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THE HAMILTONIAN EQUATIONS IN SOME MATHEMATICS AND PHYSICS PROBLEMS *

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Abstract: *Some new Hamiltonian canonical system are discussed for a series of partial differential equations in Mathematics and Physics. It includes the Hamiltonian formalism for the symmetry second-order equation with the variable coefficients, the new nonhomogeneous Hamiltonian representation for fourth-order symmetry equation with constant coefficients, the one of MKdV equation and KP equation.*

Key words: infinite dimensional Hamiltonian system; Hamiltonian canonical system; Hamiltonian operator; MKdV (Modified Korteweg-de Vries) equation; KP (Kadomtsev-Petviashvili) equation

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Introduction

Hamiltonian systems are very important in mathematics, physics and mechanics. Specially after Feng Kang[1] first proposed the symplectic numerical method for finite dimensional Hamiltonian system in 1980s, the study of Hamiltonian system has been further improved in both theory and application. It is therefore natural to consider the problem about which kind of partial differential equations can be rewritten or represented by Hamiltonian system. Such kind of problem has now been warmly studied by many mathematicians and physicians such as in Refs. [2 - 5]. Until now it is known that many important equations such as wave equation, Schrödinger equation, Maxwell equations and KdV equation have been found to have various Hamiltonian structure.

Generalize the studies in early[6] about the canonical Hamiltonian representation of the problems in mathematics and mechanics, the further discussions for the cases of linear PDE in

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variable coefficients and the cases of nonlinear problems in mathematics and physics will be given in Section 2 after some introduction of the necessary concepts, notions and basic results in Section 1.

1 The Infinite Dimensional Hamiltonian

Historically Hamiltonian system came from the Lagrangian system in classical mechanics. Consider now the Lagrangian system on an n dimensional smooth manifold M, then in the local coordinate (U, q) of some point $q \in M$, it can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L(q,\dot{q})}{\partial \dot{q}}\right) = \frac{\partial L(q,\dot{q})}{\partial q},$$

in which $L(q, \dot{q})$ is a smooth function, called the Lagrange density. This equations can be equivalently rewritten as the equations on the tangent bundle TM.

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L(q,q')}{\partial q'}\right) = \dfrac{\partial L(q,q')}{\partial q}, \\ q' = \dfrac{\mathrm{d}}{\mathrm{d}t}q, \end{cases}$$

According to the Legendre transformation from TM to TM^*:

$$p = \frac{\partial L}{\partial q'},$$

$$H(q,p) = pq' - L(q,q'),$$

where L is nondegenerate for q'. Thus the Lagrangian system can be equivalently represented by the Hamiltonian system on the cotangent bundle TM^*.

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p}, \\ \dot{p} = -\dfrac{\partial H}{\partial q}. \end{cases}$$

Similary, it can be generalized to cases of infinite dimension. Let $u = (u_1, \cdots, u_{2m})^{\mathrm{T}}$ be the vector function of $(t, x_1, \cdots, x_{n-1}) \in \Omega \subset \mathbf{R}^n$. Denote

$$\dot{u} = \frac{\partial u}{\partial t} = \left(\frac{\partial u_1}{\partial t}, \cdots, \frac{\partial u_{2m}}{\partial t}\right)^{\mathrm{T}}, \quad \mathrm{D}^i u = (\mathrm{D}^i u_1, \cdots, \mathrm{D}^i u_{2m})^{\mathrm{T}},$$

$$\mathrm{D}^i u_j = \left(\frac{\partial^i u_j}{\partial x_1^i}, \cdots, \frac{\partial^i u_j}{\partial x_{k_1} \cdots \partial x_{k_i}}, \cdots\right)^{\mathrm{T}}, \quad H[u] = H(x, u, D^1 u, \cdots, D^k u),$$

for some $k \in \mathbf{N}$, and H is always smooth function. For any such function H, there corresponds to a functional $\mathscr{H} = \int_\Omega H[u]\,\mathrm{d}t\mathrm{d}x$. Denote the set of all such functional as $\mathscr{F}$.

Definition 1.1 The infinite dimensional Hamiltonian system or equations is the following partial differential equations

$$\dot{u} = \mathscr{D}\frac{\delta \mathscr{H}}{\delta u},$$

in which $\mathscr{D}$ is a Hamiltonian operator, which is a linear operator on functions that can be used to define the following Poisson structure,

$$\{\mathscr{H}, \mathscr{L}\} = \int_\Omega \left(\frac{\delta \mathscr{H}}{\delta u}\right)^{\mathrm{T}} \mathscr{D} \left(\frac{\delta \mathscr{L}}{\delta u}\right) \mathrm{d}x,$$

for any $\mathscr{K}$, $\mathscr{L} \in \mathscr{F}$. It can be proved that the operator

$$J = \begin{pmatrix} 0 & I_{m\times m} \\ -I_{m\times m} & 0 \end{pmatrix}$$

is an Hamiltonian operator. When $\mathscr{D} = J$, the corresponding Hamiltonian system is called infinite dimensional Hamiltonian canonical system. Applying the Vainberg Theorem in the theory of variational calculus, it is easy to prove that if and only if the following form of linear PDEs can be an infinite dimensional linear Hamiltonian canonical equations

$$\frac{\mathrm{d}}{\mathrm{d}t}u = \begin{pmatrix} F & Q \\ P & -F^* \end{pmatrix} u,$$

where F, P, Q are linear partial differential operator matrixes independent of ∂_t, and $P^* = P$, $Q^* = Q$, F^*, P^* and Q^* are the dual operators of F, P, Q respectively.

2 The Representation of Hamiltonian Canonical System

Example 2.1 Consider following general second-order symmetric equation with the smooth variable coefficients,

$$\sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}x_i}\left(\sum_{j=1}^{n} a_i \frac{\partial u}{\partial x_j}\right) + \frac{1}{2}\sum_{1\leqslant i<j\leqslant n}^{n} \frac{\mathrm{d}}{\mathrm{d}x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \frac{\mathrm{d}}{\mathrm{d}x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + cu = 0. \tag{1}$$

This equation is just the Euler-Lagrangian equation of the functional $\mathscr{L}$,

$$\mathscr{L} = \int_{\Omega}\left(-\frac{1}{2}\sum_{i=1}^{n} a_i\left(\frac{\partial u}{\partial x_i}\right)^2 - \frac{1}{2}\sum_{1\leqslant i<j\leqslant n} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + \frac{1}{2}cu^2\right)\mathrm{d}x.$$

Suppose that $a_{k0} \neq 0$, for some $1 \leqslant k_0 \leqslant n$. Denote $\tilde{a}_{ij} = \frac{a_{ij}}{a_{k0}}$, $\partial_{x_i} = \frac{\partial}{\partial x_i}$, $\partial^2_{x_ix_j} = \frac{\partial^2}{\partial x_i \partial x_j}$.

Define:

$$A = \frac{1}{2}(\alpha_1 + \alpha_2), \quad B = -\frac{1}{\alpha_{k_0}}, \quad C = -\frac{1}{4}\gamma + \omega - \frac{1}{4}\lambda,$$

$$A^* = -A + \frac{1}{2}\beta, \quad \beta = \sum_{i=1}^{k_0-1}\frac{\partial \tilde{a}_{ik_0}}{\partial x_i} + \sum_{i=k_0+1}^{n}\frac{\partial \tilde{a}_{k_0 i}}{\partial x_i},$$

$$\alpha_1 = \sum_{i=1}^{k_0-1}\tilde{a}_{ik_0}\partial_{x_i}, \quad \alpha_2 = \sum_{i=k_0+1}^{n}\tilde{a}_{k_0 i}\partial_{x_i},$$

$$\gamma = (r_1 r_2 - 2r_3)\partial^2_{x_ix_j}, \quad \omega = \sum_{i\neq k_0} a_i \partial^2_{x_ix_i},$$

$$\lambda = \left(\sum_{i=1}^{4}\lambda_i - 4\lambda_5 - 2\lambda_6\right)\partial_{x_i},$$

$$r_1 = \sum_{i=1}^{k_0-1} a_{ik_0} + \sum_{i=k_0+1}^{n} a_{k_0 i}, \quad r_2 = \sum_{i=1}^{k_0-1}\tilde{a}_{ik_0} + \sum_{i=k_0+1}^{n}\tilde{a}_{k_0 i}, \quad r_3 = \sum_{i,j\neq k_0} a_{ij},$$

$$\lambda_1 = \sum_{i=1}^{k_0-1}\sum_{j=1}^{k_0-1}\frac{\partial(a_{ik_0}\tilde{a}_{jk_0})}{\partial x_j}, \quad \lambda_2 = \sum_{i=1}^{k_0-1}\sum_{j=k_0+1}^{n}\frac{\partial(a_{ik_0}\tilde{a}_{k_0 j})}{\partial x_j},$$

$$\lambda_3 = \sum_{i=k_0+1}^{n}\sum_{j=1}^{k_0-1}\frac{\partial(a_{k_0 i}\tilde{a}_{jk_0})}{\partial x_j}, \quad \lambda_4 = \sum_{i=k_0+1}^{n}\sum_{j=k_0+1}^{n}\frac{\partial(a_{k_0 i}\tilde{a}_{k_0 j})}{\partial x_j},$$

$$\lambda_5 = \sum_{i \neq k_0} \frac{\partial a_i}{\partial x_i}, \quad \lambda_6 = \sum_{i,j \neq k_0} \frac{\partial a_{ij}}{\partial x_j}.$$

When we define $v = a_{k_0}(Au - \partial u / \partial x_{k_0})$, then this equation is equivalent to following Hamiltonian system,

$$\frac{\partial}{\partial x_{k0}} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Specially when all the coefficients of Eq. (1) are constants, this Hamiltonian formalism is just the one of Ex. 2.1 in Ref. [6].

Example 2.2 As in Ref. [6], consider also following fourth-order equation with constant coefficients,

$$\sum_{i+j=4} a_{ij} \frac{\partial^4 u}{\partial x^i \partial y^j} + \sum_{i+j=2} a_{ij} \frac{\partial^2 u}{\partial x^i \partial y^j} + eu = 0. \tag{2}$$

This equation is the Euler-Lagrange equation of functional $\mathscr{F}$:

$$\mathscr{F}[u] = \int_\Omega K \mathrm{d}x,$$

$$K = \frac{a_{40}}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{a_{04}}{2}\left(\frac{\partial^2 u}{\partial y^2}\right)^2 + \frac{a_{22}}{2}\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2} + \frac{a_{31}}{2}\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 u}{\partial y^2} - \frac{a_{20}}{2}\left(\frac{\partial u}{\partial x}\right)^2 - \frac{a_{02}}{2}\left(\frac{\partial u}{\partial y}\right)^2 - \frac{a_{11}}{2}u_x u_y + \frac{eu^2}{2}.$$

Assume that $a_{40} \neq 0$. Then we define

$$\alpha_1 = -\frac{a_{22}}{2a_{40}}\partial_y^2, \quad \alpha_2 = -\frac{a_{31}}{2a_{40}}\partial_y, \quad \alpha_3 = \frac{1}{a_{40}},$$

$$\alpha_4 = \left(-\frac{a_{22}^2}{4a_{40}} + a_{04}\right)\partial_y^4 + a_{02}\partial_y^2 + e, \quad \alpha_5 = \left(-\frac{a_{31}a_{22}}{4_{40}} + \frac{a_{13}}{2}\right)\partial_y^3 + \frac{a_{11}}{2}\partial_y,$$

$$\alpha_6 = \frac{a_{31}^2}{4a_{40}}\partial_y^2 - a_{20}, \quad \alpha_7 = \frac{a_{22}}{2a_{40}}\partial_y^2, \quad \alpha_8 = -\frac{a_{31}}{2a_{40}}\partial_y.$$

Define $\theta = \phi(x, y, u) + \frac{\partial u}{\partial x_{40}}$, $\phi(x, y)$ is any smooth function, then this equation can be equivalently written in the following Hamiltonian system:

$$\frac{\partial}{\partial x_{40}} \begin{pmatrix} u \\ \theta \\ \lambda \\ m \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha_1 & \alpha_2 & 0 & \alpha_3 \\ \alpha_4 & \alpha_5 & 0 & \alpha_7 \\ -\alpha_5 & \alpha_6 & -1 & \alpha_8 \end{pmatrix} \begin{pmatrix} u \\ \theta \\ \lambda \\ m \end{pmatrix} - \boldsymbol{V},$$

where $\boldsymbol{V}^{\mathrm{T}} = \left(-\phi, \ \phi_x + \frac{a_{31}}{2a_{40}}\phi_y, \ -\frac{1}{2}\left(a_{13} - \frac{a_{23}a_{31}}{a_{40}}\right)\phi_{yyy} \frac{a_{11}}{2}\phi_y, \ -\frac{a_{31}^2}{4a_{40}}\phi_{yy} + a_{20}\phi\right)$.

Compare this result with the one in Ref. [6], we see that this is another case of the nonhomogeneous Hamiltonian system. Specially if $\phi = 0$, this is just the case in Ref. [6].

Let's now discuss the following nonlinear problems in Mathematics and Physics.

Example 2.3 Consider the MKdV equation

$$\phi_t = \phi_{xxx} - 6\phi^2\phi_x. \tag{3}$$

Define $\phi = u_x$, then we get the following equation of u,

$$u_{xxxx} - 6u_x^2 u_{xx} - u_{xt} = 0, \tag{4}$$

which is the Euler-Lagrange equation of functional $\mathscr{L}$

$$\mathscr{L} = \int_\Omega \left(\frac{1}{2}(u_{xx})^2 + \frac{1}{2}u_x u_t + \frac{1}{2}u_x^4 \right) \mathrm{d}x\mathrm{d}t .$$

Denote $v = u_x + \phi(t,x)$, where ϕ is any smooth function of x and t. Then the following Hamiltonian system is equivalent to Eq.(4).

$$\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \\ \lambda \\ m \end{pmatrix} = \begin{pmatrix} \delta\mathscr{H}/\delta\lambda \\ \delta\mathscr{H}/\delta m \\ -\delta\mathscr{H}/\delta u \\ -\delta\mathscr{H}/\delta v \end{pmatrix},$$

or

$$\begin{cases} u_x = v - \phi, \\ v_x = m + \phi_x, \\ \lambda_x = -\dfrac{1}{2}v_t + \dfrac{1}{2}\phi_t, \\ m_x = \dfrac{1}{2}u_t + 2(v-\phi)^3 - \lambda, \end{cases}$$

where

$$\mathscr{H} = \int_\Omega H \mathrm{d}x\mathrm{d}t \qquad (\Omega \subset \mathbf{R}^2),$$

and

$$H[u,v,\lambda,m] = \frac{1}{2}m^2 + m\phi_x - \frac{1}{2}(v-\phi)u_t - \frac{1}{2}(v-\phi)^4 + \lambda(v-\phi).$$

Finally consider the following Kadomtstev-Petvishvili equation, which describes the motion of water wave in two-dimensional medium and is usually named by KP equation.

Example 2.4

$$v_{xt} = v_{xxxx} + 6v_x v_{xx} + 3a^2 v_{yy}, \tag{5}$$

where a can be generally any constant, specially $a = 1, -1, i, -i$. Suppose function $u = v_x$ for any solution v of Eq.(5), then u satisfies the following equation:

$$u_{xt} = (u_{xxxx} + 6uu_x)_x + 3a^2 u_{yy}, \tag{6}$$

which is originally supported by Kadomtstev-Petvishvili. Eq.(5) is the Euler-Lagrange equation of functional $\mathscr{L}$

$$\mathscr{L} = \int_\Omega \left(\frac{1}{2}(v_{xx})^2 + \frac{1}{2}v_x v_t - (v_x)^3 - \frac{3a^2}{2}(v_y)^2 \right) \mathrm{d}t\mathrm{d}x\mathrm{d}y .$$

Assume $w = v_x + \phi(t,x,y)$, where ϕ is any smooth function of (t,x,y). Then it can be proved that Eq.(5) can be equivalently written by the following Hamiltonian equation:

$$\frac{\partial}{\partial x}\begin{pmatrix} v \\ w \\ \lambda \\ m \end{pmatrix} = \begin{pmatrix} \delta\mathscr{H}/\delta\lambda \\ \delta\mathscr{H}/\delta m \\ -\delta\mathscr{H}/\delta v \\ -\delta\mathscr{H}/\delta w \end{pmatrix},$$

or

$$\begin{cases} v_x = w - \phi, \\ w_x = m + \phi_x, \\ \lambda_x = -\frac{1}{2} w_t + \frac{1}{2}\phi_t + 3a^2 v_{yy}, \\ m_x = \frac{1}{2} v_t - 3(w - \phi)^2 - \lambda, \end{cases}$$

where $\mathscr{H} = \int_\Omega H \mathrm{d}x\mathrm{d}t$, and

$$H[v, w, \lambda, m] = \frac{1}{2}m^2 + m\phi_x - \frac{1}{2}(w - \phi)v_t + (w - \phi)^3 + \frac{3a^2}{2}(v_y)^2 + \lambda(w - \phi).$$

Remarks Generalize the above results in Example 2.1 and Example 2.2, we can also get the equivalent Hamiltonian system for the general symmetry linear PDE in subsequent paper, which asserts that the following general linear Euler-Lagrange partial differential equation on $\Omega \subset \mathbf{R}^m$,

$$f(u, u^{(1)}, \cdots, u^{(2n)}) = 0$$

with the variable coefficients, satisfying the condition that the coefficient $a_i(x)$ of $\frac{\partial^{2n} u}{\partial x_i^{2n}}$ being nonzero for some $1 \leqslant i \leqslant r$, $\forall x \in \Omega$, then the equation can also be equivalently represented by finite or even infinite many Hamiltonian canonical systems.

Meanwhile according to the above equivalent Hamiltonian systems, we can also get some conservation energy in the direction of the corresponding variable x_{k_0} (such as x_{04} when $x_{04} \neq 0$ in Example 2.2) under some suitable boundary conditions of the domain $\Omega \subset \mathbf{R}^m$, which will perhaps support some new suggestions and help in the study of the problem, such as the study of integrability.

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