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Travelling wave solutions for generalized symmetric regularized long-wave equations with high-order nonlinear terms*

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Applying the general projective Riccati equations method, we consider the exact travelling wave solutions for generalized symmetric regularized long-wave equations with high-order nonlinear terms using symbolic computation. From our results, we not only can successfully recover some previously known travelling wave solutions found by using various tanh methods, but also can obtain some new formal solutions. The solutions obtained include kink-shaped solitons, bell-shaped solitons, singular solitons and periodic solutions.

Keywords: symbolic computation, projective Riccati equations method, generalized symmetric regularized long-wave equation, soliton

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1. Introduction

In this paper, we consider a generalized symmetric regularized long-wave (SRLW) equation with high-order nonlinear terms and a dissipation term

$$\begin{cases} u_{xxt} - u_t = ru_{xx} + (b_1\rho + b_2u^{p+1} + b_3u^{2p+1})_x, \\ b_i = \text{constant}, \quad p > 0, \\ \rho_t + u_x = 0. \end{cases} \tag{1}$$

Some special cases of Eq.(1) have been investigated by both mathematicians and physicists in the last few decades.[1−6] In 1984, Seyler and Fantermmacler[1] studied the following SRLW equation:

$$\begin{cases} u_{xxt} - u_t = \left(\rho + \dfrac{1}{2}u^2\right)_x, \\ \rho_t + u_x = 0, \end{cases} \tag{2}$$

which was proposed as a model for the promulgation of plasmas and sound waves with a weak nonlinearity. In 1987, 1989 and 1992, Guo[3,4] and Zheng *et al*[5] found and analysed the global and numerical solutions of the SRLW equation. Recently, by means of proper transformation and undetermined assumption method, Zhang[6] studied two special cases of Eq.(1):

$$\begin{cases} u_{xxt} - u_t = (b_1\rho + b_2u^{p+1} + b_3u^{2p+1})_x, \\ b_i = \text{constant}, \quad p > 0, \\ \rho_t + u_x = 0, \end{cases} \tag{3}$$

$$\begin{cases} u_{xxt} - u_t = ru_{xx} + (b_1\rho + b_2u^{p+1})x, \\ b_i = \text{constant}, \quad p > 0, \\ \rho_t + u_x = 0, \end{cases} \tag{4}$$

and obtained many solitary wave solutions, which include the bell-type and kink-type solitary wave solutions of Eqs.(3) and (4). By travelling wave transformation, Feng[7] reduced the typical equations of (3) to a Lienard equation and obtained the general solutions for them, which generalize the corresponding solutions by Zhang *et al.*[6,8]

In 1992, Conte and Musette,[9] based on a projective Riccati system,[10] presented the projective Riccati equation method. Recently, Yan[11] developed the Conte–Musette method, named the general projective Riccati equation method. However, it is necessary to point out that there are some errors in the solutions (11) and (12) of the projective Riccati equation

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in Ref.[11]. In this paper, we correct these errors. The present work is motivated by the hope of finding some new and more formal solutions for Eq.(1) by means of the general projective Riccati equation method. As a result, we not only successfully recover some previously known solitary wave solutions of Eq.(1) but also obtain abundant new and more general solutions of Eq.(1).

This paper is arranged as follows. In Section 2, we apply the general projective Riccati equation method to Eq.(1) and find many solutions. A short conclusion follows.

2. Exact solutions of the generalized SRLW equation (1)

In this section, we will apply the general projective Riccati equation method to find the travelling wave solutions for Eq.(1).

After making the travelling wave transformation

$$\begin{cases} u(x,t) = u(\xi) = u(x-\lambda t), \\ \rho(x,t) = \rho(\xi) = \rho(x-\lambda t), \end{cases} \tag{5}$$

and integrating the resultant equation, Eq.(1) is changed into the following form:

$$\rho(\xi) = \frac{1}{\lambda}u(\xi) + c, \tag{6}$$

$$u''(\xi) + \frac{r}{\lambda}u'(\xi) + \frac{b_1-\lambda^2}{\lambda^2}u(\xi) + \frac{b_2}{\lambda}u^{p+1}(\xi) + \frac{b_3}{\lambda}u^{2p+1}(\xi) = 0. \tag{7}$$

By use of the transformation

$$u(\xi) = [v(\xi)]^{\frac{1}{p}}, \tag{8}$$

Eq.(7) reduces to

$$\lambda^2[p\varphi(\xi)\varphi''(\xi) + (1-p){\varphi'}^2(\xi)] + pr\lambda\varphi(\xi)\varphi'(\xi) + p^2[(b_1-\lambda^2)\varphi^2(\xi) + b_3\lambda\varphi^3(\xi) + b_3\lambda\varphi^4(\xi)] = 0. \tag{9}$$

By balancing $\varphi(\xi)\varphi''(\xi)$ (or $\varphi'^2(\xi)$) and $\varphi^4(\xi)$ in Eq.(9), we obtain the balance constant $m = 1$. Therefore, according to the idea of the general projective Riccati equation method, we suppose that Eq.(9) has the following formal solutions:

$$\varphi(\xi) = A_0 + A_1\sigma(\xi) + B_1\tau(\xi), \tag{10}$$

where A_0, A_1 and B_1 are all constants to be determined, and $\sigma(\xi)$ and $\tau(\xi)$ satisfy the following constraints:

$$\sigma'(\xi) = \epsilon\sigma(\xi)\tau(\xi), \quad \tau'(\xi) = R + \epsilon\tau^2(\xi) - \mu\sigma(\xi),$$
$$\epsilon = \pm 1, \quad R, \mu = \text{constant}, \tag{11}$$

$$\tau^2(\xi) = -\epsilon\left[R - 2\mu\sigma(\xi) + \frac{\mu^2-1}{R}\sigma^2(\xi)\right], \quad (R \neq 0). \tag{12}$$

With the aid of *Maple*, substituting Eq.(10) along with Eqs.(11), (12) into Eq.(9) yields a set of algebraic equations for $\sigma^j(\xi)\tau^i(\xi)(j = 0, 1, \cdots; \quad i = 0, 1)$. Setting the coefficients of these terms $\sigma^j\tau^i$ to zero yields a set of over-determined algebraic equations with respect to A_0, A_1, B_1, b_1, b_2, b_3, and λ. Because the set is complex, for simplicity, we do not list them in the paper. At the same time, for simplicity, we only consider the case of $\epsilon = -1$.

By use of the *Maple* soft package "charsets" by Dongming Wang, which is based on the Wu-elimination method,[12] solving the above set, we have the following non-trivial results.

Case 1

$$\begin{aligned} A_1 &= \pm\frac{\sqrt{R(-1+\mu^2)}B_1}{R}, \\ A_0 &= -\sqrt{R}B_1, \quad b_3 = -\frac{(1+p)\lambda}{4B_1^2p^2}, \\ b_2 &= -\frac{1}{2}\frac{2\lambda\sqrt{R} - pr + \lambda p\sqrt{R}}{p^2B_1}, \\ b_1 &= -\frac{\lambda(-\lambda p^2\sqrt{R} + \lambda R^{\frac{3}{2}} - rRp)}{p^2\sqrt{R}}. \end{aligned} \tag{13}$$

Case 2

$$\begin{aligned} A_1 &= \pm\frac{\sqrt{R(-1+\mu^2)}B_1}{R}, \\ A_0 &= \sqrt{R}B_1, \quad b_3 = -\frac{(1+p)\lambda}{4B_1^2p^2}, \\ b_2 &= \frac{1}{2}\frac{2\lambda\sqrt{R} + pr + \lambda p\sqrt{R}}{p^2B_1}, \\ b_1 &= -\frac{\lambda(-\lambda p^2\sqrt{R} + \lambda R^{\frac{3}{2}} + rRp)}{p^2\sqrt{R}}. \end{aligned} \tag{14}$$

Case 3

$$A_1=-\frac{\sqrt{R(-1+\mu^2)}B_1}{R},\quad A_0=\frac{B_1\mu R}{\sqrt{R(-1+\mu^2)}},\quad b_3=-\frac{(1+p)\lambda}{4B_1^2p^2},$$
$$\lambda=\lambda,\quad b_2=\frac{1}{2}\frac{\lambda p\mu R+2R\lambda\mu+pr\sqrt{R(-1+\mu^2)}}{p^2B_1\sqrt{R(-1+\mu^2)}},$$
$$b_1=-\frac{1}{2}\frac{\lambda(-2p^2\lambda\mu^2+2\lambda\mu^2R+2\mu pr\sqrt{R(-1+\mu^2)}+R\lambda+2p^2\lambda)}{p^2(-1+\mu^2)}. \tag{15}$$

Case 4

$$A_1=\frac{\sqrt{R(-1+\mu^2)}B_1}{R},\quad A_0=-\frac{B_1\mu R}{\sqrt{R(-1+\mu^2)}},\quad b_3=-\frac{(1+p)\lambda}{4B_1^2p^2},$$
$$b_2=\frac{1}{2}\frac{(-\lambda p\mu R-2R\lambda\mu+pr\sqrt{R(-1+\mu^2)})}{p^2B_1\sqrt{R(-1+\mu^2)}},\quad \lambda=\lambda,$$
$$b_1=\frac{1}{2}\frac{\lambda(2p^2\mu^2-2\lambda\mu^2R+2\mu pr\sqrt{R(-1+\mu^2)}-R\lambda-2p^2\lambda)}{p^2(-1+\mu^2)}. \tag{16}$$

Case 5

$$A_0=-\sqrt{R}B_1,$$
$$b_2=-\frac{(-pr+2\lambda\sqrt{R}p+4\lambda\sqrt{R})}{p^2B_1},\quad A_1=0,$$
$$b_1=\frac{\lambda(2pRr-4\lambda R^{\frac{3}{2}}+\lambda\sqrt{R}p^2)}{\sqrt{R}p^2},$$
$$b_3=-\frac{\lambda(p+1)}{B_1^2p^2}. \tag{17}$$

Case 6

$$A_0=\sqrt{R}B_1,\quad A_1=0,$$
$$b_1=\frac{\lambda(-2pRr-4\lambda R^{\frac{3}{2}}+\lambda\sqrt{R}p^2)}{\sqrt{R}p^2},$$
$$b_3=-\frac{\lambda(p+1)}{B_1^2p^2},\quad b_2=\frac{pr+2\lambda\sqrt{R}p+4\lambda\sqrt{R}}{p^2B_1}. \tag{18}$$

We know that Eqs.(11) and (12) admit the following solutions:

Type 1 When $\epsilon=-1, R\neq 0$,

$$\begin{cases}\sigma_1(\xi)=\dfrac{R\text{sech}(\sqrt{R}\xi)}{\mu\text{sech}(\sqrt{R}\xi)+1},\\ \tau_1(\xi)=\dfrac{\sqrt{R}\tanh(\sqrt{R}\xi)}{\mu\text{sech}(\sqrt{R}\xi)+1},\\ \sigma_2(\xi)=\dfrac{R\text{csch}(\sqrt{R}\xi)}{\mu\text{csch}(\sqrt{R}\xi)+1},\\ \tau_2(\xi)=\dfrac{\sqrt{R}\coth(\sqrt{R}\xi)}{\mu\text{csch}(\sqrt{R}\xi)+1}.\end{cases} \tag{19}$$

Type 2 When $\epsilon=1, R\neq 0$,

$$\begin{cases}\sigma_3(\xi)=\dfrac{R\sec(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi)+1},\\ \tau_3(\xi)=\dfrac{\sqrt{R}\tan(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi)+1},\\ \sigma_4(\xi)=\dfrac{R\csc(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi)+1},\\ \tau_4(\xi)=-\dfrac{\sqrt{R}\cot(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi)+1}.\end{cases} \tag{20}$$

Type 3 When $R=\mu=0$,

$$\sigma_5(\xi)=\frac{C}{\xi}=C\epsilon\tau_5(\xi),\quad \tau_5(\xi)=\frac{1}{\epsilon\xi}, \tag{21}$$

where C is a constant.

Thus from Eqs.(5), (8), (10), (19) and Cases 1–6, we can obtain some exact solutions of $u(x,t)$ in Eq.(1) as follows. (Note: the solutions of $\rho(x,t)$ can be obtained using Eq.(6); for simplicity we omit them).

Family 1

$$u_{11}=\Big[-\sqrt{R}B_1\pm\sqrt{R(\mu^2-1)}B_1\frac{\text{sech}(\sqrt{R}\xi)}{\mu\text{sech}(\sqrt{R}\xi)+1}+B_1\frac{\sqrt{R}\tanh(\sqrt{R}\xi)}{\mu\text{sech}(\sqrt{R}\xi)+1}\Big]^{\frac{1}{p}}, \tag{22}$$

$$u_{12}=\Big[-\sqrt{R}B_1\pm\sqrt{R(\mu^2-1)}B_1\frac{\text{csch}(\sqrt{R}\xi)}{\mu\text{csch}(\sqrt{R}\xi)+1}+B_1\frac{\sqrt{R}\coth(\sqrt{R}\xi)}{\mu\text{csch}(\sqrt{R}\xi)+1}\Big]^{\frac{1}{p}}, \tag{23}$$

where

$$\lambda = -\frac{1}{2}\frac{(rR \pm \sqrt{R(r^2R+4b_1p^2-4Rb_1)})p}{\sqrt{R}(p^2-R)},$$

$$R = \frac{p^2(2b_2pB_1-r)^2}{\lambda^2(p+2)^2},$$

$$B_1^2 = -\frac{1}{4}\frac{(1+p)\lambda}{b_3p^2}.$$

Family 2

$$u_{21} = \left[\sqrt{R}B_1 \pm \sqrt{R(\mu^2-1)}B_1\frac{\operatorname{sech}(\sqrt{R}\xi)}{\mu\operatorname{sech}(\sqrt{R}\xi)+1} + B_1\frac{\sqrt{R}\tanh(\sqrt{R}\xi)}{\mu\operatorname{sech}(\sqrt{R}\xi)+1}\right]^{\frac{1}{p}}, \tag{24}$$

$$u_{22} = \left[\sqrt{R}B_1 \pm \sqrt{R(\mu^2-1)}B_1\frac{\operatorname{csch}(\sqrt{R}\xi)}{\mu\operatorname{csch}(\sqrt{R}\xi)+1} + B_1\frac{\sqrt{R}\coth(\sqrt{R}\xi)}{\mu\operatorname{csch}(\sqrt{R}\xi)+1}\right]^{\frac{1}{p}}, \tag{25}$$

where

$$\lambda = \frac{1}{2}\frac{(rR \pm \sqrt{R(r^2R+4b_1p^2-4Rb_1)})p}{\sqrt{R}(p^2-R)},$$

$$R = \frac{p^2(2b_2pB_1-r)^2}{\lambda^2(p+2)^2},$$

$$B_1^2 = -\frac{1}{4}\frac{(1+p)\lambda}{b_3p^2}.$$

Family 3

$$u_{31} = \left[-\frac{B_1\mu R}{\sqrt{R(-1+\mu^2)}} \pm \sqrt{R(\mu^2-1)}B_1\frac{\operatorname{sech}(\sqrt{R}\xi)}{\mu\operatorname{sech}(\sqrt{R}\xi)+1} + B_1\frac{\sqrt{R}\tanh(\sqrt{R}\xi)}{\mu\operatorname{sech}(\sqrt{R}\xi)+1}\right]^{\frac{1}{p}}, \tag{26}$$

$$u_{32} = \left[-\frac{B_1\mu R}{\sqrt{R(-1+\mu^2)}} \pm \sqrt{R(\mu^2-1)}B_1\frac{\operatorname{csch}(\sqrt{R}\xi)}{\mu\operatorname{csch}(\sqrt{R}\xi)+1} + B_1\frac{\sqrt{R}\coth(\sqrt{R}\xi)}{\mu\operatorname{csch}(\sqrt{R}\xi)+1}\right]^{\frac{1}{p}}, \tag{27}$$

where

$$\lambda = \frac{(\mu r\sqrt{R(\mu^2-1)} \pm \sqrt{(\mu^2-1)(\mu^2r^2R+4b_1p^2\mu^2-4\mu^2Rb_1-4b_1p^2-2Rb_1)})p}{-2\mu^2R+2p^2\mu^2-R-2p^2},$$

$$B_1^2 = -\frac{1}{4}\frac{1+p}{b_3p^2}, \quad R = \frac{(2b_2pB_1-r)(2b_2pB_1\mu^2-2b_2pB_1-r\mu^2+r)p^2}{(p+2)^2\lambda^2\mu^2}.$$

Family 4

$$u_{41} = \left[\frac{B_1\mu R}{\sqrt{R(-1+\mu^2)}} \pm \sqrt{R(\mu^2-1)}B_1\frac{\operatorname{sech}(\sqrt{R}\xi)}{\mu\operatorname{sech}(\sqrt{R}\xi)+1} + B_1\frac{\sqrt{R}\tanh(\sqrt{R}\xi)}{\mu\operatorname{sech}(\sqrt{R}\xi)+1}\right]^{\frac{1}{p}}, \tag{28}$$

$$u_{42} = \left[\frac{B_1\mu R}{\sqrt{R(-1+\mu^2)}} \pm \sqrt{R(\mu^2-1)}B_1\frac{\operatorname{csch}(\sqrt{R}\xi)}{\mu\operatorname{csch}(\sqrt{R}\xi)+1} + B_1\frac{\sqrt{R}\coth(\sqrt{R}\xi)}{\mu\operatorname{csch}(\sqrt{R}\xi)+1}\right]^{\frac{1}{p}}, \tag{29}$$

where

$$\lambda = -\frac{(\mu r\sqrt{R(\mu^2-1)} \pm \sqrt{(\mu^2-1)(\mu^2r^2R+4b_1p^2\mu^2-4\mu^2Rb_1-4b_1p^2-2Rb_1)})p}{-2\mu^2R+2p^2\mu^2-R-2p^2},$$

$$B_1^2 = -\frac{1}{4}\frac{1+p}{b_3p^2}, \quad R = \frac{(2b_2pB_1-r)(2b_2pB_1\mu^2-2b_2pB_1-r\mu^2+r)p^2}{(p+2)^2\lambda^2\mu^2}.$$

Family 5

$$u_{51} = \left[\sqrt{R}B_1 + B_1\sqrt{R}\tanh(\sqrt{R}\xi)\right]^{\frac{1}{p}}, \tag{30}$$

$$u_{52} = \left[\sqrt{R}B_1 + B_1\sqrt{R}\coth(\sqrt{R}\xi)\right]^{\frac{1}{p}}, \tag{31}$$

where

$$\lambda = \frac{(-rR \pm \sqrt{R(r^2R-4Rb_1+b_1p^2)})p}{\sqrt{R}(-p^2+4R)},$$

$$B_1^2 = -\frac{1}{4}\frac{(1+p)\lambda}{b_3p^2}, \quad R = \frac{1}{4}\frac{p^2(b_2\lambda B_1p-r)^2}{\lambda^2(p+2)^2}.$$

Family 6

$$u_{51} = \Big[-\sqrt{R}B_1 + B_1\sqrt{R}\tanh(\sqrt{R}\xi)\Big]^{\frac{1}{p}}, \qquad (32)$$

$$u_{52} = \Big[-\sqrt{R}B_1 + B_1\sqrt{R}\coth(\sqrt{R}\xi)\Big]^{\frac{1}{p}}, \qquad (33)$$

where

$$\lambda = \frac{(rR \pm \sqrt{R(r^2R - 4Rb_1 + b_1p^2)})p}{\sqrt{R}(-p^2+4R)},$$

$$B_1^2 = -\frac{1}{4}\frac{(1+p)\lambda}{b_3p^2}, \quad R = \frac{1}{4}\frac{p^2(b_2\lambda B_1 p - r)^2}{\lambda^2(p+2)^2}.$$

Remarks

1) In this paper, for simplicity, we only consider the case of $\epsilon = -1$ in Eqs.(11) and (12). In respect to the case of $\epsilon = 1$ and rational solution, the process is similar to the case of $\epsilon = -1$, therefore we omit them.

2) The solutions obtained in this paper recover all of the solutions obtained by Zhang.[6] For example, from solutions (30) and (32), if setting $r = 0$, the solutions (21) by Zhang [6] can be recovered; if setting $b_2 = 0$, $p \to \frac{p}{2}$ and $b_3 \to b_2$, the solutions $(28)'$, $(29)'$ obtained by Zhang [6] can be reproduced.

3) In respect to solutions (11) and (12) by Zhang,[6] they can be obtained by use of the results in Ref.[7]. For simplicity, we do not list them here.

4) To our knowledge, the other solutions (22)–(29) have not been found before.

5) The general projective Riccati equation method is more powerful, such that it can be used to obtain more types of solutions, which recover not only some results by various other methods, such as various tanh methods,[13−18] and the homogeneous balance method,[19−22] but also other types of solutions.

3. Summary and conclusions

In summary, based upon the coupled Riccati equation, we have obtained many families of exact travelling wave solutions of a generalized SRLW equation with high-order nonlinear terms. The present method evidently can be generalized to any subequations, which must be defined in reduced form: e.g., the Riccati or elliptic equation (see e.g., Ref.[9] for detail). We will extend this method to seek some soliton-like solutions for some partial differential equations in forthcoming works.

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