_5 = \frac{a_5b_6}{a_6}. \tag{50}$$

(ii) $a_3, a_4, a_5, a_6, b_3, b_5$, and b_6 are arbitrary but with

$$a_1 = \frac{1}{2}\frac{a_4a_5}{a_6}, a_2 = \frac{1}{4}\frac{a_4^2}{a_6}, b_1 = \frac{1}{2}\frac{a_4b_5}{a_6}, b_2 = \frac{1}{4}\frac{a_4^2b_6}{a_6^2}, b_4 = \frac{a_4b_6}{a_6}, b_5 \neq \frac{a_5b_6}{a_6}. \tag{51}$$

(iii) $a_1, a_2, a_3, a_4, a_5, a_6, b_3$, and b_6 are arbitrary but with

$$a_1 \neq \frac{1}{2}\frac{a_4a_5}{a_6} \quad \text{or} \quad a_2 \neq \frac{1}{4}\frac{a_4^2}{a_6}, b_1 = \frac{a_1b_6}{a_6}, b_2 = \frac{a_2b_6}{a_6}, b_4 = \frac{a_4b_6}{a_6}, b_5 = \frac{a_5b_6}{a_6}. \tag{52}$$

Substituting the above three conditions into Eqs. (41) and (42), we find that $\phi \equiv constant$, i.e., there is no invariant. Then for each case, select the corresponding representative element $\{w_1', w_2'\}$ and verify whether Eqs. (49) have the solution.

For case (i), select a representative element $\{w_1', w_2'\} = \{v_6, v_3\}$. Substituting (50) and $w_1' = v_6$, $w_2' = v_3$ into Eqs. (49), we obtain the solution

$$\begin{aligned} &k_1' = a_6 e^{-2\epsilon_4}, \quad k_2' = \frac{4a_3a_6 + 2a_4a_6 + a_5^2}{4a_6}, \quad k_3' = b_6 e^{-2\epsilon_4}, \\ &k_4' = \frac{4b_3a_6^2 + b_6a_5^2 + 2a_4a_6b_6}{4a_6^2}, \quad \epsilon_1 = \frac{a_5}{2a_6}, \quad \epsilon_2 = \frac{a_4}{4a_6}. \end{aligned} \tag{53}$$

Hence case (i) is equivalent to $\{v_6, v_3\}$.

For case (ii), there exist three circumstances in terms of the following expression:

$$\Lambda_1 \equiv 2a_6[a_4(a_5b_6 - b_5a_6)^2 - 2a_6(a_3b_6 - b_3a_6)^2 - 2a_6(a_5b_6 - b_5a_6)(a_3b_5 - b_3a_5)]. \tag{54}$$

(iia) When $\Lambda_1 > 0$, case (ii) is equivalent to $\{v_3 + v_6, v_5\}$. After the substitution of (51) with $w_1' = v_3 + v_6, w_2' = v_5$, Eqs. (49) hold for

$$\begin{aligned} &k_1' = \frac{\Lambda_1}{4a_6(a_5b_6 - b_5a_6)^2}, \quad \epsilon_4 = \ln \frac{2|a_6(a_5b_6 - b_5a_6)|}{\sqrt{\Lambda_1}}, \\ &k_2' = \frac{a_5(a_5b_6 - b_5a_6) + 2a_6(a_3b_6 - b_3a_6)}{2|a_6|(a_5b_6 - b_5a_6)^2}\sqrt{\Lambda_1}, \quad \epsilon_1 = -\frac{a_3b_6 - b_3a_6}{a_5b_6 - b_5a_6}, \\ &k_3' = \frac{b_6}{a_6}k_1, \quad \epsilon_2 = \frac{a_4}{4a_6}, \quad k_4' = \frac{b_5(a_5b_6 - b_5a_6) + 2b_6(a_3b_6 - b_3a_6)}{2|a_6|(a_5b_6 - b_5a_6)^2}\sqrt{\Lambda_1}. \end{aligned}$$

(iib) When $\Lambda_1 < 0$, case (ii) is equivalent to $\{-v_3 + v_6, v_5\}$. The solution for Eqs. (49) is

$$\begin{aligned} &k_1' = -\frac{\Lambda_1}{4a_6(a_5b_6 - b_5a_6)^2}, k_2' = \frac{a_5(a_5b_6 - b_5a_6) + 2a_6(a_3b_6 - b_3a_6)}{2|a_6|(a_5b_6 - b_5a_6)^2}\sqrt{-\Lambda_1}, \\ &k_3' = \frac{b_6}{a_6}k_1, \quad k_4' = \frac{b_5(a_5b_6 - b_5a_6) + 2b_6(a_3b_6 - b_3a_6)}{2|a_6|(a_5b_6 - b_5a_6)^2}\sqrt{-\Lambda_1}, \\ &\epsilon_1 = -\frac{a_3b_6 - b_3a_6}{a_5b_6 - b_5a_6}, \quad \epsilon_2 = \frac{a_4}{4a_6}, \quad \epsilon_4 = \ln \frac{2|a_6(a_5b_6 - b_5a_6)|}{\sqrt{-\Lambda_1}}. \end{aligned}$$

(iic) When $\Lambda_1 = 0$, case (ii) is equivalent to $\{v_6, v_5\}$. By solving Eqs. (49), we obtain

$$\begin{aligned} &k_1' = a_6e^{-2\epsilon_4}, \quad k_2' = \frac{a_5(a_5b_6 - b_5a_6) + 2a_6(a_3b_6 - b_3a_6)}{(a_5b_6 - b_5a_6)e^{\epsilon_4}}, \\ &k_3' = b_6e^{-2\epsilon_4}, \quad k_4' = \frac{a_5(a_5b_6 - b_5a_6) + 2a_6(a_3b_6 - b_3a_6)}{(a_5b_6 - b_5a_6)e^{\epsilon_4}}, \\ &\epsilon_1 = -\frac{a_3b_6 - b_3a_6}{a_5b_6 - b_5a_6}, \quad \epsilon_2 = \frac{(a_5b_6 - b_5a_6)(a_3b_5 - a_5b_3) + (a_3b_6 - b_3a_6)^2}{2(a_5b_6 - b_5a_6)^2}. \end{aligned}$$

For case (iii), it can be divided into the following several types.

(iiia) $4a_2a_6 - a_4^2 > 0$. Select a representative element $\{v_2 + v_6, v_3\}$. After substituting (52) into Eqs. (49), we have

$$\begin{aligned} &k_1' = (a_2 - 2a_4\epsilon_2 + 4a_6\epsilon_2^2)e^{2\epsilon_4}, \quad k_2' = a_3 + \frac{a_4}{2} + \frac{a_1^2a_6 - a_1a_4a_5 + a_2a_5^2}{4a_2a_6 - a_4^2}, \\ &k_3' = \frac{b_6(a_2 - 2a_4\epsilon_2 + 4a_6\epsilon_2^2)}{a_6}e^{2\epsilon_4}, k_4' = b_3 + \frac{b_6}{a_6}\left(\frac{a_4}{2} + \frac{a_1^2a_6 - a_1a_4a_5 + a_2a_5^2}{4a_2a_6 - a_4^2}\right), \\ &\epsilon_1 = \frac{2\epsilon_2(2a_1a_6 - a_4a_5) + 2a_2a_5 - a_1a_4}{4a_2a_6 - a_4^2}, \quad \epsilon_5 = \frac{a_4a_5 - 2a_1a_6}{(4a_2a_6 - a_4^2)e^{\epsilon_4}}, \\ &\epsilon_6 = \frac{4a_6\epsilon_2 - a_4}{4e^{2\epsilon_4}(a_2 - 2a_4\epsilon_2 + 4a_6\epsilon_2^2)}, \quad e^{2\epsilon_4} = \frac{\sqrt{4a_2a_6 - a_4^2}}{2|a_2 - 2a_4\epsilon_2 + 4a_6\epsilon_2^2|}. \end{aligned}$$

(iiib) $4a_2a_6 - a_4^2 < 0$. Choose a representative element $\{-v_2 + v_6, v_3\}$. Now Eqs. (49) have the solution

$$k_1' = -(a_2 - 2a_4\epsilon_2 + 4a_6\epsilon_2^2)e^{2\epsilon_4}, \quad k_2' = a_3 + \frac{a_4}{2} + \frac{a_1^2a_6 - a_1a_4a_5 + a_2a_5^2}{4a_2a_6 - a_4^2},$$
$$k_3' = -\frac{b_6(a_2 - 2a_4\epsilon_2 + 4a_6\epsilon_2^2)}{a_6}e^{2\epsilon_4}, \quad \epsilon_1 = \frac{2\epsilon_2(2a_1a_6 - a_4a_5) + 2a_2a_5 - a_1a_4}{4a_2a_6 - a_4^2},$$
$$k_4' = b_3 + \frac{b_6}{a_6}\left(\frac{a_4}{2} + \frac{a_1^2a_6 - a_1a_4a_5 + a_2a_5^2}{4a_2a_6 - a_4^2}\right), \quad \epsilon_5 = \frac{a_4a_5 - 2a_1a_6}{(4a_2a_6 - a_4^2)e^{\epsilon_4}},$$
$$\epsilon_6 = \frac{4a_6\epsilon_2 - a_4}{4e^{2\epsilon_4}(a_2 - 2a_4\epsilon_2 + 4a_6\epsilon_2^2)}, \quad e^{2\epsilon_4} = \frac{\sqrt{-(4a_2a_6 - a_4^2)}}{2|a_2 - 2a_4\epsilon_2 + 4a_6\epsilon_2^2|}.$$

(iiic) $4a_2a_6 - a_4^2 = 0$. In this case, two conditions should be considered.

When $2a_1a_6 - a_4a_5 > 0$, adopt the representative element $\{v_1 + v_6, v_3\}$. Then the solution for Eqs. (49) is

$$k_1' = \frac{a_6}{Z^2}, \quad k_2' = -\frac{(\epsilon_6^2 + \epsilon_5)(2a_1a_6 - a_4a_5)}{2a_6}Z + a_3 + \frac{a_4}{2} + \frac{a_5^2}{4a_6},$$
$$k_3' = \frac{b_6(2a_1a_6 - a_4a_5)}{2a_6^2}Z, \quad \epsilon_2 = \frac{a_4}{4a_6}, \quad \epsilon_4 = \ln(Z),$$
$$k_4' = \frac{b_6}{4a_6^2}[-2(\epsilon_6^2 + \epsilon_5)(2a_1a_6 - a_4a_5)Z + (a_5^2 + 2a_4a_6)] + b_3,$$
$$\epsilon_1 = \frac{\epsilon_6(2a_1a_6 - a_4a_5)}{2a_6^2}Z^2 + \frac{a_5}{2a_6}. \quad \left(Z = \left(\frac{2a_6^2}{2a_1a_6 - a_4a_5}\right)^{\frac{1}{3}}\right)$$

When $2a_1a_6 - a_4a_5 < 0$, adopt the representative element $\{-v_1 + v_6, v_3\}$. By solving Eqs. (49), we find

$$k_1' = \frac{a_6}{Z'^2}, \quad k_2' = -\frac{(\epsilon_5 - \epsilon_6^2)(2a_1a_6 - a_4a_5)}{2a_6}Z' + a_3 + \frac{a_4}{2} + \frac{a_5^2}{4a_6},$$
$$k_3' = -\frac{b_6(2a_1a_6 - a_4a_5)}{2a_6^2}Z', \quad \epsilon_2 = \frac{a_4}{4a_6}, \quad \epsilon_4 = \ln(Z'),$$
$$k_4' = \frac{b_6}{4a_6^2}[-2(\epsilon_5 - \epsilon_6^2)(2a_1a_6 - a_4a_5)Z' + (a_5^2 + 2a_4a_6)] + b_3,$$
$$\epsilon_1 = \frac{\epsilon_6(2a_1a_6 - a_4a_5)}{2a_6^2}Z'^2 + \frac{a_5}{2a_6}. \quad \left(Z' = \left(-\frac{2a_6^2}{2a_1a_6 - a_4a_5}\right)^{\frac{1}{3}}\right).$$

Case 2: $a_2 = a_4 = a_6 = b_2 = b_4 = b_6 = 0$.

Now determined equations (17) become

$$-a_1b_5 + a_5b_1 = 0. \tag{55}$$

Here we just need consider not all a_1, a_5, b_1, and b_5 are zeroes. Without loss of generality, let $a_5 \neq 0$. Similarly, if one of a_1, a_5, b_1, b_5 is not zero, one can choose appropriate adjoint transformation to transform it into the case $a_5 \neq 0$. By solving Eq. (55), we obtain

$$b_1 = \frac{a_1b_5}{a_5}. \tag{56}$$

Adopt a representative element $\{v_5, v_3\}$. Then one can verify that all the $\{a_1v_1 + a_3v_3 + a_5v_5, b_1v_1 + b_3v_3 + b_5v_5\}$ with condition (56) are equivalent to $\{v_5, v_3\}$ since the solution for Eqs. (49) is

$$k_1' = a_5e^{-\epsilon_4}, \quad k_2' = a_3 + a_5\epsilon_1, \quad k_3' = b_5e^{-\epsilon_4}, \quad k_4' = b_3 + b_5\epsilon_1, \quad \epsilon_2 = \frac{a_1}{2a_5}.$$

(b) The case of $\delta = 1$ in restrictive equations (17)

Case 3: Not all a_2, a_4, and a_6 are zeroes.

Without loss of generality, we adopt $a_6 \neq 0$ and it can also guarantee $b_6 \neq 0$ through transformation (9). Hence, Let $b_6 \neq 0$ first and it leads Eqs. (17) to

$$\begin{aligned} a_1 &= -\frac{a_6(2b_4+1)(4b_1b_6-b_5-2b_4b_5)}{4b_6^2}, \quad a_2 = \frac{a_6(2b_4+1)^2}{16b_6^2}, \\ a_3 &= -4a_6b_1^2 + \frac{a_6(2b_4+1)(8b_1b_5-1)}{4b_6} - \frac{a_6b_5^2(2b_4+1)^2}{4b_6^2}, \\ a_4 &= \frac{a_6(2b_4+1)}{2b_6}, \quad a_5 = \frac{a_6(b_5+2b_4b_5-4b_1b_6)}{b_6}, \quad b_2 = \frac{4b_4^2-1}{16b_6}. \end{aligned} \tag{57}$$

Substituting (57) into Eqs. (41), it yields an invariant for $\{w_1, w_2\}$,

$$\phi = \Delta_1 \equiv 16b_6b_1^2 - 2b_4 - 16b_1b_4b_5 - 4b_3 + \frac{b_5^2(4b_4^2-1)}{b_6}. \tag{58}$$

In condition of (57) and $\Delta_1 = c$, choose the corresponding representative element $\{v_6, (\frac{1}{4} - \frac{c}{4})v_3 - \frac{1}{2}v_4 + v_6\}$ and Eqs. (49) have the solution

$$\begin{aligned} &k_1' = a_6, \quad k_2' = 0, \quad k_3' = b_6 - 1, \quad k_4' = 1, \quad \epsilon_1 = \frac{b_5 + 2b_4b_5 - 4b_1b_6}{2b_6}, \\ &\epsilon_2 = \frac{2b_4+1}{8b_6}, \quad \epsilon_5 = 8b_1b_6 - 4b_4b_5, \quad \epsilon_4 = \epsilon_6 = 0. \end{aligned}$$

For simplicity, one can take $\{v_6, v_4 + \beta v_3\}(\beta \in \mathbb{R})$ instead of $\{v_6, (\frac{1}{4} - \frac{c}{4})v_3 - \frac{1}{2}v_4 + v_6\}$.

Case 4: $a_2 = a_4 = a_6 = 0$.

Not all a_1 and a_5 are zeroes, or else there must be $a_3 = 0$ shown in Eqs. (17). Let $a_5 \neq 0$, Taking $a_2 = a_4 = a_6 = 0$ with $a_5 \neq 0$ into (17), we have

$$a_3 = b_1a_5 - a_1b_5, \quad b_2 = \frac{a_1(a_1b_6 - a_5)}{a_5^2}, \quad b_4 = \frac{2a_1a_6 - a_5}{a_5}. \tag{59}$$

Substituting relations (59) into Eqs. (41), we obtain an invariant for $\{w_1, w_2\}$, i.e.,

$$\phi = \Delta_2 \equiv b_1b_5 + b_6b_1^2 - b_3 - \frac{a_1(b_5^2 + b_6 + 2b_1b_5b_6)}{a_5} + \frac{b_6b_5^2a_1^2}{a_1^5}. \tag{60}$$

In this case, choose a representative element $\{v_5, -cv_3 - v_4)\}$ for $\Delta_2 = c$. Then solving Eqs. (49), one get

$$\begin{aligned} &k_1' = a_5, \quad k_2' = 0, \quad k_3' = b_5 + 2b_1b_6 - \frac{2a_1b_5b_6}{a_5}, \quad k_4' = 1, \\ &\epsilon_1 = -b_1 + \frac{a_1b_5}{a_5}, \quad \epsilon_2 = \frac{a_1}{2a_5}, \quad \epsilon_6 = \frac{b_6}{2}, \quad \epsilon_4 = \epsilon_5 = 0. \end{aligned}$$

In summary, we have completed the construction of the two-dimensional optimal system O_2,

$$\begin{aligned} &\mathfrak{g}_1 = \{v_6, v_3\}, \quad \mathfrak{g}_2 = \{v_3 + v_6, v_5\}, \quad \mathfrak{g}_2 = \{-v_3 + v_6, v_5\}, \\ &\mathfrak{g}_4 = \{v_6, v_5\}, \quad \mathfrak{g}_5 = \{v_2 + v_6, v_3\}, \quad \mathfrak{g}_6 = \{-v_2 + v_6, v_3\}, \\ &\mathfrak{g}_7 = \{v_1 + v_6, v_3\}, \quad \mathfrak{g}_8 = \{-v_1 + v_6, v_3\}, \quad \mathfrak{g}_9 = \{v_5, v_3\}, \\ &\mathfrak{g}_{10} = \{v_6, v_4 + \beta v_3\}, \quad \mathfrak{g}_{11} = \{v_5, v_4 + \beta v_3\}, \quad (\beta \in \mathbb{R}). \end{aligned} \tag{61}$$

Remark 4. The process of the construction ensures that all $\mathfrak{g}_i (i = 1 \ldots 11)$ are mutually inequivalent since each case is closed. One can also easily find this inequivalence from the incompatibility of Eqs. (49). In Ref. 10, Chou *et al.* gave a two-parameter optimal system $\{M_i\}_1^{10}$ for the same Lie algebra (15) of the heat equation and showed their inequivalences by sufficient numerical invariants. One can see that $\{M_i\}_1^{10}$ in Ref. 10 are equivalent to our $\{\mathfrak{g}_{11}, \mathfrak{g}_{10}, \mathfrak{g}_6, \mathfrak{g}_4, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, \mathfrak{g}_8, \mathfrak{g}_5, \mathfrak{g}_9\}$, respectively. Furthermore, we realize that $\mathfrak{g}_7$ is inequivalent to any of the elements in $\{M_i\}_1^{10}$. Hence, here the two-dimensional optimal system O_2 given by (61) is complete and really optimal.

2. Two-dimensional optimal system of the Novikov equation

(a) The case of $\delta = 0$ in restrictive equations (20)

Case 1: Not all a_5 and b_5 are zeroes. Let $a_5 \neq 0$.

Case 1.1: Not all a_2, a_3, a_4, b_2, b_3, and b_4 are zeroes.

One can make $a_4 \neq 0$ and it leads restrictive equations (20) to

$$b_1 = \frac{a_1 b_5}{a_5}, \quad b_2 = \frac{a_2 b_4}{a_4}, \quad b_3 = \frac{a_3 b_4}{a_4}. \tag{62}$$

Denote

$$\Lambda_2 \equiv a_2^2 - 4a_3 a_4. \tag{63}$$

For $\Lambda_2 = 0$, select $\{v_4 + v_5, v_4\}$ and one solution for Eqs. (49) is

$$k_1' = a_5, \quad k_2' = a_4 - a_5, \quad k_3' = b_5, \quad k_4' = b_4 - b_5, \quad \epsilon_1 = -\frac{a_1}{2a_5}, \quad \epsilon_2 = 0, \quad \epsilon_3 = -\frac{a_2}{4a_4}.$$

For $\Lambda_2 > 0$, select $\{v_2 + v_4 + v_5, v_5\}$ and Eqs. (49) have the solution

$$k_1' = \sqrt{\Lambda_2}, \quad k_2' = a_5 - \sqrt{\Lambda_2}, \quad k_3' = \frac{b_4\sqrt{\Lambda_2}}{a_4}, \quad k_4' = \frac{a_4 b_5 - b_4\sqrt{\Lambda_2}}{a_4},$$
$$\epsilon_1 = -\frac{a_1}{2a_5}, \quad \epsilon_2 = 0, \quad \epsilon_3 = \frac{\sqrt{\Lambda_2} - a_2}{4a_4}, \quad \epsilon_4 = -\frac{\sqrt{\Lambda_2} - a_4}{2\sqrt{\Lambda_2}}.$$

For $\Lambda_2 < 0$, choose $\{v_3 + v_4 + v_5, v_5\}$ and Eqs. (49) have the solution

$$k_1' = a_4 e^{2\epsilon_2}, \quad k_2' = a_5 - a_4 e^{2\epsilon_2}, \quad k_3' = b_4 e^{2\epsilon_2}, \quad k_4' = b_5 - b_4 e^{2\epsilon_2},$$
$$\epsilon_1 = -\frac{a_1}{2a_5}, \quad e^{2\epsilon_2} = \frac{1}{2}\sqrt{\frac{-\Lambda_2}{a_4^2}}, \quad \epsilon_3 = -\frac{a_2}{4a_4} e^{-2\epsilon_2}, \quad \epsilon_4 = 0.$$

Case 1.2: $a_2 = a_3 = a_4 = b_2 = b_3 = b_4 = 0$.

This case is meaningless for $w_2 = 0$ or $w_2 = c w_1$ in terms of Eqs. (20).

Case 2: $a_5 = b_5 = 0$.

Case 2.1: Not all a_1 and b_1 are zeroes.

Let $a_1 \neq 0$. Then not all a_2, a_3, a_4, b_2, b_3, and $b_4 = 0$ are zeros and we take $a_4 \neq 0$. Restrictive equations (20) become

$$b_2 = \frac{a_2 b_4}{a_4}, \quad b_3 = \frac{a_3 b_4}{a_4}. \tag{64}$$

In accordance with $\Lambda_2 = 0$, $\Lambda_2 > 0$, and $\Lambda_2 < 0$, we adopt $\{v_1 + v_4, v_4\}$, $\{v_1 + v_2 + v_4, v_1\}$, and $\{v_1 + v_3 + v_4, v_1\}$, respectively, and the corresponding solutions of Eqs. (49) read

$$k_1' = a_1, \quad k_2' = a_4 - a_1, \quad k_3' = b_1, \quad k_4' = b_4 - b_1, \quad \epsilon_2 = 0, \quad \epsilon_3 = -\frac{a_2}{4a_4}, \quad \epsilon_5 = 0,$$
$$k_1' = \sqrt{\Lambda_2}, \quad k_2' = a_1 - \sqrt{\Lambda_2}, \quad k_3' = \frac{b_4\sqrt{\Lambda_2}}{a_4}, \quad k_4' = \frac{a_4 b_1 - b_4\sqrt{\Lambda_2}}{a_4},$$
$$\epsilon_2 = \epsilon_5 = 0, \quad \epsilon_3 = \frac{\sqrt{\Lambda_2} - a_2}{4a_4}, \quad \epsilon_4 = -\frac{\sqrt{\Lambda_2} - a_4}{2\sqrt{\Lambda_2}},$$

and

$$k_1' = a_4 e^{2\epsilon_2}, \quad k_2' = a_1 - a_4 e^{2\epsilon_2}, \quad k_3' = b_4 e^{2\epsilon_2}, \quad k_4' = b_1 - b_4 e^{2\epsilon_2},$$
$$e^{2\epsilon_2} = \frac{1}{2}\sqrt{\frac{-\Lambda_2}{a_4^2}}, \quad \epsilon_3 = -\frac{a_2}{4a_4} e^{-2\epsilon_2}, \quad \epsilon_4 = \epsilon_5 = 0.$$

Case 2.2: $a_1 = b_1 = 0$.

Now not all a_2, a_3, a_4, b_2, b_3, and $b_4 = 0$ are zeros and let $a_4 \neq 0$. By solving Eqs. (20), we get

$$b_2 = \frac{a_2 b_4}{a_4}, \quad b_3 = \frac{a_3 b_4}{a_4}. \tag{65}$$

We just need consider $\Lambda_2 > 0$ and select $\{v_2 + v_4, v_4\}$.

(b) The case of $\delta = 1$ in restrictive equations (20)

Solving Eqs. (20), it first requires $a_5 = 0$ and $b_5 = -\frac{1}{2}$.

Case 1: $a_1 \neq 0$.

Case 1.1: Not all a_2, a_3, a_4 are zeroes and then one can make $b_4 \neq 0$, which leads Eqs. (20) to

$$a_2 = \frac{a_4(2b_2 - 1)}{2b_4}, \quad a_3 = \frac{a_4(2b_2 - 1)^2}{16b_4^2}, \quad b_3 = \frac{4b_2^2 - 1}{16b_4}. \tag{66}$$

For $a_1 a_4 > 0$, select $\{v_1 + v_4, \frac{1}{2}v_2 + v_4 - \frac{1}{2}v_5\}$ and Eqs. (49) have the solution

$$k_1' = a_4, \quad k_2' = 0, \quad k_3' = \frac{a_4 b_1}{a_1}, \quad k_4' = 1, \quad \epsilon_1 = \epsilon_2 = 0,$$
$$\epsilon_3 = \frac{1 - 2b_2}{8b_4}, \quad \epsilon_4 = b_4 - \frac{a_4 b_1}{a_1} - 1, \quad \epsilon_5 = -\frac{1}{2}\ln\left(\frac{a_4}{a_1}\right).$$

For $a_1 a_4 < 0$, select $\{v_1 - v_4, \frac{1}{2}v_2 + v_4 - \frac{1}{2}v_5\}$ and Eqs. (49) have the solution

$$k_1' = -a_4, \quad k_2' = 0, \quad k_3' = -\frac{a_4 b_1}{a_1}, \quad k_4' = 1, \quad \epsilon_1 = \epsilon_2 = 0,$$
$$\epsilon_3 = \frac{1 - 2b_2}{8b_4}, \quad \epsilon_4 = b_4 - \frac{a_4 b_1}{a_1} - 1, \quad \epsilon_5 = -\frac{1}{2}\ln\left(-\frac{a_4}{a_1}\right).$$

Case 1.2: $a_2 = a_3 = a_4 = 0$.

Case 1.2.1: Not all b_2, b_3, b_4 are zeros and let $b_4 \neq 0$. Now we have an invariant by solving Eqs. (46), saying

$$K_1 \equiv b_2^2 - 4b_3 b_4. \tag{67}$$

For $K_1 = c$, we choose $\{v_1, \frac{c}{4}v_3 - v_4 - \frac{1}{2}v_5\}$ and $\{v_1, -\frac{c}{4}v_3 + v_4 - \frac{1}{2}v_5\}$.

Case 1.2.2: $b_2 = b_3 = b_4 = 0$.

In this case, it only remains $\{v_1, -\frac{1}{2}v_5\}$.

Case 2: $a_1 = 0$.

Now not all a_2, a_3, a_4 are zeros and this can yield $b_4 \neq 0$. Solving Eqs. (20), we obtain

$$a_2 = \frac{a_4(2b_2 - 1)}{2b_4}, \quad a_3 = \frac{a_4(2b_2 - 1)^2}{16b_4^2}, \quad b_3 = \frac{4b_2^2 - 1}{16b_4}. \tag{68}$$

Choose $\{v_4, \frac{1}{2}v_2 + v_4 - \frac{1}{2}v_5\}$ and one solution of Eqs. (49) is

$$k_1' = a_4, \quad k_2' = 0, \quad k_3' = b_4 - 1, \quad k_4' = 1, \quad \epsilon_2 = \epsilon_4 = 0, \quad \epsilon_1 = b_1, \quad \epsilon_3 = \frac{1 - 2b_2}{8b_4}.$$

Recapitulating, a two-dimensional optimal system O_2 of the Novikov equation contains thirteen elements,

$$\begin{aligned}
&\mathfrak{g}_1 = \{v_5, v_4\}, \quad \mathfrak{g}_2 = \{v_2 + v_4, v_5\}, \quad \mathfrak{g}_2 = \{v_3 + v_4, v_5\}, \quad \mathfrak{g}_4 = \{v_1, v_4\},\\
&\mathfrak{g}_5 = \{v_2 + v_4, v_1\}, \quad \mathfrak{g}_6 = \{v_3 + v_4, v_1\}, \quad \mathfrak{g}_7 = \{v_2, v_4\}, \quad \mathfrak{g}_8 = \{v_1 + v_4, v_2 + 2v_4 - v_5\},\\
&\mathfrak{g}_9 = \{v_1 - v_4, v_2 + 2v_4 - v_5\}, \quad \mathfrak{g}_{10} = \{v_1, \beta v_3 - 2v_4 - v_5\},\\
&\mathfrak{g}_{11} = \{v_1, \beta v_3 + 2v_4 - v_5\}. \quad \mathfrak{g}_{12} = \{v_1, v_5\}, \quad \mathfrak{g}_{13} = \{v_4, v_2 + 2v_4 - v_5\}, \quad (\beta \in \mathbb{R}).
\end{aligned} \tag{69}$$

IV. TWO-DIMENSIONAL OPTIMAL SYSTEM AND INVARIANT SOLUTIONS OF (2+1)-DIMENSIONAL NAVIER-STOKES EQUATION

One of the most important open problems in fluid is the existence and smoothness problem of the Navier-Stokes (NS) equation, which has been recognized as the basic equation and the

TABLE V. Commutator table of the NS equation.

	v_1	v_2	v_3	v_4
v_1	0	$-v_2$	v_3	0
v_2	v_2	0	v_4	0
v_3	$-v_3$	$-v_4$	0	0
v_4	0	0	0	0

very starting point of all problems in fluid physics.[16,17] In Ref. 18, by means of the classical Lie symmetry method, we investigated the (2+1)-dimensional Navier-Stokes equation,

$$\begin{aligned} &\omega = \psi_{xx} + \psi_{yy}, \\ &\omega_t + \psi_x \omega_y - \psi_y \omega_x - \gamma(\omega_{xx} + \omega_{yy}) = 0. \end{aligned} \tag{70}$$

One can rewrite Eq. (70) into

$$\begin{aligned} &\psi_{xxt} + \psi_{yyt} + \psi_x \psi_{xxy} + \psi_x \psi_{yyy} - \psi_y \psi_{xxx} - \psi_y \psi_{xyy} \\ &- \gamma(\psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy}) = 0. \end{aligned} \tag{71}$$

The associated vector fields for the one-parameter Lie group of NS equation (71) are given by

$$\begin{aligned} &v_1 = \frac{x}{2}\partial_x + \frac{y}{2}\partial_y + t\partial_t, \quad v_2 = \partial_t, \quad v_3 = -yt\partial_x + xt\partial_y + \frac{x^2 + y^2}{2}\partial_\psi, \\ &v_4 = -y\partial_x + x\partial_y, \quad v_5 = f(t)\partial_x - f'(t) y \partial_\psi, \\ &v_6 = g(t)\partial_y + g'(t) x \partial_\psi, \quad v_7 = h(t)\partial_\psi. \end{aligned} \tag{72}$$

Here, ignoring the discussion of the infinite dimensional subalgebra, we apply the new approach to construct the two-dimensional optimal system and the corresponding invariant solutions for the four-dimensional Lie algebra spanned by v_1, v_2, v_3, v_4 in (72).

The commutator table and the adjoint representation table for $\{v_1, v_2, v_3, v_4\}$ are given in Tables V and VI, respectively.

A. Adjoint transformation matrix and the invariant equations

Applying the adjoint action of v_1 to $w_1 = \sum_{i=1}^{4} a_i v_i$, we have

$$Ad_{\exp(\epsilon_1 v_1)}(a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4) = a_1 v_1 + a_2 e^{\epsilon_1} v_2 + a_3 e^{-\epsilon_1} v_3 + a_4 v_4.$$

Hence the corresponding adjoint transformation matrix A_1 is saying

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\epsilon_1} & 0 & 0 \\ 0 & 0 & e^{-\epsilon_1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{73}$$

TABLE VI. Adjoint representation table of the NS equation.

Ad	v_1	v_2	v_3	v_4
v_1	v_1	$e^{\epsilon} v_2$	$e^{-\epsilon} v_3$	v_4
v_2	$v_1 - \epsilon v_2$	v_2	$v_3 - \epsilon v_4$	v_4
v_3	$v_1 + \epsilon v_3$	$v_2 + \epsilon v_4$	v_3	v_4
v_4	v_1	v_2	v_3	v_4

Similarly, one can get

$$A_2 = \begin{pmatrix} 1 & -\epsilon_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\epsilon_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & \epsilon_3 & 0 \\ 0 & 1 & 0 & \epsilon_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_4 = E.$$

Then the most general adjoint matrix A can be taken as

$$A = A_1 A_2 A_3 A_4 = \begin{pmatrix} 1 & -\epsilon_2 & \epsilon_3 & -\epsilon_2\epsilon_3 \\ 0 & e^{\epsilon_1} & 0 & e^{\epsilon_1}\epsilon_3 \\ 0 & 0 & e^{-\epsilon_1} & -e^{-\epsilon_1}\epsilon_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{74}$$

Let

$$w_1 = \sum_{i=1}^{4} a_i v_i, \quad w_2 = \sum_{j=1}^{4} b_j v_j. \tag{75}$$

For the general two-dimensional subalgebra $\{w_1, w_2\}$, the corresponding invariant ϕ is a real function of $a_1, \ldots, a_4, b_1, \ldots, b_4$. Let $v = \sum_{k=1}^{4} c_k v_k$ be a general element of $\mathcal{G}$, then in conjunction with Table V, we have

$$\begin{aligned} Ad_g(w_1) &= Ad_{\exp(\epsilon v)}(w_1) \\ &= w_1 - \epsilon[v, w_1] + \frac{1}{2!}\epsilon^2[v,[v,w_1]] - \cdots \\ &= (a_1 v_1 + \cdots + a_4 v_4) - \epsilon[c_1 v_1 + \cdots + c_4 v_4, a_1 v_1 + \cdots + a_4 v_4] + O(\epsilon^2) \\ &= a_1 v_1 + (a_2 - \epsilon(c_2 a_1 - c_1 a_2))v_2 + (a_3 - \epsilon(c_1 a_3 - c_3 a_1))v_3 \\ &\quad + (a_4 - \epsilon(c_2 a_3 - c_3 a_2))v_4 + O(\epsilon^2). \end{aligned} \tag{76}$$

Similarly, applying the same adjoint action $v = \sum_{k=1}^{4} c_k v_k$ to w_2, we get

$$\begin{aligned} Ad_g(w_2) = b_1 v_1 + (a_2 - \epsilon(c_2 b_1 - c_1 b_2))v_2 + (b_3 - \epsilon(c_1 b_3 - c_3 b_1))v_3 \\ + (b_4 - \epsilon(c_2 b_3 - c_3 b_2))v_4 + O(\epsilon^2). \end{aligned} \tag{77}$$

Two cases are considered in the follows.

(a) When $[w_1, w_2] = 0$, the invariant function $\phi = \phi(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ is determined by seven equations,

$$\begin{aligned} &a_1\phi_{a_1} + a_2\phi_{a_2} + a_3\phi_{a_3} + a_4\phi_{a_4} = 0, \quad a_1\phi_{b_1} + a_2\phi_{b_2} + a_3\phi_{b_3} + a_4\phi_{b_4} = 0, \\ &a_2\phi_{a_2} - a_3\phi_{a_3} + b_2\phi_{b_2} - b_3\phi_{b_3} = 0, \quad a_1\phi_{a_2} + a_3\phi_{a_4} + b_1\phi_{b_2} + b_3\phi_{b_4} = 0, \\ &a_1\phi_{a_3} + a_2\phi_{a_4} + b_1\phi_{b_3} + b_2\phi_{b_4} = 0, \end{aligned} \tag{78}$$

and

$$b_1\phi_{b_1} + b_2\phi_{b_2} + b_3\phi_{b_3} + b_4\phi_{b_4} = 0, \quad b_1\phi_{a_1} + b_2\phi_{a_2} + b_3\phi_{a_3} + b_4\phi_{a_4} = 0. \tag{79}$$

(b) When $[w_1, w_2] = w_1$, the invariant function $\phi = \phi(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ only needs to meet Eqs. (78) in terms of $a_{12} = 0$ and $a_{22} = 0$.

B. Construction of two-dimensional optimal system for the NS equation

Substituting (75) into $[w_1, w_2] = \delta w_1$, the determined equations are found as follows:

$$\delta a_1 = 0, \quad a_2 b_1 - a_1 b_2 = \delta a_2, \quad a_1 b_3 - a_3 b_1 = \delta a_3, \quad a_2 b_3 - a_3 b_2 = \delta a_4. \tag{80}$$

1. The case of $\delta = 0$ in restrictive equations (80)

We consider two cases.

Case 1: Not all a_1 and b_1 are zeroes.

Without loss of generality, we adopt $a_1 \neq 0$. Solving (80), we get

$$b_2 = \frac{a_2 b_1}{a_1}, \quad b_3 = \frac{a_3 b_1}{a_1}, \tag{81}$$

with a_1, a_2, a_3, a_4, b_1, and b_4 being arbitrary.

Substituting condition (81) into Eqs. (78) and (79), we find that $\phi = constant$. Hence according to (81), select the corresponding representative element $\{v_1, v_4\}$. Since Eqs. (49) have the solution

$$k_1' = a_1, \quad k_2' = \frac{a_1 a_4 - a_2 a_3}{a_1}, \quad k_3' = b_1,$$
$$k_4' = \frac{b_4 a_1^2 - a_2 a_3 b_1}{a_1^2}, \quad \epsilon_2 = \frac{e^{\epsilon_1} a_2}{a_1}, \quad \epsilon_3 = -\frac{a_3}{a_1 e^{\epsilon_1}},$$

case (1) is equivalent to $\{v_1, v_4\}$.

Case 2: $a_1 = b_1 = 0$. Now determined equations (80) become

$$a_2 b_3 - a_3 b_2 = 0. \tag{82}$$

Case 2.1: Not all a_2 and b_2 are zeroes and we let $a_2 \neq 0$.

From Eq. (82), we get $b_3 = \frac{a_3 b_2}{a_2}$. Then by solving Eqs. (78) and (79), we find $\phi \equiv constant$. In this case, there exist three circumstances in terms of the sign of $a_2 a_3$.

(i). When $a_3 = 0$, choose the representative element $\{v_2, v_4\}$ and Eqs. (49) have the solution

$$k_1' = e^{\epsilon_1} a_2, \quad k_2' = e^{\epsilon_1}\epsilon_3 a_2 + a_4, \quad k_3' = e^{\epsilon_1} b_2, \quad k_4' = e^{\epsilon_1}\epsilon_3 b_2 + b_4.$$

(ii). For $a_2 a_3 > 0$, we select $\{v_2 + v_3, v_4\}$ as a representative element. Eqs. (49) hold for

$$k_1' = \sqrt{\frac{a_3}{a_2}} a_2, \quad k_2' = \sqrt{\frac{a_3}{a_2}}(\epsilon_3 - \epsilon_2) a_2 + a_4,$$
$$k_3' = \sqrt{\frac{a_3}{a_2}} b_2, \quad k_4' = \sqrt{\frac{a_3}{a_2}}(\epsilon_3 - \epsilon_2) b_2 + b_4.$$

(iii). For $a_2 a_3 < 0$, we select $\{v_2 - v_3, v_4\}$ as a representative element and Eqs. (49) have the solution

$$k_1' = \sqrt{-\frac{a_3}{a_2}} a_2, \quad k_2' = \sqrt{-\frac{a_3}{a_2}}(\epsilon_3 + \epsilon_2) a_2 + a_4,$$
$$k_3' = \sqrt{-\frac{a_3}{a_2}} b_2, \quad k_4' = \sqrt{\frac{a_3}{a_2}}(\epsilon_3 + \epsilon_2) b_2 + b_4.$$

Case 2.2: For $a_2 = b_2 = 0$, Eqs. (49) always stand up and the general two-dimensional Lie algebra becomes $\{a_3 v_3 + a_4 v_4, b_3 v_3 + b_4 v_4\}$. Then if not all a_3 and b_3 are zeroes (and let $a_3 \neq 0$), it will equivalent to $\{v_3, v_4\}$ since that Eqs. (49) have the solution

$$k_1' = e^{-\epsilon_1} a_3, \quad k_2' = -e^{-\epsilon_1}\epsilon_2 a_3 + a_4, \quad k_3 = e^{-\epsilon_1} b_3, \quad k_4' = -e^{-\epsilon_1}\epsilon_2 b_3 + b_4.$$

For the case of $a_3 = b_3 = 0$, the general two-dimensional Lie algebra $\{a_4 v_4, b_4 v_4\}$ is trivial.

2. The case of $\delta = 1$ in restrictive equations (80)

Substituting $\delta = 1$ into Eqs. (80), there must be $a_1 = 0$.

Case 3: $a_2 \neq 0$.

Now, Eqs. (80) require

$$a_3 = 0, \quad a_4 = a_2 b_3, \quad b_1 = 1. \tag{83}$$

Substituting (83) into Eqs. (78), it leads to an invariant for $\{w_1, w_2\}$,

$$\phi = \Delta_3 \equiv b_4 - b_2 b_3. \tag{84}$$

In condition of (83) and $\Delta_3 = c$, choose the corresponding representative element $\{v_2, v_1 + cv_4\}$ and Eqs. (49) have the solution

$$k_1' = a_2, \quad k_2' = 0, \quad k_3' = b_2, \quad k_4' = 1, \quad \epsilon_3 = -b_3, \quad \epsilon_1 = \epsilon_2 = 0. \tag{85}$$

Case 4: $a_2 = 0$.
By solving Eqs. (80), we get

$$b_1 = -1, \quad a_4 = -a_3 b_2. \tag{86}$$

Substituting (86) with $a_1 = a_2 = 0$ into Eqs. (78), one can obtain an invariant as follows:

$$\phi = \Delta_4 \equiv b_4 + b_2 b_3. \tag{87}$$

In condition of (86) and $\Delta_4 = c$, select a representative element $\{v_3, -v_1 + cv_4\}$ and Eqs. (49) have the solution

$$k_1' = a_3, \quad k_2' = 0, \quad k_3' = b_3, \quad k_4' = 1, \quad \epsilon_2 = -b_2, \quad \epsilon_1 = \epsilon_3 = 0.$$

In summary, a two-dimensional optimal system O_2 for the four-dimensional Lie algebra spanned by v_1, v_2, v_3, v_4 in (72) is shown as follows:

$$\begin{aligned} &\mathfrak{g}_1' = \{v_1, v_4\}, \quad \mathfrak{g}_2' = \{v_2, v_4\}, \quad \mathfrak{g}_3' = \{v_2 + v_3, v_4\}, \quad \mathfrak{g}_4' = \{v_2 - v_3, v_4\}, \\ &\mathfrak{g}_5' = \{v_3, v_4\}, \quad \mathfrak{g}_6' = \{v_2, v_1 + cv_4\}, \quad \mathfrak{g}_7' = \{v_3, -v_1 + cv_4\}, \quad (c \in \mathbb{R}). \end{aligned} \tag{88}$$

C. Two-dimensional reductions for the NS equation

Using two-dimensional optimal system (88), one can reduce the (2+1)-dimensional NS equation to some ordinary differential equations and further get rich group invariant solutions. For the case of $\mathfrak{g}_1' = \{v_1, v_4\}$ and $\mathfrak{g}_2' = \{v_2, v_4\}$, one can refer to Ref. 18. The case of $\mathfrak{g}_5'$ leads to no group invariant solutions. Then we just consider the rest elements in (88).

(a) $\mathfrak{g}_3' = \{v_2 + v_3, v_4\}$ and $\mathfrak{g}_4' = \{v_2 - v_3, v_4\}$. By solving $(v_2 \pm v_3)(\psi) = 0$ and $v_4(\psi) = 0$, we have $\psi = F(x^2 + y^2) \pm \frac{1}{2} t(x^2 + y^2)$. Substituting it into Eq. (71), one can get

$$8\gamma[\xi^2 F^{(4)}(\xi) + 4\xi F'''(\xi) + 2F''(\xi)] \mp 1 = 0, \tag{89}$$

with $\xi = x^2 + y^2$. By solving Eq. (89), we find $\mathfrak{g}_3'$ and $\mathfrak{g}_4'$ lead to the same group invariant solution

$$\begin{aligned} \psi = {} & c_1 + c_2(x^2 + y^2) + c_3 \ln(x^2 + y^2) + c_4(x^2 + y^2)[\ln(x^2 + y^2) - 1] \\ & + \frac{1}{32\gamma}(x^2 + y^2)^2 + \frac{1}{2} t(x^2 + y^2). \end{aligned}$$

(b) $\mathfrak{g}_6' = \{v_2, v_1 + cv_4\}$. From $v_2(\psi) = 0$ and $(v_1 + cv_4)(\psi) = 0$, one can get $\psi = F(\arctan(\frac{y}{x}) - c\ln(x^2 + y^2))$. Substituting it into Eq. (71) and integrating the reduced equation once, we have

$$\gamma[(4c^2 + 1)G''(\xi) + 8cG'(\xi) + 4G(\xi)] - G^2(\xi) = 0, \quad (G(\xi) = F'(\xi)), \tag{90}$$

with $\xi = \arctan(\frac{y}{x}) - c\ln(x^2 + y^2)$. Specially, when $c = 0$ in Eq. (90), there is a solution

$$G(\xi) = -6\gamma \operatorname{sech}^2(\xi + c_0) + 4\gamma. \tag{91}$$

Then it leads to the solution of the NS equation

$$\psi = -6\gamma \tanh(\arctan(\frac{y}{x}) + c_0) + 4\gamma \arctan(\frac{y}{x}) + c_1. \tag{92}$$

(c) $\mathfrak{g}_7' = \{v_3, v_1 + cv_4\}$. In this case, we have $\psi = \arctan(\frac{x}{y}) + c\ln(t) + F(\frac{x^2+y^2}{t})$. The reduced equation for Eq. (71) is

$$4\gamma Z^2 G''(Z) + Z(Z + 8\gamma - 2)G'(Z) + (Z - 2)G(Z) = 0, (F'(Z) = G(Z)), \tag{93}$$

with $Z = \frac{x^2+y^2}{t}$.

In particular, for $\gamma = \frac{1}{4}$, we obtain a solution

$$\psi = \arctan\left(\frac{x}{y}\right) + c\ln(t) + c_1 + c_2\ln(Z) + c_3(2\ln(Z) + 3e^{-Z} + Ze^{-Z} + 2\mathrm{Ei}(1,Z) + \frac{4}{3}), \quad (94)$$

for $\gamma = 1$, there is

$$\psi = \arctan\left(\frac{x}{y}\right) + c\ln(t) + c_1 + c_2\ln(Z) + c_3\Big(\sqrt{\pi}\mathrm{erf}(\frac{\sqrt{Z}}{2}) - \sqrt{Z}\mathrm{hypergeom}([\frac{1}{2},\frac{1}{2}],[\frac{3}{2},\frac{3}{2}],-\frac{Z}{4})\Big). \quad (95)$$

Here, the special function "Ei" in (94) is the exponential integral, described by

$$\mathrm{Ei}(1,z) = \int_1^\infty \frac{1}{e^{xz}x}dx. \quad (96)$$

In (95), the error function "erf" is defined by

$$\mathrm{erf}(x) = \frac{2\int_0^x e^{-t^2}dt}{\sqrt{\pi}}, \quad (97)$$

while the "hypergeom(n,d,z)" calling sequence with $n = [n_1, n_2, \ldots, n_p]$ and $d = [d_1, d_2, \ldots, d_q]$ is the generalized hypergeometric function,

$$\mathrm{hypergeom}(n,d,z) = \sum_{k=0}^{\infty} \frac{z^k \prod_{i=1}^{p} \mathrm{pochhammer}(n_i,k)}{k! \prod_{j=1}^{q} \mathrm{pochhammer}(d_j,k)}, \quad (98)$$

where the "pochhammer" function is defined for a positive integer k as

$$\mathrm{pochhammer}(z,k) = z(z+1)(z+2)\cdots(z+k-1). \quad (99)$$

V. SUMMARY AND DISCUSSION

Since many important equations arising from physics are of low dimensions, only the determination of small parameter optimal systems can reduce them to ODEs which often lead to inequivalent group invariant solutions. In this paper, we give an elementary algorithm for constructing two-dimensional optimal system which only depends on fragments of the theory of Lie algebras. The intrinsic idea of our method is that every element in the optimal system corresponds to different values of invariants, the definition of which have been refined in this paper. Thanks to these invariants which are often overlooked except the Killing form in the almost existing methods, all the elements in the two-dimensional optimal system are found one by one and their inequivalences are evident, with no further proof. Moreover, the construction of two-dimensional optimal system in this paper starts from the algebra directly, which does not require the prior one-dimensional optimal system as usual.

Before manipulating the given algorithm to construct two-dimensional optimal system, one should make a refinement for the two-dimensional algebra and compute the general adjoint transformation matrix with the invariants equations, which seem much complicated but in fact can all be carried out in mechanization with the compute software *"Maple."* A new method is shown to provide all the invariants for the two-dimensional subalgebras, which is based on the idea of "invariant" under the meaning of both adjoint transformation and combination act. Applying the algorithm to the heat equation, Novikov equation, and NS equation, we obtain their two-dimensional optimal systems, respectively. For the heat equation, the obtained two-dimensional optimal system contains eleven elements, which are discovered more comprehensive than that in Ref. 10 after a detailed comparison. For the NS equation, all the reduced ordinary differential equations and some exact group invariant solutions which come from the obtained two-dimensional optimal system are found.

The algorithm considered in this paper is elementary and practical, without too much algebraic knowledge. Since the designed algorithm essentially starts from the algebra of the differential equations rather than the equations themselves, the method can also be applied to ODEs and systems of differential equations. Due to the programmatic process, to give a Maple package on the computer for two-dimensional optimal system is necessary and under our consideration. How to apply all the invariants to construct r-parameter ($r > 2$) optimal systems is also an interesting job.

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[1] L. V. Ovsiannikov, *Group Analysis of Differential Equations* (Academic Press, New York, 1982).
[2] F. Galas and E. W. Richet, Physica D **50**, 297 (1991).
[3] N. H. Ibragimov, *Lie Group Analysis of Differential Equations* (CRC Press, Boca Raton, 1994).
[4] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York, 1993).
[5] J. Patera, P. Winternitz, and H. Zassenhaus, J. Math. Phys. **16**, 1597 (1975).
[6] J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus, J. Math. Phys. **17**, 986 (1976).
[7] J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus, J. Math. Phys. **17**, 977 (1976).
[8] J. Patera and P. Winternitz, J. Math. Phys. **18**, 1449 (1977).
[9] S. V. Coggeshalla and J. Meyer-ter-Vehn, J. Math. Phys. **33**, 3585 (1992).
[10] K. S. Chou, G. X. Li, and C. Z. Qu, J. Math. Anal. Appl. **261**, 741 (2001).
[11] K. S. Chou and C. Z. Qu, Acta Appl. Math. **83**, 257 (2004).
[12] K. S. Chou and G. X. Li, Commun. Anal. Geom. **10**(2), 241 (2002).
[13] X. R. Hu, Y. Q. Li, and Y. Chen, J. Math. Phys. **56**, 053504 (2015).
[14] V. S. Novikov, J. Phys. A: Math. Theor. **42**, 342002 (2009).
[15] Y. Bozhkov, I. L. Freire, and N. H. Ibragimov, Comput. Appl. Math. **33**, 193 (2014).
[16] D. Sundkvist, V. Krasnoselskikh, P. K. Shukla, A. Vaivads, M. André, S. Buchert, and H. Réme, Nature **436**, 825 (2005).
[17] G. Pedrizzetti, Phys. Rev. Lett. **94**, 194502 (2005).
[18] X. R. Hu, Z. Z. Dong, F. Huang, and Y. Chen, Z. Naturforsch. **65a**, 504 (2010), http://www.znaturforsch.com/s65a/s65a0504.pdf.