

Rogue wave solutions of AB system



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ABSTRACT

In this paper, the generalized Darboux transformation is established to the AB system, which mainly describes marginally unstable baroclinic wave packets in geophysical fluids and ultra-short pulses in nonlinear optics. A unified formula of N th-order rogue wave solution for the AB system is found by the direct iterative rule. In particular, rogue wave solutions possessing several free parameters from first to second order are calculated. The dynamic properties of rogue waves in the AB system are shown through some figures.

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1. Introduction

In geophysical fluid dynamics, the process by which the available potential energy of a rotating stratified fluid may be converted into the kinetic energy of a growing disturbance is called Baroclinic instability [1–5]. Some important baroclinic wave packets equations have attracted widespread attention in recent decades, such as the two and the three-layer Phillips models [4,5], the $(2+1)$ -dimensional two-layer equations [6,7], the $(2+1)$ and $(3+1)$ -dimensional baroclinic potential vorticity equations [8,9] and the AB system [1]. Among them, the AB system serves as model equations to describe marginally unstable baroclinic wave packets in geophysical fluids [10]. It was firstly proposed by Pedlosky through using the singular perturbation theory [1], and is also important to describe ultra-short optical pulse propagation in nonlinear optics [2], to illustrate mesoscale gravity current transmission on a sloping bottom in the problem of cold gravity current [3]. So far, there has been surge of interest in studying the dynamic properties of the AB system, such as the single-phase periodic solution depending on a complete set of four complex parameters [11], the envelope solitary waves and periodic waves [10], the Painlevé analysis and conservation laws of the variable-coefficient AB system [12], the soliton and breather solutions through the classical Darboux transformation [13], and the N -soliton solutions by using the dressing method [14].

To our knowledge, there are no reports on rogue wave solutions of the AB system up to the present. While in the past five years, rogue waves (also known as freak waves, monster waves, killer waves, rabid-dog waves and similar names) have become a hot spot in many research fields [15–18]. Rogue waves were first observed in deep ocean, they constitute a frightening phenomenon which can lead to water walls as tall as 20–30 m and represent a threat for large boats and mariners [19]. A wave can be classified into this category when it has height and steepness much greater than the average crest, and appears from nowhere and disappears without a trace [20]. So far, rogue waves have also appeared in the fields of optics [21], atmosphere [22], Bose–Einstein condensates [23], superfluid [24], capillary flow [25] and even finance [26]

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and so on [27]. Rogue waves are localized in both space and time, and it is known that the rational form solution can describe rogue waves well in mathematics [20]. Many nonlinear Schrödinger (NLS)-type equations, for instance, the standard NLS equation [17,28–32], the Hirota equation [33], the Sasa-Satsuma equation [34], the high-order dispersive generalized NLS equation [35], the variable coefficient NLS equation [36], the discrete NLS equation [37], the Manakov equations [27,38,39], the coupled Hirota equations [40], the three-component NLS equations [41] have been confirmed to possess lower or high-order rogue waves of diverse structures. Nevertheless, a complete understanding of the mysterious and catastrophic rogue wave phenomenon is still far from been achieved, due to the difficult and hazardous observational conditions [20]. Therefore, it is of great interest to investigate rogue wave solutions of the AB system, which may be helpful to better understand the dynamic properties of the complicated rogue wave phenomenon in fluid mechanics and atmosphere.

In this paper, we take the AB system in canonical form [11]

$$A_{xt} = AB, \tag{1}$$

$$B_x = -\frac{1}{2}(|A|^2)_t, \tag{2}$$

where x and t are semi-characteristic normalized coordinates, A and B are the wave amplitudes yielding the normalization condition

$$|A_t|^2 + B^2 = 1. \tag{3}$$

When A is the real value, Eqs. (1) and (2) can be transformed into the sine–Gordon equation, and when A is the complex value, the self-induced transparency system [2,11].

The aim of the present paper is to research Eqs. (1) and (2) through the so-called generalized Darboux transformation (DT) proposed by Guo, Ling and Liu [29], which is a powerful tool to derive general rogue wave solutions of many nonlinear equations, the NLS equation [29], the derivative NLS equation [42], the Manakov equations [43], etc. Based on the generalized DT, we construct a unified formula of N th-order rogue wave solution for Eqs. (1) and (2) by the direct iterative rule. As application, rogue waves in Eqs. (1) and (2) from first to second order are studied. The first-order rogue wave of fundamental pattern, the second-order rogue waves of fundamental and triangular patterns are displayed by choosing different parameters, respectively.

Our paper is organized as follows. In Section 2, we construct the generalized DT to Eqs. (1) and (2) under the normalization condition (3), then a unified formula of N th-order rogue wave solution for Eqs. (1) and (2) is obtained by the direct iterative rule. In Section 3, the dynamic properties of rogue waves in Eqs. (1) and (2) from first to second order are illustrated through some figures. In Section 4, we give the conclusion.

2. Generalized Darboux transformation

In this section, we start from the Lax pair of Eqs. (1) and (2), which reads [11]

$$\Psi_x = U\Psi, \quad U = \begin{pmatrix} -i\lambda & \frac{1}{2}A \\ -\frac{1}{2}A^* & i\lambda \end{pmatrix}, \tag{4}$$

$$\Psi_t = V\Psi, \quad V = \frac{1}{4i\lambda} \begin{pmatrix} -B & A_t \\ A_t^* & B \end{pmatrix}, \tag{5}$$

where $\Psi = (\psi(x, t), \phi(x, t))^T$ is the vector eigenfunction, λ is the spectral parameter, and asterisk denotes the complex conjugation. It could be easily verified that the compatibility condition $U_t - V_x + UV - VU = 0$ gives rise to Eqs. (1) and (2).

Next, let $\Psi_1 = (\psi_1, \phi_1)^T$ be a basic solution of the Lax pair (4) and (5) with $A = A[0]$, $B = B[0]$ and $\lambda = \lambda_1$. Thus, on basis of the above lax pair, the classical DT [44,45] of Eqs. (1) and (2) can be built [13]

$$\Psi[1] = T[1]\Psi, \quad T[1] = \lambda I - H[0]\Lambda_1 H[0]^{-1}, \tag{6}$$

$$A[1] = A[0] - 4i(\lambda_1 - \lambda_1^*) \frac{\psi_1[0]\phi_1[0]^*}{(|\psi_1[0]|^2 + |\phi_1[0]|^2)}, \tag{7}$$

$$B[1] = B[0] - 4i(\lambda_1 - \lambda_1^*) \frac{[|\psi_1[0]|^2(|\phi_1[0]|^2)_t - |\phi_1[0]|^2(|\psi_1[0]|^2)_t]}{(|\psi_1[0]|^2 + |\phi_1[0]|^2)^2}, \tag{8}$$

where $\psi_1[0] = \psi_1$, $\phi_1[0] = \phi_1$,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H[0] = \begin{pmatrix} \psi_1[0] & \phi_1[0]^* \\ \phi_1[0] & -\psi_1[0]^* \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^* \end{pmatrix}.$$

In the following, suppose $\Psi_l = (\psi_l, \phi_l)^T$, $1 \leq l \leq N$ be a basic solution of the Lax pair (4) and (5) with $A = A[0]$, $B = B[0]$ and $\lambda = \lambda_l$. Then the N -step classical DT of Eqs. (1) and (2) can be naturally given as follows

$$\Psi[N] = T[N]T[N - 1] \cdots T[1]\Psi, \quad T[l] = \lambda I - H[l - 1]\Lambda_l H[l - 1]^{-1}, \tag{9}$$

$$A[N] = A[N - 1] - 4i(\lambda_N - \lambda_N^*) \frac{\psi_N[N - 1]\phi_N[N - 1]^*}{(|\psi_N[N - 1]|^2 + |\phi_N[N - 1]|^2)}, \tag{10}$$

$$B[N] = B[N - 1] - 4i(\lambda_N - \lambda_N^*) \frac{[|\psi_N[N - 1]|^2(|\phi_N[N - 1]|^2)_t - |\phi_N[N - 1]|^2(|\psi_N[N - 1]|^2)_t]}{(|\psi_N[N - 1]|^2 + |\phi_N[N - 1]|^2)^2}, \tag{11}$$

where

$$H[l - 1] = \begin{pmatrix} \psi_l[l - 1] & \phi_l[l - 1]^* \\ \phi_l[l - 1] & -\psi_l[l - 1]^* \end{pmatrix}, \quad \Lambda_l = \begin{pmatrix} \lambda_l & 0 \\ 0 & \lambda_l^* \end{pmatrix},$$

with $(\psi_l[l - 1], \phi_l[l - 1])^T = \Psi_l[l - 1]$, and

$$\Psi_l[l - 1] = T_l[l - 1]T_l[l - 2] \cdots T_l[1]\Psi_l, \quad T_l[k] = T[k]|_{\lambda=\lambda_l}, \quad 1 \leq l \leq N, \quad 1 \leq k \leq l - 1.$$

According to the above facts, the generalized DT can be derived for Eqs. (1) and (2). To this end, let $\Psi_1(\lambda_1 + \delta)$ be a special solution of the Lax pair (4) and (5) with $A[0]$, $B[0]$ and $\lambda = \lambda_1 + \delta$, and it can be expanded as Taylor series at $\delta = 0$, that is

$$\Psi_1 = \Psi_1^{[0]} + \Psi_1^{[1]}\delta + \Psi_1^{[2]}\delta^2 + \Psi_1^{[3]}\delta^3 + \cdots + \Psi_1^{[N]}\delta^N + o(\delta^N), \tag{12}$$

where $\Psi_1^{[k]} = (\psi_1^{[k]}, \phi_1^{[k]})^T = \lim_{\delta \rightarrow 0} \frac{1}{k!} \frac{\partial^k \Psi_1}{\partial \delta^k}$, $k = 0, 1, 2, \dots$

Afterwards, it is easy to find that $\Psi_1^{[0]}$ is a special solution of the Lax pair (4) and (5) with $A = A[0]$, $B = B[0]$ and $\lambda = \lambda_1$. Hence, by means of the formulas (6)–(8), the first-step generalized DT of Eqs. (1) and (2) can be directly given.

(1) The first-step generalized DT

$$\Psi[1] = T[1]\Psi, \quad T[1] = \lambda I - H[0]\Lambda_1 H[0]^{-1}, \tag{13}$$

$$A[1] = A[0] - 4i(\lambda_1 - \lambda_1^*) \frac{\psi_1[0]\phi_1[0]^*}{(|\psi_1[0]|^2 + |\phi_1[0]|^2)}, \tag{14}$$

$$B[1] = B[0] - 4i(\lambda_1 - \lambda_1^*) \frac{[|\psi_1[0]|^2(|\phi_1[0]|^2)_t - |\phi_1[0]|^2(|\psi_1[0]|^2)_t]}{(|\psi_1[0]|^2 + |\phi_1[0]|^2)^2}, \tag{15}$$

where $\psi_1[0] = \psi_1^{[0]}$, $\phi_1[0] = \phi_1^{[0]}$,

$$H[0] = \begin{pmatrix} \psi_1[0] & \phi_1[0]^* \\ \phi_1[0] & -\psi_1[0]^* \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^* \end{pmatrix}.$$

(2) The second-step generalized DT

It is clear that $T[1]\Psi_1$ is a basic solution of the Lax pair (4) and (5) with $A[1]$, $B[1]$ and $\lambda = \lambda_1 + \delta$. So that, by using the identity $T_1[1]\Psi_1^{[0]} = 0$, the following limit process

$$\lim_{\delta \rightarrow 0} \frac{T[1]|_{\lambda=\lambda_1+\delta} \Psi_1}{\delta} = \lim_{\delta \rightarrow 0} \frac{(\delta + T_1[1])\Psi_1}{\delta} = \Psi_1^{[0]} + T_1[1]\Psi_1^{[1]} \equiv \Psi_1[1]$$

provides a nontrivial solution of the Lax pair (4) and (5) with $A[1]$, $B[1]$, $\lambda = \lambda_1$, and can be adopted to do the second-step generalized DT, i.e.

$$\Psi[2] = T[2]T[1]\Psi, \quad T[2] = \lambda I - H[1]\Lambda_2 H[1]^{-1}, \tag{16}$$

$$A[2] = A[1] - 4i(\lambda_1 - \lambda_1^*) \frac{\psi_1[1]\phi_1[1]^*}{(|\psi_1[1]|^2 + |\phi_1[1]|^2)}, \tag{17}$$

$$B[2] = B[1] - 4i(\lambda_1 - \lambda_1^*) \frac{[|\psi_1[1]|^2(|\phi_1[1]|^2)_t - |\phi_1[1]|^2(|\psi_1[1]|^2)_t]}{(|\psi_1[1]|^2 + |\phi_1[1]|^2)^2}, \tag{18}$$

where $(\psi_1[1], \phi_1[1])^T = \Psi_1[1]$,

$$H[1] = \begin{pmatrix} \psi_1[1] & \phi_1[1]^* \\ \phi_1[1] & -\psi_1[1]^* \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^* \end{pmatrix}.$$

(3) The third-step generalized DT In the same way, with the aid of the identities

$$T_1[1]\Psi_1^{[0]} = 0, \quad T_1[2](\Psi_1^{[0]} + T_1[1]\Psi_1^{[1]}) = 0,$$

we get the following limit process

$$\lim_{\delta \rightarrow 0} \frac{[T[2]T[1]]|_{\lambda=\lambda_1+\delta} \Psi_1}{\delta^2} = \lim_{\delta \rightarrow 0} \frac{(\delta + T_1[2])(\delta + T_1[1])\Psi_1}{\delta^2} = \Psi_1^{[0]} + (T_1[2] + T_1[1])\Psi_1^{[1]} + T_1[2]T_1[1]\Psi_1^{[2]} \equiv \Psi_1[2],$$

which is a nontrivial solution of the Lax pair (4) and (5) with $A[2]$, $B[2]$, $\lambda = \lambda_1$, and can lead to the third-step generalized DT, namely,

$$\Psi[3] = T[3]T[2]T[1]\Psi, \quad T[3] = \lambda I - H[2]\Lambda_3 H[2]^{-1}, \tag{19}$$

$$A[3] = A[2] - 4i(\lambda_1 - \lambda_1^*) \frac{\psi_1[2]\phi_1[2]^*}{(|\psi_1[2]|^2 + |\phi_1[2]|^2)}, \tag{20}$$

$$B[3] = B[2] - 4i(\lambda_1 - \lambda_1^*) \frac{[|\psi_1[2]|^2(|\phi_1[2]|^2)_t - |\phi_1[2]|^2(|\psi_1[2]|^2)_t]}{(|\psi_1[2]|^2 + |\phi_1[2]|^2)^2}, \tag{21}$$

where $(\psi_1[2], \phi_1[2])^T = \Psi_1[2]$,

$$H[2] = \begin{pmatrix} \psi_1[2] & \phi_1[2]^* \\ \phi_1[2] & -\psi_1[2]^* \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^* \end{pmatrix}.$$

(4) The N -step generalized DT.

Iterating the above process N times, we arrive at the N -step generalized DT of Eqs. (1) and (2)

$$\Psi_1[N-1] = \Psi_1^{[0]} + \sum_{l=1}^{N-1} T_1[l]\Psi_1^{[1]} + \sum_{l=1}^{N-1} \sum_{k=1}^{l-1} T_1[l]T_1[k]\Psi_1^{[2]} + \dots + T_1[N-1]T_1[N-2] \dots T_1[1]\Psi_1^{[N-1]},$$

$$\Psi[N] = T[N]T[N-1] \dots T[1]\Psi, \quad T[N] = (\lambda I - H[N-1]\Lambda_N H[N-1]^{-1}), \tag{22}$$

$$A[N] = A[N-1] - 4i(\lambda_1 - \lambda_1^*) \frac{\psi_1[N-1]\phi_1[N-1]^*}{(|\psi_1[N-1]|^2 + |\phi_1[N-1]|^2)}, \tag{23}$$

$$B[N] = B[N-1] - 4i(\lambda_1 - \lambda_1^*) \frac{[|\psi_1[N-1]|^2(|\phi_1[N-1]|^2)_t - |\phi_1[N-1]|^2(|\psi_1[N-1]|^2)_t]}{(|\psi_1[N-1]|^2 + |\phi_1[N-1]|^2)^2}, \tag{24}$$

where $(\psi_1[N-1], \phi_1[N-1])^T = \Psi_1[N-1]$,

$$H[l-1] = \begin{pmatrix} \psi_1[l-1] & \phi_1[l-1]^* \\ \phi_1[l-1] & -\psi_1[l-1]^* \end{pmatrix}, \quad \Lambda_l = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^* \end{pmatrix}, \quad 1 \leq l \leq N.$$

For better applying the above formulas to generate rogue wave solutions of the AB system, we rewrite (23) and (24) as

$$A[N] = A[0] - 4i(\lambda_1 - \lambda_1^*) \sum_{j=0}^{N-1} \frac{\psi_1[j]\phi_1[j]^*}{(|\psi_1[j]|^2 + |\phi_1[j]|^2)}, \tag{25}$$

$$B[N] = B[0] - 4i(\lambda_1 - \lambda_1^*) \sum_{j=0}^{N-1} \frac{[|\psi_1[j]|^2(|\phi_1[j]|^2)_t - |\phi_1[j]|^2(|\psi_1[j]|^2)_t]}{(|\psi_1[j]|^2 + |\phi_1[j]|^2)^2}. \tag{26}$$

Hence, according to the generalized DT of the NLS equation [29], we know that (25) and (26) give rise to a unified formula of N th-order rogue wave solution for Eqs. (1) and (2) by the direct iterative rule, and can be converted into the $2N \times 2N$ determinant representation. But, to avoid the calculation of the determinant of a matrix of very high order, we prefer to use Darboux transformations of degree one successively instead of a Darboux transformation of higher degree with determinant representation. In the next section, the formulas (25) and (26) will be applied to work out the explicit rogue wave solutions of Eqs. (1) and (2), the dynamic properties of rogue waves in Eqs. (1) and (2) from first to second order are illustrated through some figures.

3. Rogue wave solutions

From the above section, we can observe that acquiring the adequate initial eigenfunction under the seed solutions enables us to generate rogue wave solutions of Eqs. (1) and (2). To this end, we start from the periodic plane waves

$$A[0] = e^{i\theta}, \quad B[0] = -\frac{a}{\sqrt{1+a^2}}, \tag{27}$$

where $\theta = \frac{(a\sqrt{1+a^2}x+t)}{\sqrt{1+a^2}}$, and a is a real constant. After that, inserting (27) into the Lax pair (4) and (5) and solving it, we have

$$\Psi_1 = \begin{pmatrix} (C_1 e^M - C_2 e^{-M}) e^{\frac{i}{2}\theta} \\ (C_1 e^{-M} - C_2 e^M) e^{-\frac{i}{2}\theta} \end{pmatrix}, \tag{28}$$

where

$$C_1 = \frac{(2\lambda + a - \sqrt{4\lambda^2 + 4a\lambda + 1 + a^2})^{\frac{1}{2}}}{\sqrt{4\lambda^2 + 4a\lambda + 1 + a^2}}, \quad C_2 = \frac{(2\lambda + a + \sqrt{4\lambda^2 + 4a\lambda + 1 + a^2})^{\frac{1}{2}}}{\sqrt{4\lambda^2 + 4a\lambda + 1 + a^2}},$$

and

$$M = \frac{i}{4\sqrt{1+a^2}\lambda} \sqrt{4\lambda^2 + 4a\lambda + 1 + a^2} \left(2\sqrt{1+a^2}\lambda x + t + \sum_{k=1}^N s_k f^{2k} \right).$$

Here f is a small real parameter, $s_k = m_k + in_k$, ($m_k, n_k \in \mathbb{R}$). Next, we fix $\lambda_1 = -\frac{a}{2} + \frac{i}{2}$, and set $\lambda = -\frac{a}{2} + \frac{i}{2} + f^2$ in (28). Then, the vector function Ψ_1 can be expanded as Taylor series at $f = 0$, that is

$$\Psi_1(f) = \Psi_1^{[0]} + \Psi_1^{[1]}f^2 + \Psi_1^{[2]}f^4 + \dots, \tag{29}$$

here we firstly present the explicit expression of $\Psi_1^{[0]}$

$$\psi_1^{[0]} = -\frac{\sqrt{2}}{2\sqrt{1+a^2}(i-a)} p_1^{[0]} e^{\frac{1}{2}\theta}, \quad \phi_1^{[0]} = \frac{\sqrt{2}}{2\sqrt{1+a^2}(i-a)} p_2^{[0]} e^{-\frac{1}{2}\theta}, \tag{30}$$

where

$$p_1^{[0]} = (1-i)\sqrt{1+a^2}(i-a)x + (1-i)t + (1-i)\sqrt{1+a^2}(i-a),$$

$$p_2^{[0]} = (1-i)\sqrt{1+a^2}(i-a)x + (1-i)t - (1-i)\sqrt{1+a^2}(i-a).$$

With the aid of the symbolic computation tool Maple, it is easy to verify that $\Psi_1^{[0]} = (\psi_1^{[0]}, \phi_1^{[0]})^T$ is a nontrivial solution of the Lax pair (4) and (5) with the seed solutions (27) and the fixed spectral parameter $\lambda_1 = -\frac{a}{2} + \frac{i}{2}$. So that, by means of the formulas (25) and (26) with $N = 1$, we have

$$A[1] = e^{i\theta} \left(1 + \frac{F_1 + iH_1}{D_1} \right), \quad B[1] = \frac{1}{\sqrt{1+a^2}} \frac{G_1}{D_1^2}, \tag{31}$$

where

$$F_1 = (2a^4 + 4a^2 + 2)x^2 - 4\sqrt{1+a^2}axt + 2t^2 - 2a^4 - 4a^2 - 2, \quad H_1 = 4\sqrt{1+a^2}t,$$

$$D_1 = -(a^4 + 2a^2 + 1)x^2 + 2a\sqrt{1+a^2}xt - t^2 - a^4 - 2a^2 - 1,$$

$$G_1 = -a(a^2 + 1)^4 x^4 + 4a^2(a^2 + 1)^{5/2} tx^3 - 2a(a^2 + 1)((3a^2 + 1)t^2 + a^6 + 5a^4 + 7a^2 + 3)x^2$$

$$+ 4\sqrt{1+a^2}(a^2 t^2 + a^6 + 4a^4 + 5a^2 + 2)xt - at^4 - (2a^5 + 8a^3 + 6a)t^2 - a^9 + 6a^5 + 8a^3 + 3a,$$

which is nothing but the first-order rogue wave solution of the AB system. Now we discuss dynamic properties of the solution (31), see Figs. 1 and 2.

For A component, from Figs. 1(a) and 2(a), we see that the first-order rogue wave in A component is just the standard eye-shaped Peregrine soliton like the NLS equation [17], there is one highest peak at the center (0,0), the maximum value of the peak is three times higher than the background crest.

However, for B component the situation is quite different. From Figs. 1(b) and 2(b), we see that the first-order rogue wave in B component is actually a four peaky-shaped rogue wave, in contrast to the standard eye-shaped Peregrine soliton in NLS equation. There are four highest peaks around the center (0,0), whose coordinates are (0.6669,0.6736), (-0.7384,0.7458), (-0.6669,-0.6736) and (0.7384,-0.7458). And the maximum value of these four highest peaks is uniformly 1, which is ten times higher than the background crest. The reason for this interesting structure can also be theoretically explained

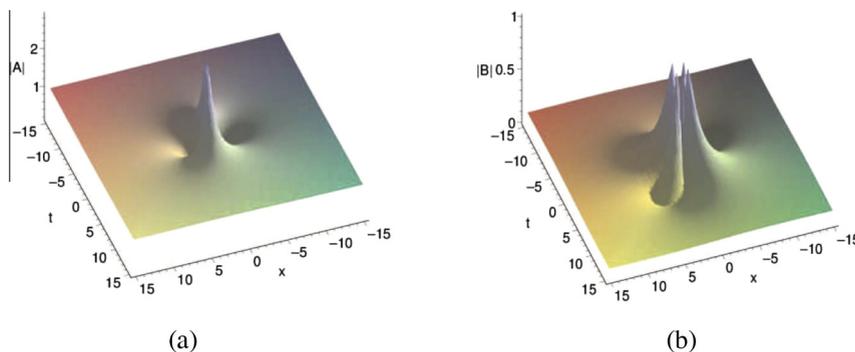


Fig. 1. The first-order rogue waves in AB system. (a) Rogue wave in A component; (b) Rogue wave in B component. The parameters are $a = 1/10$.

by the concrete expression of $B[1]$, since orders of the numerator and the denominator in $B[1]$ are both four. This is different from the first-order rational solution of the NLS equation, and the similar circumstances can also be found in the p and q components of the Hirota and the Maxwell-Bloch (H-MB) system [46].

Next, in order to obtain the second-order rogue wave solution of Eqs. (1) and (2), $\Psi_1^{[1]}$ should be used to construct the generating function,

$$\psi_1^{[1]} = \frac{\sqrt{2}(1+i)}{12(i-a)^3(1+a^2)^{3/2}} p_1^{[1]} e^{i\theta}, \quad \phi_1^{[1]} = -\frac{\sqrt{2}(1+i)}{12(i-a)^3(1+a^2)^{3/2}} p_2^{[1]} e^{-i\theta},$$

where

$$\begin{aligned} p_1^{[1]} &= \sqrt{1+a^2}(i-a)[(a^4-2ia^3-2ia-1)x^3+(3a^4-6ia^3-6ia-3)x^2+(3a^4-6ia^3-6ia-3)x \\ &\quad +(3x+3)t^2-3a^4+6ia^3+6ia+3]+t^3+((3a^4-6ia^3-6ia-3)x^2 \\ &\quad +(6a^4-12ia^3-12ia-6)x+3a^4+6ia^3+12a^2+6ia+9)t \\ &\quad +6im_1a^4-6n_1a^4+12in_1a^3+12m_1a^3+12in_1a+12m_1a-6im_1+6n_1, \\ p_2^{[1]} &= \sqrt{1+a^2}(i-a)[(a^4-2ia^3-2ia-1)x^3-(3a^4-6ia^3-6ia-3)x^2+(3a^4-6ia^3-6ia-3)x \\ &\quad +(3x-3)t^2+3a^4-6ia^3-6ia-3]+t^3+((3a^4-6ia^3-6ia-3)x^2-(6a^4-12ia^3-12ia-6)x+3a^4 \\ &\quad +6ia^3+12a^2+6ia+9)t+6im_1a^4-6n_1a^4+12in_1a^3+12m_1a^3+12in_1a+12m_1a-6im_1+6n_1. \end{aligned}$$

By using the following limit process

$$\lim_{f \rightarrow 0} \frac{T^{[1]}|_{z=-a/2+i/2+f^2} \Psi_1}{f^2} = \lim_{f \rightarrow 0} \frac{(f^2+T_1[1])\Psi_1}{f^2} = \Psi_1^{[0]} + T_1[1]\Psi_1^{[1]} \equiv \Psi_1[1],$$

we have

$$\psi_1[1] = \frac{\sqrt{2}(-1+i)}{6(i-a)^2(1+a^2)^{3/2}D_1} \rho_1 e^{i\theta}, \quad \phi_1[1] = \frac{\sqrt{2}(-1+i)}{6(i-a)^2(1+a^2)^{3/2}D_1} \rho_2 e^{-i\theta}, \tag{32}$$

where

$$\begin{aligned} \rho_1 &= \sqrt{1+a^2}[(-a^8+2ia^7-2a^6+6ia^5+6ia^3+2a^2+2ia+1)x^4 \\ &\quad +(-2a^8+4ia^7-4a^6+12ia^5+12ia^3+4a^2+4ia+2)x^3+(-6a^8+12ia^7-12a^6+36ia^5+36ia^3+12a^2+12ia+6)x \\ &\quad -t^4+((-6a^4+6ia^3-6a^2+6ia)x^2+(-6a^4+12ia^3+12ia+6)x+6ia^3+6a^2+6ia+6)t^2 \\ &\quad +(3im_1a^4-3n_1a^4+6m_1a^3+6in_1a^3+6in_1a+6m_1a-3im_1+3n_1)t-3a^8+6ia^7-6a^6+18ia^5+18ia^3+6a^2+6ia+3] \\ &\quad +(3n_1a^7-3im_1a^7-3m_1a^6-3in_1a^6+9n_1a^5-9im_1a^5-9m_1a^4-9in_1a^4-9im_1a^3+9n_1a^3 \\ &\quad -9in_1a^2-9m_1a^2+3n_1a-3im_1a-3m_1-3in_1)x+((4a^3-2ia^2+4a-2i)x+2a^3-4ia^2+2a-4i)t^3 \\ &\quad +((4a^7-6ia^6+8a^5-14ia^4+4a^3-10ia^2-2i)x^3+(6a^7-12ia^6+6a^5-24ia^4-6a^3-12ia^2-6a)x^2 \\ &\quad -(6ia^6+18ia^4+18ia^2+6i)x+6a^7+18a^5+18a^3+6a)t+3im_1a^7-3n_1a^7+3m_1a^6+3in_1a^6-9n_1a^5 \\ &\quad +9im_1a^5+9m_1a^4+9in_1a^4+9im_1a^3-9n_1a^3+9in_1a^2+9m_1a^2-3n_1a+3im_1a+3m_1+3in_1, \\ \rho_2 &= \sqrt{1+a^2}[(-a^8+2ia^7-2a^6+6ia^5+6ia^3+2a^2+2ia+1)x^4+(2a^8-4ia^7+4a^6-12ia^5-12ia^3-4a^2-4ia-2)x^3 \\ &\quad +(6a^8-12ia^7+12a^6-36ia^5-36ia^3-12a^2-12ia-6)x-t^4+((-6a^4+6ia^3-6a^2+6ia)x^2 \\ &\quad +(6a^4-12ia^3-12ia-6)x+6ia^3+6a^2+6ia+6)t^2+(3im_1a^4-3n_1a^4+6in_1a^3+6m_1a^3+6m_1a+6in_1a3n_1-3im_1)t \\ &\quad -3a^8+6ia^7-6a^6+18ia^5+18ia^3+6a^2+6ia+3]+(3n_1a^7-3im_1a^7-3in_1a^6-3m_1a^6-9im_1a^5+9n_1a^5-9in_1a^4 \\ &\quad -9m_1a^4-9im_1a^3+9n_1a^3-9m_1a^2-9in_1a^2-3im_1a+3n_1a-3m_1-3in_1)x \\ &\quad +((4a^3-2ia^2+4a-2i)x-2a^3+4ia^2-2a+4i)t^3+((4a^7-6ia^6+8a^5-14ia^4+4a^3-10ia^2-2i)x^3 \\ &\quad +(-6a^7+12ia^6-6a^5+24ia^4+6a^3+12ia^2+6a)x^2-(6ia^6+18ia^4+18ia^2+6i)x-6a^7-18a^5-18a^3-6a)t \\ &\quad -3im_1a^7+3n_1a^7-3m_1a^6-3in_1a^6+9n_1a^5-9im_1a^5-9m_1a^4-9in_1a^4-9im_1a^3+9n_1a^3-9in_1a^2 \\ &\quad -9m_1a^2+3n_1a-3im_1a-3m_1-3in_1, \end{aligned}$$

then a special solution of the Lax pair (4) and (5) with $A[1]$, $B[1]$ and $\lambda = \lambda_1 = -a/2 + i/2$ can be obtained. Hence, the second-order rogue wave solution of Eqs. (1) and (2) can be given by substituting (27), (30) and (32) into (25) and (26) with $N = 2$. Here we omit the explicit expressions of $A[2]$ and $B[2]$ because it is rather tedious and inconvenient to write them down here, but it is not difficult to verify that they satisfy Eqs. (1)–(3) with the help of Maple. Finally, we show some interesting structures and dynamic properties of the second-order rogue waves, see Figs. 3–6.

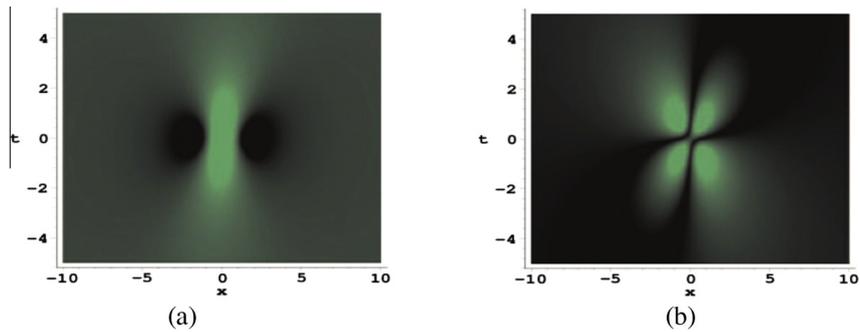


Fig. 2. Density plot of the first-order rogue waves in AB system. (a) Rogue wave in A component; (b) Rogue wave in B component. The parameters are $a = 1/10$.

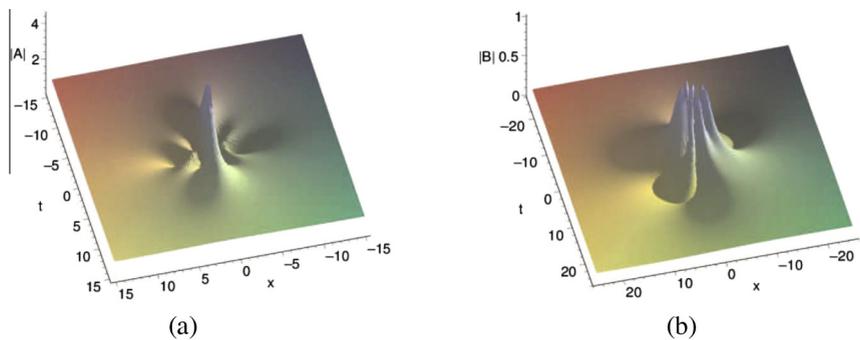


Fig. 3. The second-order rogue waves in AB system. (a) Rogue wave in A component; (b) Rogue wave in B component. The parameters are $a = 1/10$, $m_1 = 0$, $n_1 = 0$.

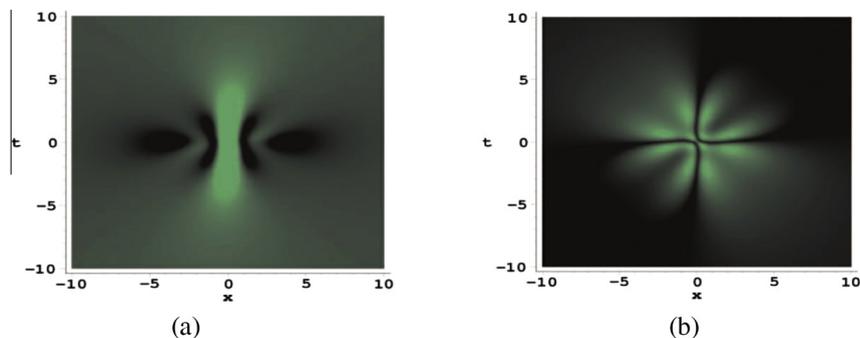


Fig. 4. Density plot of the second-order rogue waves in AB system. (a) Rogue wave in A component; (b) Rogue wave in B component. The parameters are $a = 1/10$, $m_1 = 0$, $n_1 = 0$.

For A component, like the first-order case, we say that it has absolutely identical shape with the second-order rogue wave of NLS equation. When $m_1 = 0$, $n_1 = 0$, the fundamental second-order rogue wave can be presented, see Figs. 3(a) and 4(a). Here, the maximum amplitude is arrived at the center $(0,0)$, and its value is five times higher than the background crest. When $m_1 = 500$, $n_1 = 0$, the second-order rogue wave in A component of triangular pattern can be generated, see Fig. 5(a) and 6(a). The maximum amplitude 3 is achieved at $(11.4405, 0.3733)$, $(-4.9500, 9.7731)$ and $(-6.2283, -10.1196)$.

For B component, the situation becomes complicated. When $m_1 = 0$, $n_1 = 0$, we obtain the fundamental second-order rogue wave, see Figs. 3(b) and 4(b). In this case, The maximum value of the peak is also 1 like the first-order case, while instead of four highest peaks, there are twelve highest peaks around the center $(0,0)$. The coordinates are $(0.3943, 0.3983)$, $(-0.3750, 0.3787)$, $(0.5101, 2.0165)$, $(-0.5845, 2.1537)$, $(1.9965, 0.5152)$, $(-2.1323, 0.5903)$ and their symmetric points about the original point. When $m_1 = 500$, $n_1 = 0$, we get the second-order rogue wave in B component of triangular pattern, see Fig. 5(b) and 6(b). It is clear that there are three basic four peaky-shaped rogue waves distributed around the original point, the maximum value is 1, and every basic four peaky-shaped rogue wave has four highest peaks around its

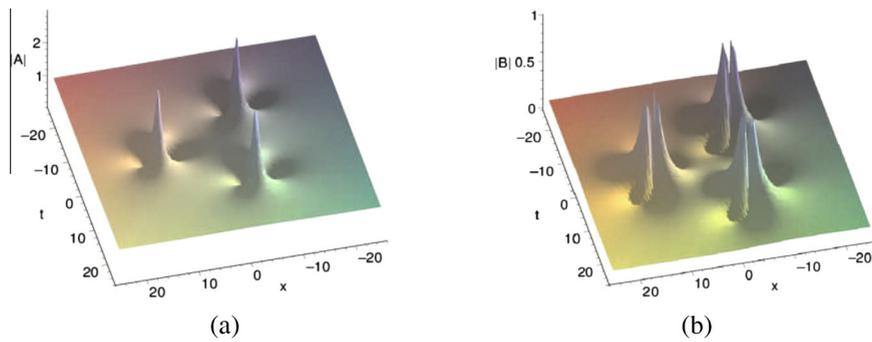


Fig. 5. The second-order rogue waves of triangular pattern in AB system. (a) Rogue wave in A component; (b) Rogue wave in B component. The parameters are $a = 1/10$, $m_1 = 500$, $n_1 = 0$.

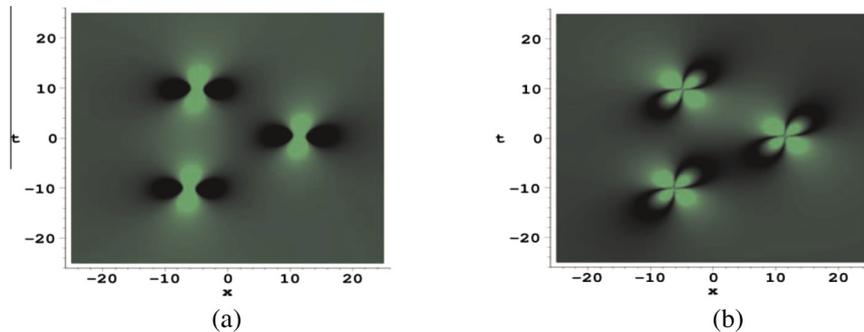


Fig. 6. Density plot of the second-order rogue waves of triangular pattern in AB system. (a) Rogue wave in A component; (b) Rogue wave in B component. The parameters are $a = 1/10$, $m_1 = 500$, $n_1 = 0$.

center. Through direct numerical calculation, we can also determine the concrete positions of the highest peaks, here we omit the lengthy data of them.

4. Conclusion

In summary, based on the generalized DT the AB system (1) and (2) is investigated, which is important to describe marginally unstable baroclinic wave packets in geophysical fluids, ultra-short optical pulse propagation in nonlinear optics, and mesoscale gravity current transmission on a sloping bottom in the problem of cold gravity current. We find a unified formula to construct N th-order rogue wave solution for Eqs. (1) and (2) by the direct iterative rule. As application, rogue wave solutions from first to second order are obtained. With the help of some free parameters, the first-order rogue wave of fundamental pattern, the second-order rogue waves of fundamental and triangular patterns are shown, respectively. The results further reveal and enrich the dynamical properties of Eqs. (1) and (2), and we hope our results will be verified in real experiments in the future. Besides, on the one hand, continuing the generalized DT one by one, the higher-order rogue wave solutions of Eqs. (1) and (2) can be generated, and they are likely to possess the more abundant dynamic properties, such as the “claw”, “claw-line”, and “claw-arc” structures like the high-order rogue waves in the standard NLS equation [30]. On the other hand, motivated by the remarkable work of Baronio and Guo et al. for the Manakov equations [27,38], the interactions between the rogue waves and the solitons or the breathers in Eqs. (1) and (2) may also be obtained by the Darboux transformation. In addition, based on the recent work of symmetry theory developed by us [47,48], the exact interactional solutions among solitons and other kinds of complicated waves such as cnoidal waves and Painlevé waves for Eqs. (1) and (2), and the rogue wave solutions of the variable-coefficient AB system may also be derived through the localization procedure of nonlocal symmetries and the similarity transformation method, which are useful to further reveal the dynamical properties of Eqs. (1) and (2). All of these problems are interesting, and we will investigate them in our future papers.

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