Exact analytical solutions of the generalized Calogero–Bogoyavlenskii–Schiff equation using symbolic computation

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By means of generalized Riccati equation expansion method and symbolic computation, some exact analytical solutions, which contain soliton-like solutions and periodic-like solutions to the generalized Calogero–Bogoyavlenskii–Schiff (GCBS) equation, are obtained. From our results, the solitary-wave solutions and previously known soliton-like solutions of the special cases of GCBS equation can be recovered.

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1 Introduction

One of the most efficient and direct method for constructing solitary wave solutions of nonlinear evolution equations (NEEs) is tanh method (see Refs., e.g., [1,2]). Recent years, much work have been done on the various extensions and applications of tanh method [3–19]. In particular, because of the striking success of the symbolic computation discipline, symbolic systems, such as *Maple, Mathematica*, become more and more useful in the investigation of some NEEs [20–22]. Recently, based on the tanh method and its various extended forms [1–19], using symbolic computation we presented a generalized Riccati equation expansion (GREE) method for constructing exact analytical solutions, which include soliton-like solutions and periodic-like solutions for some NEEs [23,24].

In this paper, using the GREE method, we investigate the existence of more general analytical solutions for the generalized Calogero–Bogoyavlenskii–Schiff (GCBS)

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$$\alpha u_{xt}+\beta u_x u_{xy}+\delta u_y u_{xx}+u_{xxxy}=0, \tag{1}$$

where α, β and δ are constants.

Some special cases of Eq.(1) have been studied by many authors. When setting $\alpha=4$, $\beta=4$, $\delta=2$, Eq. (1) becomes the Bogoyavlenskii–Schiff equation [25,26]:

$$4u_{xt}+4u_x u_{xy}+2u_{xx}u_y+u_{xxxy}=0. \tag{2}$$

When setting $\alpha=1$, $\beta=-4$ and $\delta=-2$, Eq. (1) becomes the breaking soliton equation [23,27–31],

$$u_{xt}-4u_x u_{xy}-2u_y u_{xx}+u_{xxxy}=0. \tag{3}$$

Equation (3) describes the (2+1)-dimensional interaction of Riemann wave propagation with the long-wave propagation [29,31]. Some solutions of Eq. (3) are given in [27,28,30]. The self-dual Yang–Mills equation is shown to belong to the class of (2+1)-dimensional breaking soliton equations [32].

When setting $\alpha=4$, $\beta=8$ and $\delta=4$, Eq. (1) becomes the Calogero–Bogoyavlenskii–Schiff (CBS) equation [33,34],

$$4u_{xt}+8u_x u_{xy}+4u_y u_{xx}+u_{xxxy}=0. \tag{4}$$

If we make the transformations $u\to -\frac{1}{2}u$, $t\to \frac{1}{4}t$, then (4) reduces to the breaking soliton equation (3).

Now let us recall the main steps of the GREE method. Consider a given nonlinear evolution equation with one physical field $u(x,y,t)$ in three variables $\{x,\ y,\ t\}$:

$$H(u,u_t,u_x,u_y,u_{xx},u_{xt},u_{xy},u_{yt},\cdots)=0. \tag{5}$$

Step 1. We assume that Eq. (5) has the solutions in the form

$$u(x,y,t)=a_0+\sum_{i=1}^{m}[a_i\phi^i(\xi)+b_i\phi^{i-1}(\xi)\sqrt{R+\phi^2(\xi)}+k_i\phi^{-i}(\xi)], \tag{6}$$

and

$$\frac{\mathrm{d}\phi(\xi)}{\mathrm{d}\xi}=R+\phi^2(\xi), \tag{7}$$

where m is an integer to be determined by balancing the highest-order derivative terms with the nonlinear terms in Eq. (5), R is a real constant and $a_0=a_0(x,y,t)$, $a_i=a_i(x,y,t)$, $b_i=b_i(x,y,t)$, $k_i=k_i(x,y,t)$, $(i=1,\cdots,m)$, $\xi=\xi(x,y,t)$ are all differentiable functions.

Step 2. Substituting (6) and (7) into (5), multiplying the most simple common denominator in the obtained system, setting the coefficients of $\phi^j(\xi)(\sqrt{R+\phi^2(\xi)})^n$ $(j=0,1,\cdots;\ n=0,1)$ to zero, will yield a set of over-determined partial differential equations (PDEs) with regard to arbitrary functions a_0, a_i, b_i, k_i $(i=1,\cdots,m)$ and ξ.

Step 3. Solving the set of over-determined PDEs by use of the PDE tools package of *Maple*, we would end up with the explicit expressions for a_0, a_i, b_i, k_i ($i = 1, \cdots, m$) and ξ or the constrains among them.

Step 4. It is well known that the general solutions of Riccati equation (7) are

$$\phi(\xi) = \begin{cases} -\sqrt{-R}\tanh(\sqrt{-R}\xi), & R < 0, \\ -\sqrt{-R}\coth(\sqrt{-R}\xi), & R < 0, \\ \sqrt{R}\tan(\sqrt{R}\xi), & R > 0, \\ -\sqrt{R}\cot(\sqrt{R}\xi), & R > 0, \\ -\dfrac{1}{\xi}, & R = 0. \end{cases} \tag{8}$$

Thus according to Eqs. (6), (8) and the conclusions in *Step 3*, some soliton-like solutions, periodic-like solutions and rational solutions of Eq. (5) can be obtained.

Remark 1. In general, it is very difficult, sometimes impossible, to solve the set of over-determined PDEs in *Step 2*. As the calculation goes on, in order to drastically simplify the work or make the work feasible, we often choose special function forms for a_0, a_i, b_i, k_i ($i = 1, \cdots, m$) and ξ, on a trial-and-error basis. (As we do in Section 2.)

2 Soliton-like solutions and periodic-like solutions

In this section, by use of the GREE method and symbolic computation, we investigate the soliton-like solutions and periodic-like solutions for the GCBS equation (1).

By balancing the highest-order contributions from both the linear and nonlinear terms in Eq. (1), we obtain $m = 1$ in (5). Therefore we assume the solutions of Eq. (1) in the following special form

$$u(x, y, t) = a_0 + b_0 x + a_1\phi(\xi) + b_1\sqrt{R + \phi^2(\xi)} + \frac{k_1}{\phi(\xi)}, \tag{9}$$

where $a_0 = a_0(y, t)$, $b_0 = b_0(t)$, $a_1 = a_1(y, t)$, $b_1 = b_1(y, t)$, $k_1 = k_1(y, t)$, and $\xi = xp(y, t) + q(y, t)$ are all differentiable functions and $\phi(\xi)$ satisfies (6).

Substituting (9) along with (6) into (1), multiplying $\phi(\xi)^5\sqrt{R + \phi(\xi)^2}$ in the obtained system, then setting the coefficients of $\phi(\xi)^i(R + \phi(\xi)^2)^{j/2}$ ($j = 0, 1; i = 0, 1, 2, \cdots$) in the obtained system of partial differential equation to zero (Notice that a_0, b_0, a_1, b_1, k_1, p and q are independent of x), we can deduce a set of over-determined PDEs with respect to the unknown functions a_0, b_0, a_1, b_1, k_1, p and q. In order to keep the continuity of the paper, we move the set of over-determined PDEs to the Appendix at the end of the paper.

Note 1:

(1) In the rest of this paper a_{1y} denotes $\partial a_1(y, t)/\partial y$, and so on.

(2) It is necessary to point out that, when setting $b_0 = b_0(y, t)$, the corresponding set of over-determined PDEs of $\{a_0, b_0, a_1, b_1, k_1, p, q.\}$ can not be solved by *Maple*. Therefore we choose $b_0 = b_0(t)$ in the paper.

Using the powerful PDEtools package of *Maple*, solving the set of PDEs ($A1$)-($A32$), we can obtain the following non-trivial solutions.

Note 2:
(1) In the following cases, $q = q$ denotes that q is an arbitrary function of $\{y, t\}$ and $\alpha = \alpha$ denotes α is an arbitrary constant, and so on. C_1, C_2, C_3 are all arbitrary constants.
(2) $F_1(t)$ and $F_2(t)$ denote arbitrary differentiable function of t.
(3) We omit the solutions with $p = 0$ or the solutions with $q = F(t)$, where $F(t)$ is an arbitrary function of t.

Case 1.

$$\alpha = \alpha, \quad q = q, \quad p = C_1, \quad b_1 = k_1 = 0, \quad \beta = \beta, \quad b_0 = C_2, \quad \delta = \delta,$$
$$a_0 = \int \frac{4C_1^2 q_y R - \alpha q_t - \beta C_2 q_y}{C_1 \delta} \mathrm{d}y + F_1(t), \quad a_1 = -\frac{12C_1}{\delta + \beta} \,. \tag{10}$$

Case 2.

$$\alpha = \alpha, \quad q = q, \quad p = C_1, \quad k_1 = 0, \quad \beta = \beta, \quad b_1 = \pm\frac{6C_1}{\delta + \beta}, \quad a_1 = -\frac{6C_1}{\delta + \beta} \,,$$
$$\delta = \delta, \quad b_0 = C_3, \quad a_0 = \int \frac{C_1^2 q_y R - \alpha q_t - \beta C_3 q_y}{C_1 \delta} \mathrm{d}y + F_1(t) \,. \tag{11}$$

Case 3.

$$\beta = 2\delta, \quad q = q, \quad b_1 = -\frac{2p}{\delta}, \quad k_1 = 0, \quad a_1 = -\frac{2p}{\delta}, \quad \delta = \delta, \quad \alpha = \frac{2pRp_y^2 p_t}{p_y p_{tt} - p_{ty} p_t},$$

$$\begin{aligned} a_0 &= \int \frac{-2Rp_y p_t(-p_y q_t + p_t q_y)}{\delta(-p_y p_{tt} + p_{ty} p_t)} \mathrm{d}y + F_1(t) \,, \\ b_0 &= \frac{1}{2} \frac{-p^2 R p_{ty} p_t + p^2 R p_y p_{tt} - 2pRp_y p_t^2}{\delta(p_y p_{tt} - p_{ty} p_t)} \,, \end{aligned} \tag{12}$$

where

$$\begin{cases} p_{tty} = -\dfrac{(p_y^2 p_{tt} p_t^2 - p_y^2 p p_t p_{ttt} + p_y^2 p_{tt}^2 p + p_y p_{tt} p p_t p_{ty} - p_y p_{ty} p_t^3 - 2p_{ty}^2 p_t^2 p)}{p p_t^2 p_y}, \\ p_{yy} = \dfrac{2p_y p_{ty} p_t - p_{tt} p_y^2}{p_t^2}. \end{cases} \tag{13}$$

Case 4.

$$\begin{aligned} &\beta = 2\delta, \quad q = q, \quad a_1 = -\frac{4p}{\delta}, \quad b_1 = k_1 = 0, \quad \delta = \delta, \\ &a_0 = \int -\frac{8Rp_y p_t(p_y q_t - p_t q_y)}{\delta(p_y p_{tt} - p_{ty} p_t)} \mathrm{d}y + F_1(t), \\ &\alpha = -\frac{8p_y^2 p R p_t}{-p_y p_{tt} + p_{ty} p_t}, \quad b_0 = \frac{2pR(-p p_{ty} p_t + p p_y p_{tt} - 2p_y p_t^2)}{\delta(p_y p_{tt} - p_{ty} p_t)}, \end{aligned} \tag{14}$$

where

$$\begin{cases} p_{tty} = \dfrac{(-p_y^2 p_{tt} p_t^2 + p_y^2 p p_t p_{ttt} - p_y^2 p_{tt}^2 p - p_y p_{tt} p p_t p_{ty} + p_y p_{ty} p_t^3 + 2 p_{ty}^2 p_t^2 p)}{p_y p p_t^2}, \\ p_{yy} = \dfrac{(2 p_y p_{ty} p_t - p_{tt} p_y^2)}{p_t^2}. \end{cases} \tag{15}$$

Case 5.

$$\alpha = \alpha, \quad q = q, \quad p = C_1, \quad b_1 = 0, \quad \beta = \beta, \quad b_0 = C_2, \quad \delta = \delta, \quad a_1 = -\frac{12 C_1}{\delta + \beta},$$

$$k_1 = \frac{12 C_1 R}{\delta + \beta}, \quad a_0 = \int \frac{16 C_1^2 q_y R - \alpha q_t - \beta C_2 q_y}{C_1 \delta} \mathrm{d}y + F_1(t) \, . \tag{16}$$

Case 6.

$$\beta = 2\delta, \quad q = q, \quad a_1 = -\frac{4p}{\delta}, \quad b_1 = 0, \quad k_1 = \frac{4pR}{\delta}, \quad \delta = \delta, \quad \alpha = \frac{32 p R p_y^2 p_t}{p_y p_{tt} - p_{ty} p_t},$$

$$a_0 = \int \frac{32 R p_y p_t (-p_y q_t + p_t q_y)}{\delta (p_y p_{tt} - p_{ty} p_t)} \mathrm{d}y + F_1(t), \quad b_0 = -\frac{8pR(p p_{ty} p_t - p p_y p_{tt} + 2 p_y p_t^2)}{\delta (p_y p_{tt} - p_{ty} p_t)}, \tag{17}$$

where

$$\begin{cases} p_{tty} = \dfrac{(-p_y^2 p_{tt} p_t^2 + p_y^2 p p_t p_{ttt} - p_y^2 p_{tt}^2 p - p_y p_{tt} p p_t p_{ty} + p_y p_{ty} p_t^3 + 2 p_{ty}^2 p_t^2 p)}{p_y p p_t^2}, \\ p_{yy} = -\dfrac{-2 p_y p_{ty} p_t + p_{tt} p_y^2}{p_t^2}. \end{cases} \tag{18}$$

From (8), (9), and Cases 1–6, we can obtain the following families of solutions for the GCBS equation (1), which contain soliton-like solutions, singular soliton-like solutions and periodic-like solutions.

Family 1: From Case 1, we can obtain the following solutions:

$$u_{11} = a_0 + C_2 x + \frac{12 C_1}{\delta + \beta} \sqrt{-R} \tanh \left\{ \sqrt{-R} \, [C_1 x + q] \right\}, \quad R < 0, \tag{19}$$

$$u_{12} = a_0 + C_2 x + \frac{12 C_1}{\delta + \beta} \sqrt{-R} \coth \left\{ \sqrt{-R} \, [C_1 x + q] \right\}, \quad R < 0, \tag{20}$$

$$u_{13} = a_0 + C_2 x - \frac{12 C_1}{\delta + \beta} \sqrt{R} \tan \left\{ \sqrt{R} \, [C_1 x + q] \right\}, \quad R > 0, \tag{21}$$

$$u_{14} = a_0 + C_2 x + \frac{12 C_1}{\delta + \beta} \sqrt{R} \cot \left\{ \sqrt{R} \, [C_1 x + q] \right\}, \quad R > 0, \tag{22}$$

where $a_0 = \int (4 C_1^2 q_y R - \alpha q_t - \beta C_2 q_y)/(C_1 \delta) \mathrm{d}y + F_1(t)$ and q is an arbitrary function of $\{y, t\}$.

Remark 2: We call the solutions (19), (20) solitary wave solutions if they contain the variable $\xi = C_1 x + q(y,t)$ that is a linear form of $\{x, y, t\}$ and $C_2 = 0$, $a_0 =$ const. Otherwise, we call them soliton-like solutions. We call the solutions (53), (54) periodic-like solutions.

Family 2: From Case 2, we can obtain the following solutions:

$$u_{21} = a_0 + C_3 x + \frac{6C_1\sqrt{-R}}{\alpha+\beta}\Big[\tanh(\sqrt{-R}\,\xi) \pm \mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,\xi)\Big], \quad R<0, \tag{23}$$

$$u_{22} = a_0 + C_3 x + \frac{6C_1\sqrt{-R}}{\alpha+\beta}\Big[\coth(\sqrt{-R}\,\xi) \pm \mathrm{csch}(\sqrt{-R}\,\xi)\Big], \quad R<0, \tag{24}$$

$$u_{23} = a_0 + C_3 x - \frac{6C_1\sqrt{R}}{\alpha+\beta}\Big[\tan(\sqrt{R}\,\xi) \pm \sec(\sqrt{R}\,\xi)\Big], \quad R>0, \tag{25}$$

$$u_{24} = a_0 + C_3 x + \frac{6C_1\sqrt{R}}{\alpha+\beta}\Big[\cot(\sqrt{R}\,\xi) \pm \csc(\sqrt{R}\,\xi)\Big], \quad R>0, \tag{26}$$

where $a_0 = \int (C_1^2 q_y R - \alpha q_t - \beta C_3 q_y)(C_1\delta)\mathrm{d}y + F_1(t)$, $\xi = C_1 x + q$ and q is an arbitrary function of $\{y, t\}$.

Family 3: From Case 3, we can obtain the following solutions:

$$u_{31} = a_0 + b_0 x - a_1\sqrt{-R}\Big[\tanh(\sqrt{-R}\,(px+q)) \pm \mathrm{i}\,\mathrm{sech}(\sqrt{-R}\,(px+q))\Big], \quad R<0, \tag{27}$$

$$u_{32} = a_0 + b_0 x - a_1\sqrt{-R}\Big[\coth(\sqrt{-R}\,(px+q)) \pm \mathrm{csch}(\sqrt{-R}\,(px+q))\Big], \quad R<0, \tag{28}$$

$$u_{33} = a_0 + b_0 x + a_1\sqrt{R}\Big[\tan(\sqrt{R}\,(px+q)) \pm \sec(\sqrt{R}\,(px+q))\Big], \quad R>0, \tag{29}$$

$$u_{34} = a_0 + b_0 x - a_1\sqrt{R}\Big[\cot(\sqrt{R}\,(px+q)) \pm \csc(\sqrt{R}\,(px+q))\Big], \quad R>0, \tag{30}$$

where $p, q, a_0, b_0, a_1, \alpha, \beta, \delta$ are constrained by (12) and (13).

Family 4: From Case 4, we can obtain the following solutions:

$$u_{41} = a_0 + b_0 x - a_1\sqrt{-R}\tanh\Big\{\sqrt{-R}\,[px+q]\Big\}, \quad R<0, \tag{31}$$

$$u_{42} = a_0 + b_0 x - a_1\sqrt{-R}\coth\Big\{\sqrt{-R}\,[px+q]\Big\}, \quad R<0, \tag{32}$$

$$u_{43} = a_0 + b_0 x + a_1\sqrt{R}\tan\Big\{\sqrt{R}\,[px+q]\Big\}, \quad R>0, \tag{33}$$

$$u_{44} = a_0 + b_0 x - a_1\sqrt{R}\cot\Big\{\sqrt{R}\,[px+q]\Big\}, \quad R>0, \tag{34}$$

where $a_0, b_0, a_1, p, q, \alpha, \beta, \delta$ are constrained by (14) and (15).

Family 5: From Case 5, we can obtain the following solutions:

$$u_{51} = a_0 + C_2 x + \frac{12C_1}{\delta+\beta}\sqrt{-R}\Big[\tanh(\sqrt{-R}\,(C_1x+q)) \pm \coth(\sqrt{-R}\,(C_1x+q)\Big], \quad R<0, \tag{35}$$

$$u_{52} = a_0 + C_2 x + \frac{12C_1}{\delta+\beta}\sqrt{R}\Big[\tan(\sqrt{R}\,(C_1x+q)) \pm \cot(\sqrt{R}\,(C_1x+q))\Big], \quad R>0, \tag{36}$$

where $a_0 = \int (16C_1^2 q_y R - \alpha q_t - \beta C_2 q_y)/(C_1\delta)\mathrm{d}y + F_1(t)$ and α, β, δ are arbitrary constants.

Family 6: From Case 6, we can obtain the following solutions:

$$u_{61} = a_0 + b_0 x - a_1\sqrt{-R}\Big[\tanh(\sqrt{-R}\,(px+q)) \pm \coth(\sqrt{-R}\,(px+q))\Big], \quad R<0, \tag{37}$$

$$u_{62} = a_0 + b_0 - a_1\sqrt{R}\Big[\tan(\sqrt{R}\,(px+q)) \pm \cot(\sqrt{R}\,(px+q))\Big], \quad R>0. \tag{38}$$

where $a_0, b_0, a_1, p, q, \alpha, \beta, \delta$ are constrained by (17) and (18).

Remark 3: From the solution (31), when setting $\alpha = 4$, $\beta = 8$, $\delta = 4$ and $b_0 = k = \text{const}$, with the help of *Maple*, we can deduce the corresponding solutions of CBS equation (4) as follows:

$$u(x,y,t) = a_0 + kx + p\sqrt{-R}\tanh[\sqrt{-R}\,(xp + q(y,t))], \tag{39}$$

where

$$a_0 = \int \frac{p^2 R q_y - 2kq_y - q_t}{p}\mathrm{d}y + F_2(t), \quad p = \mathrm{RootOf}(F_1(\_Z) - y - tR\_Z^2 + 2tk). \tag{40}$$

It is easy to see that solution (39), (40) is just one part of solution what we got from *Maple*. This particular solution belongs to general solution (31), namely that it fulfills all four PDEs for p included in (14) and (15).

Note 3. In the above expressions, $F(\_Z)$ denotes arbitrary function with respect to $\_Z$ and $\mathrm{RootOf}(F_1(\_Z) - y - tR\_Z^2 + 2tk)$ denotes all the roots of the equation $F_1(\_Z) - y - tR\_Z^2 + 2tk = 0$ with respect to one variable $\_Z$. At the same time, $p = \mathrm{RootOf}(F_1(\_Z) - y - tR\_Z^2 + 2tk)$ is general solutions of the equation $p_t + (-Rp^2 + 2k)p_y = 0$. (With respect to these results, the reader can see **Help** of *Maple* and Refs. [20–22] for detail).

If further setting $R = -\frac{1}{4}$, $p = \Theta(y,t)$, $q = \Psi(y,t) - \ln P(y,t)$ and $a_0 = h(y,t) + \frac{1}{2}\Theta(y,t)$, the solution (39) with (40) is changed into

$$u(x,y,t) = \tfrac{1}{2}\Theta(y,t)\Big\{1 + \tanh\{\tfrac{1}{2}[\Theta(y,t)x + \Psi(y,t) - \ln P(y,t)]\}\Big\} + kx + h(y,t), \tag{41}$$

where $\Theta(y,t), \Psi(y,t), k, h(y,t)$ are constrained by the following conditions:

$$\begin{cases} 4\Theta_t + \Theta^2\Theta_y + 8k\Theta_y = 0, \\ 4P_t + P_y\Theta^2 + 8kP_y - P(4\Theta_t + \Theta^2\Psi_y + 2\Theta\Theta_y + 4h_y\Theta + 8k\Psi_y) = 0. \end{cases} \tag{42}$$

Thus it is easy to see that the solution (41) with the constraints (42) is the same as the solutions (14a) with the constraints (13b) in [33]. Also from the solutions (32) we can recover the solution (14b) with the constraints (13b) in [33] by similar analysis.

Therefore the results obtained in [33] can be recovered by solutions (31) and (32).

Remark 4. When setting $\alpha = 1$, $\beta = -4$, $\delta = -2$, $b_0 = k = \text{const}$, the corresponding solution of (31) reduces to

$$u = a_0 + kx - 2p\sqrt{-R}\tanh[\sqrt{-R}\,(xp(y,t) + q(y,t))], \tag{43}$$

where

$$a_0 = \frac{1}{2}\int \frac{-4p^2Rq_y - 4kq_y + q_t}{p}\mathrm{d}y + F_2(t),$$

$$p = \mathrm{RootOf}(4F_1(\_Z)R\_Z^2 + 4F_1(\_Z)k - y - 4tR\_Z^2 - 4tk). \tag{44}$$

If further setting the parameters to be equal to various special forms, the results in [27] can be recovered. For example, if setting $k = 0$, from (44) we can deduce

$$p = \mathrm{RootOf}(-y - 4tR\_Z^2) = \pm\sqrt{-\frac{y}{4Rt}}. \tag{45}$$

Therefore we can obtain the following exact analytical solutions for Eq. (3):

$$u(x,y,t) = \int \frac{yq_y + tq_t}{t\sqrt{-\dfrac{y}{Rt}}} + F_2(t) \mp \sqrt{\frac{y}{t}}\tanh\left[\sqrt{-R}\left(\pm\sqrt{-\frac{y}{4Rt}}x + q(y,t)\right)\right], \tag{46}$$

where $q(y,t)$ is an arbitrary function of $\{y,t\}$.

If further setting $q(y,t) = -\ln\left(c_1\sqrt{y/t} + c_2\right) - (1+\alpha)\ln(y) + \alpha\ln(t)$, the solution (46) reduce to

$$\begin{aligned} u(x,y,t) = {} & 2\sqrt{-\frac{y}{Rt}}R + F_2(t) \mp \sqrt{\frac{y}{t}}\tanh\left\{\sqrt{-R}\left[\pm\sqrt{-\frac{y}{4Rt}}x\right.\right. \\ & \left.\left. - \ln\left(c_1\sqrt{\frac{y}{t}} + c_2\right) - (1+\alpha)\ln(y) + \alpha\ln(t)\right]\right\}. \end{aligned} \tag{47}$$

It is easy to see that we only set $R = -1/4$, the solution (20a) obtained in [27] is recovered. Through similar analysis, the results obtained in [27] can be obtained by our solutions (31) and (32).

Therefore the results obtained in [27] are the special cases of our solutions (31) and (32).

Remark 5. When setting $\alpha = 1$, $\beta = -4$, $\delta = -2$, $b_0 = 0$, the results obtained in [23] and the travelling wave solutions obtained in [30] of Eq. (3) can be recovered by *Family 1–6* and by *Family 1*, *Family 5*, respectively. Here we only take the solution (35) as an explanation. Under the above conditions and $R < 0$, (35) reduce to

$$u = \int \frac{16C_1^2 q_y R - q_t}{-2C_1} \mathrm{d}y + F_1(t) - 2C_1\sqrt{-R}\Big[\tanh(\sqrt{-R}\,\xi) \pm \coth(\sqrt{-R}\,\xi\Big], \quad (48)$$

where $\xi = C_1 x + q(y,t)$.

We only set $C_1 \to -C_1/(2R)$, the solution (3.69) obtained in [23] can be reproduced.

If set $\int (16C_1^2 q_y R - q_t)/(-2C_1)\mathrm{d}y = 0$, $F_1(t) = c = \text{const}$ and $q(y,t) = \lambda_2 x + \lambda_3 t$, we deduce the solution (48) is changed into the following travelling wave solution

$$u = c - 2C_1\sqrt{-R}\Big[\tanh(\sqrt{-R}\,\xi) \pm \coth(\sqrt{-R}\,\xi\Big], \quad (49)$$

where $\xi = C_1 x + \lambda_2 y + 16\,{C_1}^2 R\lambda_2 t$. Through verification, there are some sign errors in the corresponding solution (20) given in the work in [30].

Remark 6. To our knowledge, except the special solutions discussed above, the other solutions of GBCS equation (1) are not reported before. We hope that it may be of some help to study the physical problem of GCBS equation (1) by a judicious choice of the free parameters occurring in the solutions obtained in the paper.

3 Conclusions

In summary, based on the computerized symbolic computation and a generalized Riccati equation expansion method, we study the generalized Calogero–Bogoyavlenskii–Schiff equation and obtain many families of exact analytical solutions, which include soliton-like solutions, singular soliton-like solutions and periodic-like solutions. From our results, some results previously known as travelling wave solutions and soliton-like solutions can be recovered. It is necessary to point out that, to make the work feasible (or to obtain explicit exact soliton-like solutions), how to choose the forms for a_0, a_i, b_i, k_i $(i = 1, \cdots, m)$ and ξ in the ansatz would be the key step in the computation of the generalized Riccati equation expansion method. The method may be extended to find exact soliton-like solutions of other NEEs and coupled NEEs.

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Appendix: The set of over-determined PDEs in Section 2

$$4p^2q_yb_1(a_1\beta a_1\alpha+6p\delta)=0, \tag{A1}$$

$$4p^2b_1p_y(a_1\beta a_1\alpha+6p\delta)=0, \tag{A2}$$

$$p(18p\delta b_1p_y+p\alpha a_1b_{1y}+p\alpha b_1a_{1y}+2p\beta a_{1y}b_1 \\ +2p\beta b_{1y}a_1+2\alpha a_1b_1p_y+6p^2\delta b_{1y})=0, \tag{A3}$$

$$pb_1(2q_t+2q_yb_0\alpha+9q_yRa_1\beta p-2q_yk_1\beta p+2a_{0y}\beta p \\ -2q_yk_1\alpha p+9q_yRa_1\alpha p+52q_yR\delta p^2)=0, \tag{A4}$$

$$pb_1(-2p\beta k_1p_y+9p\alpha a_1Rp_y+52p^2\delta Rp_y \\ +9p\beta a_1Rp_y+2\alpha b_0p_y-2p\alpha k_1p_y+2p_t)=0, \tag{A5}$$

$$R(15\delta p^2p_yb_1R+\alpha p^2b_1a_{1y}R+5\delta p^3b_{1y}R+\beta p^2a_{1y}b_1R+\alpha p^2a_1Rb_{1y} \\ +2\alpha pa_1Rp_yb_1+2\beta p^2b_{1y}a_1R+pb_{1t}+\alpha b_0pb_{1y}+2\beta p^2b_{1y}k_1-2\alpha p^2b_1k_{1y} \\ -4\alpha pb_1p_yk_1-2\alpha p^2k_1b_{1y}+p_tb_1+3\beta p^2k_{1y}b_1+\alpha b_0p_yb_1)=0, \tag{A6}$$

$$3pb_1R(2q_yRa_1\beta p+2q_yRa_1\alpha p+11q_yR\delta p^2 \\ +a_{0y}\beta p+q_yb_0\alpha-q_yk_1\beta p-q_yk_1\alpha p+q_t)=0, \tag{A7}$$

$$3pb_1R(2p\beta a_1Rp_y+2p\alpha a_1Rp_y+11p^2\delta Rp_y+\alpha b_0p_y-p\alpha k_1p_y-p\beta k_1p_y+p_t)=0, \tag{A8}$$

$$pR^2b_1(5q_yR\delta p^2+q_yRa_1\alpha p+q_yRa_1\beta p+a_{0y}\beta p+q_yb_0\alpha+q_t)=0, \tag{A9}$$

$$pR^2b_1(p\alpha a_1Rp_y+5p^2\delta Rp_y+p\beta a_1Rp_y+\alpha b_0p_y+p_t)=0, \tag{A10}$$

$$2\beta p^2k_{1y}b_1+p_tb_1+3\beta p^2a_{1y}b_1R+4\beta p^2b_{1y}a_1R-\alpha p^2b_1k_{1y}+2\alpha p^2a_1Rb_{1y} \\ -\alpha p^2k_1b_{1y}+4\alpha pa_1Rp_yb_1+33\delta p^2p_yb_1R+2\alpha p^2b_1a_{1y}R \\ +11\delta p^3b_{1y}R-2\alpha pb_1p_yk_1+\alpha b_0pb_{1y}+\alpha b_0p_yb_1+pb_{1t}=0, \tag{A11}$$

$$-pR^2(p\alpha b_1k_{1y}-p\beta k_{1y}b_1-4p\beta b_{1y}k_1+2\alpha b_1p_yk_1+p\alpha k_1b_{1y})=0, \tag{A12}$$

$$2pk_1R(-q_yk_1\alpha p-q_yk_1\beta p+8q_yR\delta p^2+q_yRa_1\alpha p \\ +q_yRa_1\beta p+a_{0y}\beta p+q_yb_0\alpha+q_t)=0, \tag{A13}$$

$$2pk_1R(-p\alpha k_1p_y-p\beta k_1p_y+8p^2\delta Rp_y+p\alpha a_1Rp_y+p\beta a_1Rp_y+\alpha b_0p_y+p_t)=0, \tag{A14}$$

$$-2\alpha pa_1p_yk_1+2\beta p^2k_{1y}a_1+8\delta p^3a_{1y}R+p_ta_1-\alpha p^2a_1k_{1y}+2\beta p^2a_{1y}a_1R \\ +\alpha b_0p_ya_1+pa_{1t}+\alpha pb_1^2p_yR+\alpha b_0pa_{1y}+2\alpha pa_1^2Rp_y \\ +3\beta p^2b_{1y}b_1R-\alpha p^2k_1a_{1y}+2\alpha p^2a_1Ra_{1y}+24\delta p^2p_ya_1R+\alpha p^2b_1b_{1y}R=0, \tag{A15}$$

$$\alpha b_0 p_y a_1 R + p_t a_1 R + \alpha p^2 a_1 R^2 a_{1y} + 2\delta p^3 a_{1y} R^2 + p a_{1t} R - p_t k_1 - 2\delta p^3 k_{1y} R - 2\alpha p^2 k_1 R a_{1y} + b_{0t} + 2\beta p^2 k_{1y} a_1 R + \alpha b_0 p a_{1y} R - 4\alpha p a_1 R p_y k_1 + \alpha p k_1^2 p_y + \beta p^2 b_{1y} b_1 R^2 + 2\beta p^2 a_{1y} k_1 R - \alpha b_0 p_y k_1 + \alpha p a_1^2 R^2 p_y - 6\delta p^2 p_y k_1 R + 6\delta p^2 p_y a_1 R^2 - p k_{1t} - 2\alpha p^2 a_1 R k_{1y} + \alpha p^2 k_1 k_{1y} - \alpha b_0 p k_{1y} = 0, \tag{A16}$$

$$p(3p\alpha b_1^2 q_y R + 2p\beta a_{0y} a_1 + 4p\beta a_1^2 R q_y + 4p\alpha a_1^2 R q_y + 40 p^2 \delta a_1 R q_y + 2\alpha b_0 a_1 q_y + 3p\beta b_1^2 q_y R - 2p\alpha k_1 a_1 q_y - 2p\beta k_1 q_y a_1 + 2 a_1 q_t) = 0, \tag{A17}$$

$$p(2\alpha b_0 p_y a_1 + 3\alpha p b_1^2 p_y R + 3p\beta b_1^2 p_y R + 2 p_t a_1 - 2p\beta k_1 p_y a_1 - 2\alpha p a_1 p_y k_1 + 40\delta p^2 p_y a_1 R + 4\alpha p a_1^2 R p_y + 4p\beta a_1^2 R p_y) = 0, \tag{A18}$$

$$pR(2p\alpha a_1^2 R q_y + p\beta b_1^2 q_y R + p\alpha b_1^2 q_y R + 2p\beta a_1^2 R q_y + 16 p^2 \delta a_1 R q_y + 2 a_1 q_t + 2p\beta a_{0y} a_1 - 2p\alpha k_1 a_1 q_y + 2\alpha b_0 a_1 q_y - 2p\beta k_1 q_y a_1) = 0, \tag{A19}$$

$$pR(2p\beta a_1^2 R p_y + 2\alpha p a_1^2 R p_y + \alpha p b_1^2 p_y R + p\beta b_1^2 p_y R + 16\delta p^2 p_y a_1 R + 2 p_t a_1 + 2\alpha b_0 p_y a_1 - 2\alpha p a_1 p_y k_1 - 2p\beta k_1 p_y a_1) = 0, \tag{A20}$$

$$p(18p\delta p_y a_1 + 2p\beta b_{1y} b_1 + 2p\beta a_{1y} a_1 + \alpha b_1^2 p_y + p\alpha a_1 a_{1y} + \alpha a_1^2 p_y + 6p^2 \delta a_{1y} + p\alpha b_1 b_{1y}) = 0, \tag{A21}$$

$$2p^2 q_y (b_1^2 \alpha + \beta a_1^2 + 12 p\delta a_1 + \alpha a_1^2 + b_1^2 \beta) = 0, \tag{A22}$$

$$2p^2 p_y (b_1^2 \alpha + \beta a_1^2 + 12 p\delta a_1 + \alpha a_1^2 + b_1^2 \beta) = 0, \tag{A23}$$

$$2pk_1 R^2 (20\delta q_y R p^2 - 2pk_1 \beta q_y - 2\alpha q_y k_1 p + pR\beta a_1 q_y + pR\alpha a_1 q_y + p a_{0y} \beta \alpha b_0 q_y + q_t) = 0, \tag{A24}$$

$$2pk_1 R^2 (20\delta p_y R p^2 - 2pk_1 \beta p_y - 2\alpha p_y k_1 p + pR\alpha a_1 p_y + pR\beta a_1 p_y + \alpha b_0 p_y + p_t) = 0, \tag{A25}$$

$$-R(8\delta p^3 k_{1y} R + 24\delta p^2 p_y k_1 R - 2\beta p^2 a_{1y} k_1 R + \alpha p^2 a_1 R k_{1y} + \alpha p^2 k_1 R a_{1y} + 2\alpha p a_1 R p_y k_1 - \alpha b_0 p k_{1y} + p_t k_1 + 2\beta p^2 k_{1y} k_1 + \alpha b_0 p_y k_1 + p k_{1t} - 2\alpha p^2 k_1 k_{1y} - 2\alpha p k_1^2 p_y) = 0, \tag{A26}$$

$$-pR^2 (6Rp^2 \delta k_{1y} + 18 Rp\delta p_y k_1 - \alpha k_1^2 p_y - p\alpha k_1 k_{1y} - 2p\beta k_{1y} k_1) = 0, \tag{A27}$$

$$2p^2 R^3 q_y k_1 (-\alpha k_1 - k_1 \beta 12 p\delta R) = 0, \tag{A28}$$

$$2p^2 k_1 R^3 p_y (-\alpha k_1 - k_1 \beta 12 p\delta R) = 0, \tag{A29}$$

$$2\beta p^2 b_{1y} k_1 R^3 = 0, \tag{A30}$$

$$b_1 p^2 q_y k_1 R^3 (\alpha + \beta) = 0, \tag{A31}$$

$$b_1 p^2 p_y k_1 R^3 (\alpha + \beta) = 0. \tag{A32}$$

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