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CONSTRUCTING FAMILIES TRAVELING WAVE SOLUTIONS IN TERMS OF SPECIAL FUNCTION FOR THE ASYMMETRIC NIZHNIK–NOVIKOV–VESSELOV EQUATION

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By means of a more general ansatz and the computerized symbolic system *Maple*, a generalized algebraic method to uniformly construct solutions in terms of special function of nonlinear evolution equations (NLEEs) is presented. We not only successfully recover the previously-known traveling wave solutions found by Fan's method, but also obtain some general traveling wave solutions in terms of the special function for the asymmetric Nizhnik–Novikov–Vesselov equation.

Keywords: Asymmetric Nizhnik–Novikov–Vesselov equation; soliton solution; Weierstrass and Jacobi functions; computerized symbolic system.

1. Introduction

The tanh method[1–3] is considered to be one of the most straightforward and effective algorithm to obtain solitary wave solutions for large NLEEs. Parkes and Duffy mentioned the difficulty of applying the tanh method by hand and to find more general exact solutions but simple NLEEs, due to the complicated and tedious algebraic calculation and differential computation. Recently, the application of computer algebra to science has a bright future. Parkes and Duffy automated, to some degree, the tanh method using symbolic computation software *Mathematica*.[1] In line with the development of computerized symbolic computation, much work has been concentrated on the various extensions and applications of the tanh method,

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such as the extended tanh method by Fan,[4] the improved tanh method by Yan,[5] the modified extended tanh-function method by Elwakil *et al.*,[6] and the generalized extended tanh-function method by Chen and Zheng.[7]

On the other hand, Gao and Tian[8–10] presented a generalized hyperbolic-function method by introducing coefficient functions to find soliton-like solution to NLEEs. Based on the method[8–10] by Gao and Tian and the tanh function method,[1–7] Chen and Li[11] extended tanh function method and presented the generalized Riccati equation expansion method to construct soliton-like solution of NLEEs. In Refs. 12 and 13, Fan developed a new algebraic method with symbolic computation to obtain the above-mentioned various traveling wave solutions in a unified way and provided us with new and more general traveling wave solutions in terms of special functions such as hyperbolic, rational, triangular, Weierstrass and Jacobi elliptic double periodic functions.

The present work is motivated by the desire to extend the transformation in Refs. 12 and 13 to more general transformations and use symbolic computation to solve the asymmetric Nizhnik–Novikov–Vesselov equation,[14] the (2+1)-dimensional KdV equation (also named the asymmetric NNV (ANNV) equation or BLMP (Boiti–Leon–Manna–Pempinelli) equation).[15] We not only successfully recover the previously-known traveling wave solutions found by Fan's method, but also obtain some general traveling wave solutions in terms of special function for the asymmetric Nizhnik–Novikov–Vesselov equation.

2. Summary of the Improved Method: Generalized Extended Tanh-Function Method

In the following we outline the main steps of our general method.

Step 1. Given the nonlinear partial differential equation (NPDE) system with some physical fields $u_i(x, y, t)$ in three variables x, y, t,

$$F_i(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \ldots) = 0\,, \tag{1}$$

by using the wave transformation

$$u_i(x, y, t) = u_i(\xi)\,, \qquad \xi = k(x + ly - \lambda t)\,, \tag{2}$$

where k, l and λ are constants to be determined later, then the nonlinear partial differential equation (1) is reduced to a nonlinear ordinary differential equation (ODE):

$$G_i(u_i, u_i', u_i'', \ldots) = 0\,. \tag{3}$$

Step 2. We introduce a new and more general ansatz in the forms:

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \left\{ a_{ij}\phi^j + b_{ij}\phi^{-j} + f_{ij}\phi^{j-1}\sqrt{\sum_{p=0}^{r} c_p\phi^p} + k_{ij}\frac{\sqrt{\sum_{p=0}^{r} c_p\phi^p}}{\varphi^j} \right\}\,, \tag{4}$$

where the new variable $\phi = \phi(\xi)$ satisfying

$$\phi' = \frac{d\phi}{d\xi} = \sqrt{\sum_{p=0}^{r} c_p \phi^p}\,, \tag{5}$$

and a_{i0}, a_{ij}, b_{ij}, f_{ij}, k_{ij} $(i = 1, 2, \ldots;\ j = 1, 2, \ldots, m_i)$ are constants to be determined later.

Step 3. The underlying mechanism for a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions to occur is that different effects that act to change wave forms in many nonlinear equations, i.e., dispersion, dissipation and nonlinearity, either separately or various combinations, are able to balance out. We define the degree of $u_i(\xi)$ as $D[u_i(\xi)] = n_i$, which gives rise to the degrees of other expressions as

$$D[u_i^{(\alpha)}] = n_i + \alpha\,, \qquad D[u_i^{\alpha}(u_j^{(\beta)})^s] = n_i\alpha + (\beta + n_j)s\,. \tag{6}$$

Therefore we can get the value of m_i in Eq. (4). If n_i is a non-negative integer, then we first make the transformation $u_i = \omega^{n_i}$.

Step 4. Substitute Eq. (4) into Eq. (3) along with Eq. (5) and then set all coefficients of $\phi^{\alpha}(\sqrt{\sum_{p=0}^{r} c_p\phi^p})^{\beta}$ $(\beta = 0, 1;\ \alpha = 0, 1, 2, \ldots)$ to zero to get an over-determined system of nonlinear algebraic equations with respect to λ, l, k, a_{i0}, a_{ij}, b_{ij}, f_{ij} and k_{ij} $(i = 1, 2, \ldots;\ j = 1, 2, \ldots, m_i)$.

Step 5. Solving the over-determined system of nonlinear algebraic equations by using Maple, we end up with the explicit expressions for λ, l, k, a_{i0}, a_{ij}, b_{ij}, f_{ij} and k_{ij} $(i = 1, 2, \ldots;\ j = 1, 2, \ldots, m_i)$.

Step 6. By using the results obtained in the above steps, we can derive a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions. We are interested only in solitary wave, Jacobi and Weierstrass doubly periodic solutions. The tan and cot type solutions appear in pairs with tanh and coth type solutions. The polynomial and rational triangular periodic solutions are omitted in this paper. By considering the different values of c_0, c_1, c_2, c_3 and c_4, Eq. (5) has many kinds of solitary wave, Jacobi and Weierstrass doubly periodic solutions as follows:

(i) Solitary wave solutions:

(a) Bell-shaped solitary wave solutions:

$$\phi = \sqrt{-\frac{c_2}{c_4}}\operatorname{sech}(\sqrt{c_2}\,\xi)\,, \qquad c_0 = c_1 = c_3 = 0\,, \qquad c_2 > 0\,, \qquad c_4 < 0\,, \tag{7}$$

$$\phi = -\frac{c_2}{c_3}\operatorname{sech}^2\left(\frac{\sqrt{c_2}}{2}\xi\right)\,, \qquad c_0 = c_1 = c_4 = 0\,, \qquad c_2 > 0\,. \tag{8}$$

(b) Kink-shaped solitary wave solutions:

$$\phi = \sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right)\,,$$
$$c_0 = \frac{c_2^2}{4c_4}\,, \qquad c_1 = c_3 = 0\,, \qquad c_2 < 0\,, \qquad c_4 > 0\,. \tag{9}$$

(c) Solitary wave solutions:

$$\phi = \frac{c_2\mathrm{sech}^2\left(\left(\frac{1}{2}\right)\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\left(\frac{1}{2}\right)\sqrt{c_2}\xi\right) - c_3}\,, \qquad c_0 = c_1 = 0\,, \qquad c_2 > 0\,. \tag{10}$$

(ii) Jacobi and Weierstrass doubly periodic solutions:

$$\phi = \sqrt{\frac{-c_2m^2}{c_4(2m^2-1)}}\mathrm{cn}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)\,,$$
$$c_4 < 0\,, \qquad c_2 > 0\,, \qquad c_0 = \frac{c_2^2m^2(1-m^2)}{c_4(2m^2-1)^2}\,, \tag{11}$$

$$\phi = \sqrt{\frac{-m^2}{c_4(2-m^2)}}\mathrm{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)\,,$$
$$c_4 < 0\,, \qquad c_2 > 0\,, \qquad c_0 = \frac{c_2^2(1-m^2)}{c_4(2-m^2)^2}\,, \tag{12}$$

$$\phi = \sqrt{\frac{-c_2m^2}{c_4(m^2+1)}}\mathrm{sn}\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right)\,,$$
$$c_4 > 0\,, \qquad c_2 < 0\,, \qquad c_0 = \frac{c_2^2m^2}{c_4(m^2+1)^2}\,, \tag{13}$$

where m is a modulus.

$$\phi = \wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)\,, \qquad c_2 = 0\,, \qquad c_3 > 0\,, \tag{14}$$

where $g_2 = -4(c_1/c_3)$ and $g_3 = -4(c_0/c_3)$ are called invariants of Weierstrass elliptic function. The Jacobi elliptic functions are doubly periodic and possess properties of triangular functions:

$$\mathrm{sn}^2\xi + \mathrm{cn}^2\xi = 1\,, \qquad dn^2\xi = 1 - m^2\mathrm{sn}^2\xi\,,$$

$$(\mathrm{sn}\,\xi)' = \mathrm{cn}\,\xi\mathrm{dn}\,\xi\,, \quad (\mathrm{cn}\,\xi)' = -\mathrm{sn}\,\xi\,, \qquad (\mathrm{dn}\,\xi)' = -m^2\mathrm{sn}\,\xi\mathrm{cn}\,\xi\,.$$

When $m \to 1$, the Jacobi functions degenerate to the hyperbolic functions, i.e.,

$$\mathrm{sn}\,\xi \to \tanh\xi\,, \qquad \mathrm{cn}\,\xi \to \mathrm{sech}\,\xi\,.$$

When $m \to 0$, the Jacobi functions degenerate to the triangular functions, i.e.,

$$\operatorname{sn}\xi \to \sin\xi\,, \qquad \operatorname{cn}\xi \to \cos\xi\,.$$

More detailed notations for the Weierstrass and Jacobi elliptic functions can be found in Refs. 16 and 17.

Remark 1. Compared with the method proposed by Fan,[12,13] our ansatz is more general. When $b_{ij} = f_{ij} = k_{ij} = 0$, Eq. (4) becomes the ansatz proposed by Fan.

Remark 2. The method can be extended to find soliton-like solutions and more types double periodic solutions of PDE (1). Only the restriction on $\xi(x, y, t)$ as merely a linear function x, y, t and the restrictions on the coefficients a_{i0}, a_{ij}, b_{ij}, f_{ij}, k_{ij} and c_i as constants will be removed.

3. Exact Solutions of ANNV Equation

Let us consider the ANNV equations,[14]

$$\begin{cases} u_t - u_{xxx} + \alpha(uv)_x = 0\,, \\ u_x + \beta v_y = 0\,, \end{cases} \tag{15}$$

where α, $\beta \neq 0$ are all constants. This system also named the $(2+1)$-dimensional KdV equation or BLMP (Boiti–Leon–Manna–Pempinelli) equation by Boiti *et al.*[15] using the idea of the weak Lax pair. The ANNV equation (15) can also be obtained from the inner parameter-dependent symmetry constraint of the KP equation.[18] Lou pointed out that the ANNV equation (15) is an asymmetric part of the Nizhnik–Novikov–Vesselov (NNV) equation.[19] For more detail about the results of this system, the reader is advised to see the remarkable achievements in Refs. 14, 15, and 18–21.

According to the above method, to seek the solutions of Eq. (15), we make the following transformation:

$$u(x,t) = \sigma(\xi)\,, \qquad v(\xi) = \tau(\xi)\,, \qquad \xi = x + ly - \lambda t\,, \tag{16}$$

where λ is constant to be determined later, and thus Eq. (15) becomes

$$\begin{cases} -\lambda\sigma' - \sigma''' + \alpha(\sigma\tau)' = 0\,, \\ \sigma' + \beta l\tau' = 0\,. \end{cases} \tag{17}$$

According to Step 1 in Sec. 2, if $a \neq 0$ and $\beta \neq 0$, by balancing $\sigma'''(\xi)$ and $(\sigma(\xi)\tau(\xi))'$ as well as $\tau'(\xi)$ and $\sigma'(\xi)$ in Eq. (17), we suppose that Eq. (17) has the following formal solutions:

$$\left\{\begin{aligned} \sigma &= a_0 + a_1\phi + \frac{b_1}{\phi} + f_1\sqrt{\sum_{i=0}^{4} c_i\phi^i} + k_1\frac{\sqrt{\sum_{i=0}^{4} c_i\phi^i}}{\phi} + a_2\phi^2 + \frac{b_2}{\phi^2} \\ &\quad + f_2\phi\sqrt{\sum_{i=0}^{4} c_i\phi^i} + k_2\frac{\sqrt{\sum_{i=0}^{4} c_i\phi^i}}{\phi^2}\,, \\ \tau &= A_0 + A_1\phi + \frac{B_1}{\phi} + F_1\sqrt{\sum_{i=0}^{4} c_i\phi^i} + K_1\frac{\sqrt{\sum_{i=0}^{4} c_i\phi^i}}{\phi} + A_2\phi^2 + \frac{B_2}{\phi^2} \\ &\quad + F_2\phi\sqrt{\sum_{i=0}^{4} c_i\phi^i} + K_2\frac{\sqrt{\sum_{i=0}^{4} c_i\phi^i}}{\phi^2}\,, \end{aligned}\right. \tag{18}$$

where $\phi(\xi)$ satisfies Eq. (5), where a_0, a_1, b_1, f_1, k_1, a_2, b_2, f_2, k_2, A_0, A_1, B_1, F_1, K_1, A_2, B_2, F_2 and K_2 are constants to be determined later.

With the aid of *Maple*, substituting Eq. (18) along with Eq. (5) into Eq. (17), we yield a set of algebraic equations for $\phi^p(\xi)(\sqrt{\sum_{i=0}^{4} c_i\phi^i})^q$, $(p = 0, 1, \ldots; q = 0, 1)$. Setting the coefficients of these terms $\phi^p(\xi)(\sqrt{\sum_{i=0}^{4} c_i\phi^i})^q$ to zero yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, f_1, k_1, a_2, b_2, f_2, k_2, A_0, A_1, B_1, F_1, K_1, A_2, B_2, F_2, K_2, l, and λ.

By using *Maple*, solving the over-determined algebraic equations, we get the following results.

Case 1.

$$\begin{aligned} &c_3 = -\frac{2}{3}A_1\alpha\,, \qquad c_0 = -\frac{1}{6}\alpha B_2\,, \qquad c_1 = -\frac{1}{3}B_1\alpha\,, \\ &\lambda = \frac{\alpha(-6B_2a_0 + 6A_0\beta lB_2 - B_1^2\beta l)}{6\beta lB_2}\,, \\ &c_2 = -\frac{B_1^2\alpha}{6B_2}\,, \qquad b_2 = -\beta lB_2\,, \qquad b_1 = -\beta lB_1\,, \qquad a_1 = -\beta lA_1\,, \\ &c_4 = a_2 = f_1 = f_2 = k_1 = k_2 = A_2 = F_1 = F_2 = K_1 = K_2 = 0\,. \end{aligned} \tag{19}$$

Case 2.

$$\begin{aligned} &a_0 = -\frac{\beta l(\lambda - \alpha A_0)}{\alpha}\,, \qquad k_2 = -\beta lK_2\,, \qquad c_1 = -\frac{2}{3}B_1\alpha\,, \qquad c_0 = \frac{1}{9}\alpha^2K_2^2\,, \\ &b_2 = \frac{1}{3}\beta lK_2^2\alpha\,, \qquad B_2 = -\frac{1}{3}K_2^2\alpha\,, \qquad b_1 = -\beta lB_1\,, \\ &c_2 = c_4 = a_1 = a_2 = f_1 = f_2 = k_1 = A_1 = A_2 = F_1 = F_2 = K_1 = 0\,. \end{aligned} \tag{20}$$

Case 3.

$$c_2 = -\frac{A_1^2\alpha}{6A_2}\,, \quad \lambda = \frac{\alpha(-6A_2a_0 + 6A_0\beta lA_2 - A_1^2\beta l)}{6\beta lA_2}\,, \quad c_1 = -\frac{2}{3}B_1\alpha\,,$$
$$c_4 = -\frac{1}{6}\alpha A_2\,, \quad c_3 = -\frac{1}{3}A_1\alpha\,, \quad a_2 = -\beta lA_2\,, \quad b_1 = -\beta lB_1\,, \quad a_1 = -\beta lA_1\,,$$
$$c_0 = f_1 = f_2 = b_2 = k_1 = k_2 = F_1 = F_2 = B_2 = K_1 = K_2 = 0\,. \tag{21}$$

Case 4.

$$k_2 = -\beta lK_2\,, \quad b_2 = \frac{1}{3}\beta lK_2^2\alpha\,, \quad f_1 = -\beta lF_1\,, \quad c_4 = \frac{1}{9}\alpha^2F_1^2\,, \quad A_2 = -\frac{1}{3}\alpha F_1^2\,,$$
$$a_0 = -\frac{\beta l(-3\alpha A_0 - 3c_2 + 3\lambda + 2\alpha^2K_2F_1)}{3\alpha}\,, \quad a_2 = \frac{1}{3}\beta l\alpha F_1^2\,, \quad c_0 = \frac{1}{9}\alpha^2K_2^2\,, \tag{22}$$
$$B_2 = -\frac{1}{3}K_2^2\alpha\,, \quad c_1 = c_3 = a_1 = f_2 = b_1 = k_1 = A_1 = F_2 = B_1 = K_1 = 0\,.$$

Case 5.

$$c_3 = -\frac{2}{3}A_1\alpha\,, \quad f_1 = -\beta lF_1\,, \quad \lambda = \frac{-\alpha a_0 + \alpha\beta lA_0 + \beta lc_2}{\beta l}\,,$$
$$c_4 = \frac{1}{9}\alpha^2F_1^2\,, \quad A_2 = -\frac{1}{3}\alpha F_1^2\,, \quad a_2 = \frac{1}{3}\beta l\alpha F_1^2\,, \quad a_1 = -\beta lA_1\,, \tag{23}$$
$$f_2 = b_1 = b_2 = k_1 = k_2 = F_2 = B_1 = B_2 = K_1 = K_2 = 0\,.$$

From Eqs. (16), (18) and Cases 1–5, we obtain the following solutions for Eq. (15)

Family 1. From Eq. (19), when $A_1 = 0$, we obtain the following solutions for the ANNV equations:

$$u_1 = a_0 + \frac{\beta lB_1}{\exp(\sqrt{c_2}\xi) - \frac{c_1}{2c_2}} - \frac{\beta lB_2}{(\exp(\sqrt{c_2}\xi) - \frac{c_1}{2c_2})^2}\,, \tag{24}$$

$$v_1 = A_0 - \frac{B_1}{\exp(\sqrt{c_2}\xi) - \frac{c_1}{2c_2}} + \frac{B_2}{(\exp(\sqrt{c_2}\xi) - \frac{c_1}{2c_2})^2}\,, \tag{25}$$

where $\xi = x + ly - \lambda t$, $\lambda = \alpha(-6B_2a_0 + 6A_0\beta lB_2 - B_1^2\beta l)/6\beta lB_2$, $c_2 = -B_1^2\alpha/6B_2$, a_0, A_0, l, B_1 and B_2 are arbitrary constants.

Family 2. From Eq. (20), we obtain the following solutions for the ANNV equations:

$$u_2 = -\frac{\beta l(\lambda - \alpha A_0)}{\alpha} - \frac{\beta l B_1}{\wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)} + \frac{\beta l K_2^2 \alpha}{3\wp^2\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)}$$
$$+ \frac{\beta l K_2 \sqrt{c_0 + c_1 \wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right) + c_3 \wp^3\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)}}{\wp^2\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)}, \tag{26}$$

$$v_2 = A_0 + \frac{B_1}{\wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)} - \frac{K_2^2 \alpha}{3\wp^2\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)}$$
$$+ \frac{K_2 \sqrt{c_0 + c_1 \wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right) + c_3 \wp^3\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)}}{\wp^2\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)}, \tag{27}$$

where $\xi = x + ly - \lambda t$, $g_2 = -4(c_1/c_3)$, $g_3 = -4(c_0/c_3)$, $c_0 = (1/9)\alpha^2 K_2^2$, $c_1 = -(2/3)B_1\alpha$, $c_3 > 0$, l, B_1, K_2 and λ are arbitrary constants.

Family 3. From Eq. (21), we obtain the following solutions for the ANNV equations:

$$u_3 = a_0 - \beta l A_1 \frac{c_2 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2 c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3}$$
$$- \beta l A_2 \left(\frac{c_2 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2 c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3}\right)^2, \tag{28}$$

$$v_3 = A_0 + A_1 \frac{c_2 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2 c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3}$$
$$+ A_2 \left(\frac{c_2 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2 c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3}\right)^2, \tag{29}$$

where $\xi = x + ly - \lambda t$, $\lambda = \alpha(-6A_2 a_0 + 6A_0 \beta l A_2 - A_1^2 \beta l)/6\beta l A_2$, $c_2 = -A_1^2\alpha/6A_2$, $c_3 = -(1/3)A_1\alpha$, $c_4 = -(1/6)\alpha A_2$, a_0, A_0, A_1, A_2 and l are arbitrary constants.

Family 4. From Eq. (22), we can obtain the following solutions for the ANNV equations:

$$u_4 = a_0 - \frac{a_2 c_2 m^2}{c_4(2m^2-1)} \operatorname{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right) - b_2 \frac{c_4(2m^2-1)}{c_2 m^2} \operatorname{nc}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)$$
$$+ f_1 \sqrt{c_0 - \frac{c_2^2 m^2}{c_4(2m^2-1)} \operatorname{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right) + \frac{c_2^2 m^4}{c_4(2m^2-1)^2} \operatorname{cn}^4\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}$$

$$-k_2\frac{\sqrt{c_0-\frac{c_2^2m^2\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}{c_4(2m^2-1)}+\frac{c_2^2m^4\mathrm{cn}^4\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}{c_4(2m^2-1)^2}}}{\frac{c_2m^2}{c_4(2m^2-1)}\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}, \tag{30}$$

$$\begin{aligned} v_4 = {} & A_0-\frac{A_2c_2m^2}{c_4(2m^2-1)}\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)-B_2\frac{c_4(2m^2-1)}{c_2m^2}\mathrm{nc}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)\\ & +F_1\sqrt{c_0-\frac{c_2^2m^2}{c_4(2m^2-1)}\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)+\frac{c_2^2m^4}{c_4(2m^2-1)^2}\mathrm{cn}^4\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}\\ & -K_2\frac{\sqrt{c_0-\frac{c_2^2m^2\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}{c_4(2m^2-1)}+\frac{c_2^2m^4\mathrm{cn}^4\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}{c_4(2m^2-1)^2}}}{\frac{c_2m^2}{c_4(2m^2-1)}\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}, \end{aligned} \tag{31}$$

where $\xi = x + ly - \lambda t$, $k_2 = -\beta l K_2$, $b_2 = (1/3)\beta l K_2^2\alpha$, $f_1 = -\beta l F_1$, $A_2 = -(1/3)\alpha F_1^2$, $a_0 = -\beta l(-3\alpha A_0 - 3c_2 + 3\lambda + 2\alpha^2 K_2 F_1)/3\alpha$, $a_2 = (1/3)\beta l\alpha F_1^2$, $B_2 = -(1/3)K_2^2\alpha$, $c_0 = (1/9)\alpha^2 K_2^2$, $c_4 = (1/9)\alpha^2 F_1^2$, $c_2^2 = c_0c_4(2m^2-1)^2/m^2(1-m^2)$, A_0, F_1, K_2, l and λ are arbitrary constants.

Family 5. From Eq. (22), we can obtain the following solutions for the ANNV equations:

$$\begin{aligned} u_5 = {} & a_0-\frac{a_2m^2}{c_4(2-m^2)}\mathrm{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)-b_2\frac{c_4(2-m^2)}{m^2}\mathrm{nd}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)\\ & +f_1\sqrt{c_0-\frac{c_2m^2\mathrm{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}{c_4(2-m^2)}+\frac{m^4\mathrm{dn}^4\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}{c_4(2-m^2)^2}}\\ & -k_2\frac{\sqrt{c_0-\frac{c_2m^2\mathrm{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}{c_4(2-m^2)}+\frac{m^4\mathrm{dn}^4\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}{c_4(2-m^2)^2}}}{\frac{m^2}{c_4(2-m^2)}\mathrm{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}, \end{aligned} \tag{32}$$

$$\begin{aligned} v_5 = {} & A_0-\frac{A_2m^2}{c_4(2-m^2)}\mathrm{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)-B_2\frac{c_4(2-m^2)}{m^2}\mathrm{nd}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)\\ & +F_1\sqrt{c_0-\frac{c_2m^2\mathrm{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}{c_4(2-m^2)}+\frac{m^4\mathrm{dn}^4\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}{c_4(2-m^2)^2}}\\ & -K_2\frac{\sqrt{c_0-\frac{c_2m^2\mathrm{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}{c_4(2-m^2)}+\frac{m^4\mathrm{dn}^4\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}{c_4(2-m^2)^2}}}{\frac{m^2}{c_4(2-m^2)}\mathrm{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}, \end{aligned} \tag{33}$$

where $\xi = x + ly - \lambda t$, $k_2 = -\beta l K_2$, $b_2 = (1/3)\beta l K_2^2 \alpha$, $f_1 = -\beta l F_1$, $A_2 = -(1/3)\alpha F_1^2$, $a_0 = -\beta l(-3\alpha A_0 - 3c_2 + 3\lambda + 2\alpha^2 K_2 F_1)/3\alpha$, $a_2 = (1/3)\beta l \alpha F_1^2$, $F_2 = -(1/3)K_2^2\alpha$, $c_0 = (1/9)\alpha^2 K_2^2$, $c_4 = (1/9)\alpha^2 F_1^2$, $c_2^2 = c_0 c_4 (2-m^2)^2/(1-m^2)$, A_0, F_1, K_2, l and λ are arbitrary constants.

Family 6. From Eq. (22), we can obtain the following solutions for the ANNV equations:

$$\begin{aligned} u_6 = a_0 &- \frac{a_2 c_2 m^2}{c_4(m^2+1)}\mathrm{sn}^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right) - b_2 \frac{c_4(m^2+1)}{c_2 m^2}\mathrm{ns}^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right) \\ &+ f_1\sqrt{c_0 - \frac{c_2^2 m^2 \mathrm{sn}^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}{c_4(m^2+1)} + \frac{c_2^2 m^4 \mathrm{sn}^4\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}{c_4(m^2+1)^2}} \\ &- k_2 \frac{\sqrt{c_0 - \frac{c_2^2 m^2 \mathrm{sn}^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}{c_4(m^2+1)} + \frac{c_2^2 m^4 \mathrm{sn}^4\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}{c_4(m^2+1)^2}}}{\frac{c_2 m^2}{c_4(m^2+1)}\mathrm{sn}^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}, \end{aligned} \tag{34}$$

$$\begin{aligned} v_6 = A_0 &- \frac{A_2 c_2 m^2}{c_4(m^2+1)}\mathrm{sn}^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right) - B_2 \frac{c_4(m^2+1)}{c_2 m^2}\mathrm{ns}^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right) \\ &+ F_1\sqrt{c_0 - \frac{c_2^2 m^2 \mathrm{sn}^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}{c_4(m^2+1)} + \frac{c_2^2 m^4 \mathrm{sn}^4\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}{c_4(m^2+1)^2}} \\ &- K_2 \frac{\sqrt{c_0 - \frac{c_2^2 m^2 \mathrm{sn}^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}{c_4(m^2+1)} + \frac{c_2^2 m^4 \mathrm{sn}^4\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}{c_4(m^2+1)^2}}}{\frac{c_2 m^2}{c_4(m^2+1)}\mathrm{sn}^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}, \end{aligned} \tag{35}$$

where $\xi = x + ly - \lambda t$, $k_2 = -\beta l K_2$, $b_2 = (1/3)\beta l K_2^2 \alpha$, $f_1 = -\beta l F_1$, $A_2 = -(1/3)\alpha F_1^2$, $a_0 = -\beta l(-3\alpha A_0 - 3c_2 + 3\lambda + 2\alpha^2 K_2 F_1)/3\alpha$, $a_2 = (1/3)\beta l \alpha F_1^2$, $B_2 = -(1/3)K_2^2\alpha$, $c_0 = (1/9)\alpha^2 K_2^2$, $c_4 = (1/9)\alpha^2 F_1^2$, $c_2^2 = c_0 c_4 (m^2+1)^2/m^2$, A_0, F_1, K_2, l and λ are arbitrary constants.

Family 7. From Eq. (23), when $c_0 = c_1 = 0$, then we can obtain the following solutions for the ANNV equations:

$$\begin{aligned} u_7 = a_0 &- \beta l A_1 \frac{c_2 \mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2 c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3} \\ &+ \frac{1}{3}\beta l \alpha F_1^2 \left(\frac{c_2 \mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2 c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3}\right)^2 - \left(\frac{\beta l F_1 c_2 \mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2 c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3}\right) \end{aligned}$$

$$\times\sqrt{c_2+c_3\frac{c_2\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)-c_3}+c_4\left(\frac{c_2\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)-c_3}\right)^2}\,, \tag{36}$$

$$\begin{aligned} v_7 &= A_0+A_1\frac{c_2\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)-c_3} \\ &\quad -\frac{1}{3}\alpha F_1^2\left(\frac{c_2\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)-c_3}\right)^2+\left(\frac{F_1c_2\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)-c_3}\right) \\ &\quad \times\sqrt{c_2+c_3\frac{c_2\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)-c_3}+c_4\left(\frac{c_2\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)-c_3}\right)^2}\,, \end{aligned} \tag{37}$$

where $\xi = x + ly - \lambda t$, $\lambda = (-\alpha a_0 + \alpha\beta l A_0 + \beta l c_2)/\beta l$, $c_3 = -(2/3)A_1\alpha$, $c_4 = (1/9)\alpha^2F_1^2$, $c_2 > 0$, l, a_0, A_0, A_1 and F_1 are arbitrary constants.

Remark 1. Up to now we have obtained some new forms of solutions which cannot be obtained by Fan's method, such as the solutions of Families 2, 4–7. When $b_1 = f_1 = k_1 = B_1 = B_2 = F_1 = F_2 = K_1 = K_2 = 0$, the form of the solutions of Families 2, 4–7 become the form that can be obtained by Fan's method. This further shows that our method is more general than Fan's method.

Remark 2. Some solutions derived by the generalized transformation are singular soliton solution and singular Jacobi elliptic doubly periodic wave solution. Although the wave patterns generally symmetric, sometimes some special wave patterns appear, these are the singularities in the nonlinear systems. A lot of spatio-temporal systems show this phenomena, e.g., nonlinear optical systems exhibit self-focusing effects, which may lead to the collapse of the optical power density into local divergences that may have important consequences on the integrity of optical fibers and laser system. This method is helpful to study singular solutions of partial differential equations that model nonlinear physical systems.

4. Summary and Conclusions

In summary, based on symbolic computation, by introducing a new and more general ansatz than the one in Refs. 12 and 13, we have proposed a generalized algebraic method to search for more types and general exact solutions for NPDEs. The asymmetric Nizhnik–Novikov–Vesselov equation is chosen to illustrate this algorithm such that we can successfully obtain the solutions found by the method presented by Fan[12,13] and find other new and more general solutions at the same time. The method can easily be extended to other NPDEs and is sufficient to seek more new formal solution of NPDEs.

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