# A Truncated Painlevé Expansion and Exact Analytical Solutions for the Nonlinear Schrödinger Equation with Variable Coefficients

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By using the truncated Painlevé expansion analysis an auto-Bäcklund transformation is found for the nonlinear Schrödinger equation with varying dispersion, nonlinearity, and gain or absorption. Then, based on the obtained auto-Bäcklund transformation and symbolic computation, we explore some explicit exact solutions including soliton-like solutions, singular soliton-like solutions, which may be useful to explain the corresponding physical phenomena. Further, the formation and interaction of solitons are simulated by computer. – PACS Nos.: 05.45.Yv, 02.30.Jr, 42.65.Tg

*Key words:* Truncated Painlevé Expansion; Bäcklund Transformation; Symbolic Computation; Nonlinear Schrödinger Equation; Solitons.

## 1. Introduction

The nonlinear Schrödinger equation (NLSE) is one of the most important and “universal” nonlinear models of modern science. The NLSE appears in many branches of physics and applied mathematics, including nonlinear quantum field theory, condensed matter and plasma physics, nonlinear optics and quantum electronics, fluid mechanics, theory of turbulence and phase transitions, biophysics, and star formation and so on [1]. Since solitary waves or solitons, which are the best known solutions of the NLSE, were introduced and developed in 1971 by Zakharov and Shabat [2], there have been many significant contributions, including the theoretical and numerical aspects, to the development of the NLSE solitons theory [1 – 20]. After predictions of the possibility of the existence [6] and experimental discovery by Mollenauer, Stolen, and Gordon [7], today, NLSE optical solitons are regarded as the natural data bits and as an important alternative for the next generation of ultrahigh speed optical telecommunication systems [7 – 12].

Unfortunately, due to the exponential attenuation of the pulse along the fiber in real optical fibers, the balance between nonlinearity and dispersion is broken which affects the transmission of optical solitons. Usually, by keeping the fiber parameter slowly changing along the longitudinal direction or using the heat-insulating amplification, we maintain the transmission of solitons in the actual communication system. Therefore, in real communication systems of optical solitons, the transmission of solitons is described by the NLSE with varying dispersion, nonlinearity, and gain or absorption

$$\mathrm{i}\frac{\partial u}{\partial z}+\frac{1}{2}\beta(z)\frac{\partial^2 u}{\partial t^2}+\delta(z)|u|^2u=\mathrm{i}\alpha(z)u, \qquad (1)$$

where $\beta(z)$ and $\delta(z)$ are slowly increasing dispersion and nonlinear coefficients, respectively, and $\alpha(z)$ represents the heat-insulating amplification or loss. Serkin and Hasegawa [10, 11] developed an effective mathematical algorithm to discover and investigate an infinite number of novel soliton solutions for the NLSE (1) and discussed the problem of soliton management [8 – 10] described by NLSE (1). Ruan and Chen [16] studied the NLSE (1) by a symmetry approach and reported some exact solutions. Hong and Liu [17] simulated the NLSE (1) by a novel numerical approach based on the discovery of a new and intrinsic conservation law for the NLSE (1). The finding of some mathematical algorithms to discover and investigate exact analytical solutions of nonlinear dispersive systems is important to the field, and might have significant impact on future research.

In this work, we will use the truncated Painlevé expansion method ([21–30] and references therein) to investigate the Bäcklund transformation and some exact solutions for the NLSE (1). With the help of symbolic computation, two new families of solutions of the NLSE (1) are derived. Then based on these solutions, the formation and interaction of solitons are discussed.

## 2. A Truncated Painlevé Expansion and Some Exact Solutions

In order to obtain some exact solutions of NLSE (1), first we make the transformation

$$u(z,t) = v(z,t)\exp[\mathrm{i}(\theta(z,t))]. \tag{2}$$

Then substituting (2) into (1) and setting the real and imaginary parts of the resulting equation equal to zero, we obtain the following sets of PDEs:

$$-v\theta_z + \frac{1}{2}\beta(z)(v_{tt} - v\theta_t^2) + \delta(z)v^3 = 0, \tag{3}$$

$$v_z + \frac{1}{2}\beta(z)(2v_t\theta_t + v\theta_{tt}) - \alpha(z)v = 0. \tag{4}$$

According to the idea of Painlevé truncation ([21–28] and references therein)

$$v(z,t) = \phi^{-k}(z,t)\sum_{i=0}^{\infty} v_i(z,t)\phi^i(z,t), \tag{5}$$

in (3) and (4), where $k$ is a natural number, $\phi = 0$ defines the singular manifold and balancing powers of $\phi$ at the lowest orders requires that $k = 1$. We then truncate the above Painlevé expansion at the constant level terms [21–28], i. e.,

$$v(z,t) = \frac{v_0(z,t)}{\phi(z,t)} + v_1(z,t), \tag{6}$$

in order to obtain a Bäcklund transformation and analytic solutions to (3) and (4).

Substituting (6) into (3) and (4), then setting the coefficients of $\phi$ to zero we derive the PDEs

$$2v_0(\delta v_0^2 + \phi_t^2\beta) = 0, \tag{7}$$

$$\beta(v_1)_{tt} - \beta\theta_t^2 v_1 - 2\theta_z v_1 + 2\delta v_1^3 = 0, \tag{8}$$

$$-\beta\theta_t^2 v_0 + \beta(v_0)_{tt} + 6\delta v_0 v_1^2 - 2\theta_z v_0 = 0, \tag{9}$$

$$2(v_1)_z + \beta\theta_{tt}v_1 + 2\beta\theta_t(v_1)_t - 2\alpha v_1 = 0, \tag{10}$$

$$2(v_0)_z + \beta\theta_{tt}v_0 + 2\beta\theta_t(v_0)_t - 2\alpha v_0 = 0, \tag{11}$$

$$-\beta v_0\phi_{tt} + 6\delta(v_0)^2 v_1 - 2\beta(v_0)_t\phi_t = 0, \tag{12}$$

$$-2v_0(\beta\theta_t\phi_t + \phi_z) = 0, \tag{13}$$

where $v_0 = v_0(z,t)$, $v_1 = v_1(z,t)$, $\phi = \phi(z,t)$, $\alpha = \alpha(z)$, $\beta = \beta(z)$, $\delta = \delta(z)$, $\phi_t = \frac{\partial\phi}{\partial t}$, $(v_0)_t = \frac{\partial v_0}{\partial t}$.

From (7) and (12), we obtain

$$v_0 = \pm\frac{\sqrt{-\delta\beta}}{\delta}\phi_t, \quad v_1 = -\frac{1}{2}\frac{\sqrt{-\delta\beta}\,\phi_{tt}}{\phi_t\delta}. \tag{14}$$

Substituting (14) into (7)–(13) yields a system of PDEs with respect to $\phi$, $\theta$, $\alpha$, $\beta$, $\delta$. In order to solve the system, we further assume that the forms of $\phi$, $\theta$ are as follows:

$$\phi = \psi(z) + \mathrm{e}^{p(z)t+q(z)}, \tag{15}$$

$$\theta = \lambda_0(z) + \lambda_1(z)t + \lambda_2(z)t^2, \tag{16}$$

where $\psi(z)$, $p(z)$, $q(z)$, $\lambda_0(z)$, $\lambda_1(z)$, $\lambda_2(z)$ are differentiable functions.

Then substituting (15)–(16) into (7)–(13), we can derive an ODE system as follows [notice that $\alpha(z)$, $\beta(z)$, $\delta(z)$, $\psi(z)$, $p(z)$, $q(z)$, $\lambda_0(z)$, $\lambda_1(z)$, $\lambda_2(z)$ are independent of $t$]:

$$2\beta\lambda_1^2 + \beta p^2 + 4\lambda_{0z} = 0, \tag{17}$$

$$\begin{aligned}&-p\delta_z\beta + p\delta\beta_z + 2\delta\beta p_z + 2\beta^2\lambda_2\delta p\\ &-2\alpha\delta\beta p = 0,\end{aligned} \tag{18}$$

$$\begin{aligned}&-2\beta^2\lambda_2\delta p - 2\beta^2\delta p^2\lambda_1 + p\delta_z\beta - p\delta\beta_z\\ &-2\delta\beta p_z - 2\delta\beta p q_z + 2\alpha\delta\beta p = 0,\end{aligned} \tag{19}$$

$$\psi_z = 0, \tag{20}$$

$$\lambda_1\beta p + q_z = 0, \tag{21}$$

$$2\beta p\lambda_2 + p_z = 0, \tag{22}$$

$$2\beta\lambda_1\lambda_2 + \lambda_{1z} = 0, \tag{23}$$

$$2\beta{\lambda_2}^2 + \lambda_{2z} = 0. \tag{24}$$

Solving the above ODE system by use of the symbolic computation system Maple, two solutions of it are obtained:

*Case 1.*

$$\begin{aligned}&\delta(z) = \delta(z), \quad \alpha(z) = \frac{1}{2}\frac{\delta(z)\beta'(z) - \delta'(z)\beta(z)}{\delta(z)\beta(z)},\\ &\beta(z) = \beta(z), \quad \lambda_2(z) = 0, \quad \lambda_1(z) = C_3,\\ &\lambda_0(z) = -\frac{1}{4}\int\beta(z)\mathrm{d}z{C_2}^2 - \frac{1}{2}\int\beta(z)\mathrm{d}z{C_3}^2 + C_4,\\ &\psi(z) = C_1,\\ &q(z) = -C_3C_2\int\beta(z)\mathrm{d}z + C_5, \quad p(z) = C_2,\end{aligned} \tag{25}$$

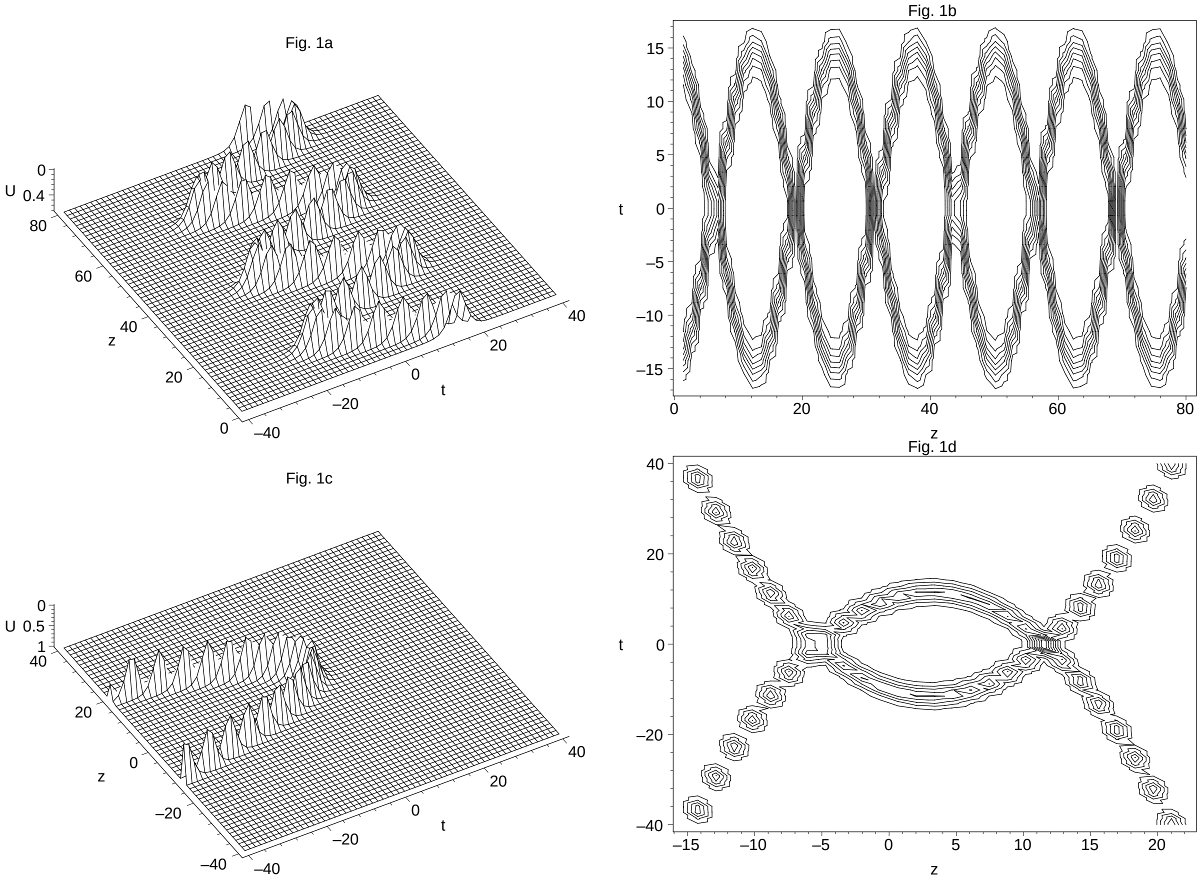


Fig. 1. Snake-solitons and boomerang-like solitons interaction scenario given by the dark soliton $u_{11}(z,t)$. (a), (b) Input conditions: $\beta(z) = -\sin(z/4)$, $\delta(z) = \sin(z/4)$, $C_2 = 0.8$, $C_3 = 4$, $C_5 = 0$. (c), (d) Input conditions: $\beta(z) = \delta(z) = 1 - 0.32z$, $C_2 = C_3 = 1, C_5 = 0$.

where $\delta(z) = \delta(z)$ denotes $\delta(z)$ being an arbitrary function of $z$, and so on; $C_i$ $(i = 1, \cdots, 5)$ are arbitrary constants.

*Case 2.*

$$\begin{aligned} &\lambda_0(z) = \frac{1}{8}\frac{p(z) + 2p(z)C_3{}^2 + 8C_4C_2}{C_2}, \quad \delta(z) = \delta(z), \quad p(z) = p(z), \quad \lambda_2(z) = C_2 p(z), \\ &\beta(z) = -\frac{1}{2}\frac{p'(z)}{C_2(p(z))^2}, \quad \lambda_1(z) = C_3 p(z), \quad q(z) = \frac{1}{2}\frac{C_3 p(z) + 2C_5C_2}{C_2}, \quad \psi(z) = C_1, \\ &\alpha(z) = -\frac{1}{2}\frac{-\delta(z)(p''(z))p(z) + \delta(z)p'(z)^2 + p'(z)\delta'(z)p(z)}{p'(z)p(z)\delta(z)}, \end{aligned} \tag{26}$$

where $\delta(z), p(z)$ are arbitrary functions of $z$; $C_i$ $(i = 1, \cdots, 5)$ are arbitrary constants.

Therefore from (2), (6), (14)–(16), (25) and (26), we obtain two families of exact analytical solutions for NLSE (1) as follows:

*Family 1.* From *Case 1*, a family of solutions of the NLSE is as follows:

$$u_1(z,t) = -\frac{\sqrt{-\delta(z)\beta(z)}}{\delta(z)}\frac{C_2\left\{C_1 - \exp[C_2 t + q(z)]\right\}}{C_1 + \exp[C_2 t + q(z)]}\exp\left[\mathrm{i}(C_3 t + \lambda_0(z))\right], \tag{27}$$

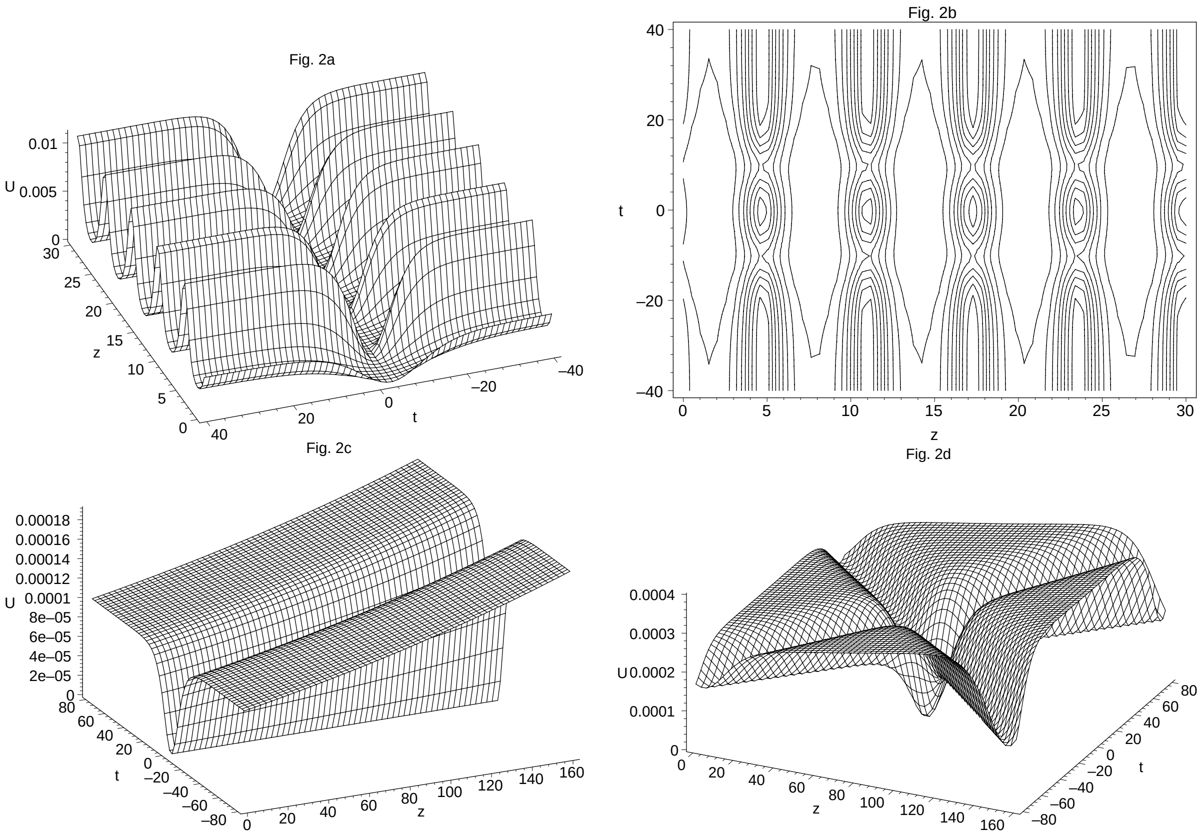


Fig. 2. (a), (b) Dark soliton propagation and two dark soliton interactions given by $u_{21}(z,t)$ with $\beta(z) = \cos(z)$, $\delta(z) = -100\cos(z)$, $C_0 = 5$, $C_2 = 1$, $C_3 = 6$, $C_5 = 0$. (c), (d) One exponentially increase dark quasi-soliton propagation and two exponentially increasing dark quasi-solitons interaction in the case of the equal amplitude and different phase with $\beta(z) = \exp(0.004z)/10000$, $\delta(z) = 1/(\int 2\exp(0.004z)/10000\mathrm{d}z + 5)^2$, $C_0 = 5, C_2 = 1, C_3 = 6, C_5 = 0$.

where $\delta(z)$, $\beta(z)$ are arbitrary functions of $z$; $\lambda_0(z)$, $q(z)$ and $\alpha(z)$ are determined by the following equations:

$$\lambda_0(z) = -\frac{1}{4}\int \beta(z)\mathrm{d}z {C_2}^2 - \frac{1}{2}\int \beta(z)\mathrm{d}z {C_3}^2 + C_4, \tag{28}$$

$$q(z) = -C_3C_2\int \beta(z)\mathrm{d}z + C_5, \quad \alpha(z) = \frac{1}{2}\frac{\delta(z)\beta'(z) - \delta'(z)\beta(z)}{\delta(z)\beta(z)}. \tag{29}$$

*Family 2.* From *Case 2*, another family of solutions of the NLSE is derived as follows:

$$u_2(z,t) = -\frac{\sqrt{-\delta(z)\beta(z)}}{\delta(z)}\frac{p(z)\left\{C_1 - \exp[p(z)(t + \frac{C_3}{2C_2}) + C_5]\right\}}{C_1 + \exp[p(z)(t + \frac{C_3}{2C_2}) + C_5]}\exp\left[\mathrm{i}(C_2p(z)t^2 + C_3p(z)t + \lambda_0)\right], \tag{30}$$

where $p(z)$, $\delta(z)$ are arbitrary functions of $z$; $\beta(z)$, $\lambda_0(z)$ and $\alpha(z)$ are determined by the following equations:

$$\beta(z) = -\frac{1}{2}\frac{p'(z)}{C_2(p(z))^2}, \quad \lambda_0(z) = \frac{1}{8}\frac{p(z) + 2p(z){C_3}^2 + 8C_4C_2}{C_2}, \tag{31}$$

$$\alpha(z) = -\frac{1}{2}\frac{-\delta(z)(p''(z))p(z) + \delta(z)p'(z)^2 + p'(z)\delta'(z)p(z)}{p'(z)p(z)\delta(z)}. \tag{32}$$

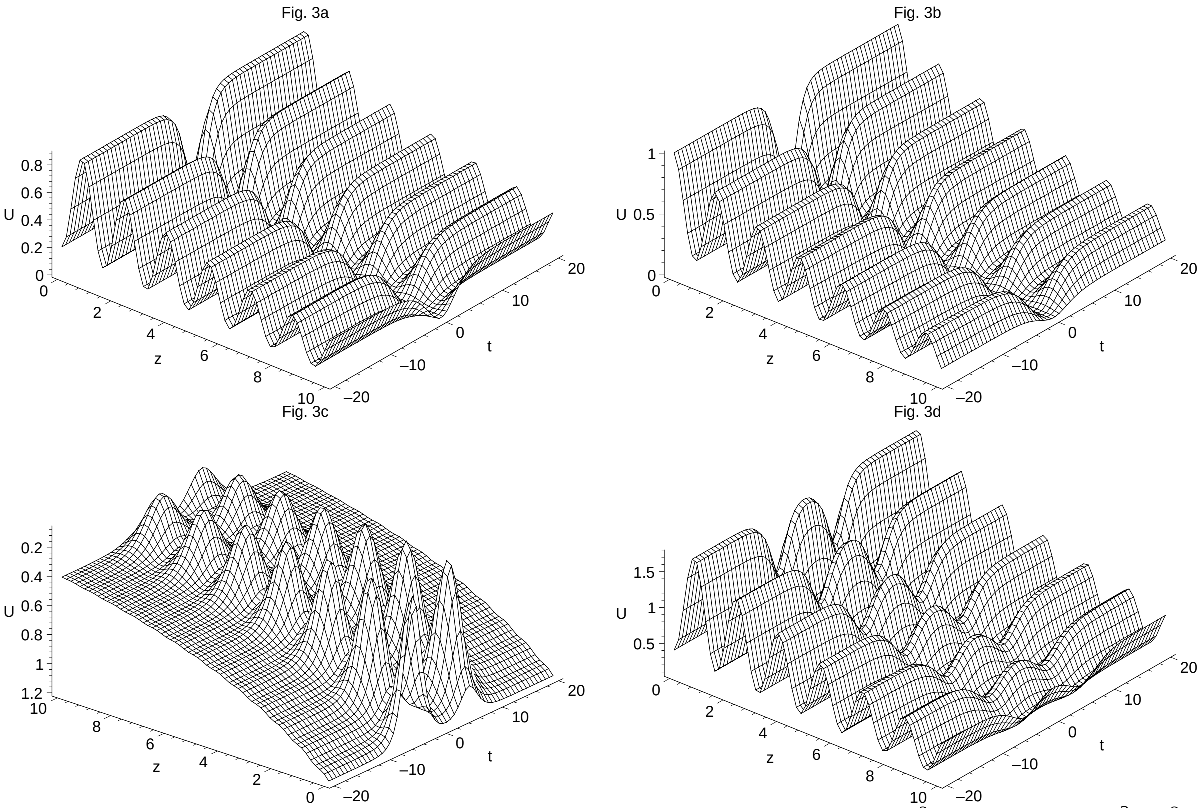


Fig. 3. (a), (b) Dispersion managed solitons propagation given by $u_{21}(z,t)$ with $\beta(z)=1-0.8\cos^2(2z)$ ($\beta=1-0.8\sin^2(2z)^2$), $\delta(z)=-1$, $C_0=1$, $C_2=0.06$, $C_3=C_5=0$. (c), (d) Interaction scenario with initial pulse separation equal to 10.

If setting $C_1=1$ and $C_1=-1$ in (27) and (30), respectively, we can obtain two soliton-like solutions and two blow-up soliton-like solutions for the NLSE:

(I)

$$u_{11}(z,t)=\frac{C_2\sqrt{-\delta(z)\beta(z)}}{\delta(z)}\tanh\left[\frac{C_2t+q(z)}{2}\right]\exp\left[\mathrm{i}(C_3t+\lambda_0(z))\right], \tag{33}$$

$$u_{12}(z,t)=\frac{C_2\sqrt{-\delta(z)\beta(z)}}{\delta(z)}\coth\left[\frac{C_2t+q(z)}{2}\right]\exp\left[\mathrm{i}(C_3t+\lambda_0(z))\right], \tag{34}$$

where $\alpha(z)$, $\beta(z)$, $\delta(z)$, $\lambda_0(z)$, $q(z)$ are determined by (28) and (29).

(II)

$$u_{21}(z,t)=\frac{\sqrt{-\delta(z)\beta(z)}}{\delta(z)}p(z)\tanh\left\{\frac{1}{2}[p(z)(t+\frac{C_3}{2C_2})+C_5]\right\}\exp\left[\mathrm{i}(C_2p(z)t^2+C_3p(z)t+\lambda_0)\right], \tag{35}$$

$$u_{22}(z,t)=\frac{\sqrt{-\delta(z)\beta(z)}}{\delta(z)}p(z)\coth\left\{\frac{1}{2}[p(z)(t+\frac{C_3}{2C_2})+C_5]\right\}\exp\left[\mathrm{i}(C_2p(z)t^2+C_3p(z)t+\lambda_0)\right], \tag{36}$$

where $\alpha(z)$, $\beta(z)$, $\delta(z)$, $\lambda_0(z)$, $q(z)$ are determined by (31) and (32).

If setting $C_3=0$ in (35), the kink-type solitons in the *Theorems 1-2* [10] and in the *Theorems 1-2* [11] can be reproduced. But to our knowledge, the other solutions are all new solutions.

In order to understand the significance of these solutions expressed by (33) – (36), eight figures for (33) and (35) are plotted to simulate the soliton propagation and soliton interaction. Here, $U = |u(z,t)|^2$ denotes the intensity of the solution, the dispersion coefficient $\beta(z)$ is taken as a periodic function and the nonlinear coefficient $\delta(z)$ is taken as either periodic function or constant and

$$p(z) = \frac{1}{\int 2\beta C_2 \mathrm{d}z + C_0} \Leftarrow \beta(z) = -\frac{1}{2}\frac{p'(z)}{C_2\,(p(z))^2}. \tag{37}$$

Figures 1a and 1c show the snake-soliton and boomerang-like soliton propagation given by $u_{11}(z,t)$, respectively. Figures 1b and 1d describe the contour plots of the snake-solitons and boomerang-like solitons interaction scenario of $u_{11}(z,t)$. The evolution of soliton $u_{21}(z,t)$ with a periodic dispersion coefficient is shown in Figure 2a. Figure 2b is the contour plot of the interaction of two solitons given by $u_{21}(z,t)$. Figures 2c and 2d represent one exponentially increasing dark quasi-soliton propagation and two exponentially increasing dark quasi-solitons interactions, respectively.

In the following, we will consider some periodical chirped soliton solution of $u_{21}(z,t)$. Suppose that the dispersion coefficients $\beta(z)$ vary periodically as

$$\beta(z) = 1 + k_1 \sin^{2n}(k_2 z) \ \text{ or } \ \beta(z) = 1 + k_1 \cos^{n}(k_2 z). \tag{38}$$

Figures 3a and 3b show the dispersion managed (DM) solitons propagation. Two neighboring DM solitons interaction scenarios are given in Figs. 3c and 3d. In Fig. 3c, two solitons are different: one soliton with $\beta = 1 - 0.8\sin^2(2z)$ and another with $\beta = 1 - 0.8\cos^2(2z)$. In Fig. 3d, the parameters of two solitons are the same.

## 3. Summary and Discussion

An auto-Bäcklund transformation and some exact analytical solutions are found for the nonlinear Schrödinger equation (NLSE) with varying dispersion, nonlinearity, and gain or absorption by using the truncated Painlevé expansion. Further, the propagation and interaction of solitons are simulated by computer. The results obtained are of general physics interest and should be readily experimentally verified. The finding of a new mathematical algorithm to discover solitary wave solutions in nonlinear dispersive systems with spatial parameter variations is important to the field, and might have significant impact on future research.

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