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# New explicit exact solutions for a generalized Hirota–Satsuma coupled KdV system and a coupled MKdV equation*

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In this paper, we make use of a new generalized ansatz in the homogeneous balance method, the well-known Riccati equation and the symbolic computation to study a generalized Hirota–Satsuma coupled KdV system and a coupled MKdV equation, respectively. As a result, numerous explicit exact solutions, comprising new solitary wave solutions, periodic wave solutions and the combined formal solitary wave solutions and periodic wave solutions, are obtained.

**Keywords:** coupled MKdV system, Riccati equation, solitary wave solution, periodic wave solution
**PACC:** 0340K, 0290, 0220, 0365G

## 1. Introduction

In recent years, search for explicit exact solutions, in particular, solitary wave solutions, of nonlinear evolution equations in mathematical physics constitutes an important part of soliton theory.[1−9,13−17] Various powerful methods, such as Bäcklund transformation, Darboux transformation, Cole–Hopf transformation, tanh method, sine-cosine method, Painlevé method, Hirota method and Rank analysis method,[22−25] have been introduced. Based upon the well-known Riccati equation, the homogenous balance method (HBM) was proposed by Wang [1,2] to find exact solutions of certain nonlinear partial differential equations (PDEs). Fan and Zhang[7,8] improved considerably the key steps of the HBM. Particularly, more general ansätze have been proposed in order to obtain new forms of solutions. Recently, Fan,[9] using an extended tanh-function method, obtained four kinds of soliton solutions for a new generalized Hirota–Satsuma coupled KdV system

$$u_t=\frac{1}{2}u_{xxx}-3uu_x+3(vw)_x,$$
$$v_t=-v_{xxx}+3uv_x,$$
$$w_t=-w_{xxx}+3uw_x, \quad (1)$$

and a new coupled MKdV equation

$$u_t=\frac{1}{2}u_{xxx}-3u^2u_x+\frac{3}{2}v_{xx}+3(uv)_x-3\lambda u_x,$$
$$v_t=-v_{xxx}-3vv_x-3u_xv_x+3u^2v_x+3\lambda v_x. \quad (2)$$

Eqs. (1) and (2) were derived recently by Wu *et al*[10] by introducing a $4\times4$ matrix spectral with three potentials. With $w=v^*$ and $w=v$, Eqs. (1) reduce respectively to a new complex coupled KdV equation[10] and the Hirota–Satsuma equation.[11,12] Eqs. (2) becomes a generalized KdV equation for $u=0$ and the MKdV equation for $v=0$, respectively. More recently, Hu[21] derived five kinds of solutions for Eqs. (1) by a delicate way of ansatz method. We have used some powerful method to seek solutions of nonlinear equations.[13−17] The motivation of the present paper is to utilize a new generalized ansatz developed by us[13] in HMB to explore more new solutions of Eqs. (1) and (2). As a result, more new exact solutions, including the solutions obtained by Fan,[9] are obtained.

## 2.Summary of our method

Our method is summed up as follows

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For the given system of nonlinear evolution equations, in two variables, say,

$$\begin{aligned}&F(u,H,u_t,H_t,u_x,H_x,u_{xt},\\&H_{xt},u_{tt},H_{tt},u_{xx},H_{xx},\ldots)=0,\\&G(u,H,u_t,H_t,u_x,H_x,u_{xt},\\&H_{xt},u_{tt},H_{tt},u_{xx},H_{xx},\ldots)=0,\end{aligned}\tag{3}$$

we seek for the formal travelling wave solutions which are of important physical significance

$$u(x,t)=u(\xi),\quad H(x,t)=v(\xi),\quad \xi=x+\lambda t+c,\tag{4}$$

where $\lambda$ is a constant to be determined later, $c$ is an arbitrary constant. Then the system (3) reduces to a system of nonlinear ordinary differential equations

$$\begin{aligned}&F_0(u,H,u',H',u'',H'',\cdots)=0,\\&G_0(u,H,u',H',u'',H'',\cdots)=0,\end{aligned}\tag{5}$$

where "$'$" denotes $\mathrm{d}/\mathrm{d}\xi$. In order to seek for the travelling wave solutions of system (3), we take the following transformations

$$\begin{aligned}u(\xi)=&\sum_{i=1}^{m}\omega^{i-1}(\xi)[A_i\omega(\xi)\\&+B_i\sqrt{\mu_1(1+\mu_2\omega^2(\xi))}]+A_0,\\H(\xi)=&\sum_{i=1}^{n}\omega^{i-1}(\xi)[a_i\omega(\xi)\\&+b_i\sqrt{\mu_1(1+\mu_2\omega^2(\xi))}]+a_0,\end{aligned}\tag{6}$$

and the new variable $\omega=\omega(\xi)$ satisfying

$$\omega'-R(1+\mu_2\omega^2)=\frac{\mathrm{d}\omega}{\mathrm{d}\xi}-R(1+\mu_2\omega^2)=0,\tag{7}$$

where $\mu_j=\pm1(j=1,2)$, $m$ and $n$ are integers to be determined and $A_i,B_i,a_j,b_j(i=0,1,2,\ldots,m;j=0,1,2,\ldots,n)$ and $R$ are constants to be determined later.

When $B_i=0,b_j=0(i=1,2,\ldots,m;j=1,2,\ldots,n)$, Eq.(6) becomes the transformation proposed by Ma *et al.*[18] But as $B_i\neq0,b_j\neq0(i=1,2,\ldots,m;j=1,2,\ldots,n)$, we would find many new exact solutions of system (3).

There are the following steps to be considered further.

**Step 1** Determine the values of $m$ and $n$ of system (6): by respectively balancing the highest-order partial derivative term and the nonlinear term in system (3) (or (5)), it is easy to find the value of $m$ and $n$.

**Step 2** With the aid of computerized symbolic computation system *Mathematica*, substituting system (6) along with the condition (7) into system (5) yields a system of algebraic equations with respect to $\omega^i(\mu_1+\mu_1\mu_2\omega^2)^{j/2}$ $(j=0,1;i=0,1,2,\ldots)$.

**Step 3** Collect all terms with the same power in $\omega^i(\mu_1+\mu_1\mu_2\omega^2)^{j/2}$ $(j=0,1;\ i=0,1,2,\ldots)$ and set the coefficients of the terms $\omega^i(\mu_1+\mu_1\mu_2\omega^2)^{j/2}$ $(j=0,1;\ i=0,1,2,\ldots)$ to zero, we obtain an over-determined system of nonlinear algebraic equations with respect to the unknown variables $\lambda,R,\ A_0,\ a_0,\ A_i,\ B_i,\ a_j,\ b_j\ (i=1,2,\ \ldots,\ m;\ j=1,2,\ldots,n)$.

**Step 4** With the aid of *Mathematica*, applying Wu-elimination method[19,20] to solve the above over-determined system of nonlinear algebraic equations obtained in step 3, yields the values of $\lambda,R,A_0,a_0,A_i,B_i,a_j,b_j(i=1,2,\ldots,m;j=1,2,\ldots,n)$.

**Step 5** It is well known that the general solutions of (7) are

(1) When $\mu_2=-1$,

$$\omega=\omega(\xi)=\frac{A-B\exp(-2R\xi)}{A+B\exp(-2R\xi)}=\begin{cases}1 & \text{for } B=0,\\-1 & \text{for } A=0,\\\tanh\left[R\xi-\dfrac{1}{2}\ln\left(\dfrac{A}{B}\right)\right] & \text{for } AB>0,\\\coth\left[R\xi-\dfrac{1}{2}\ln\left(-\dfrac{A}{B}\right)\right] & \text{for } AB<0.\end{cases}\tag{8}$$

Here $A$ and $B$ are arbitrary constants satisfying $A^2+B^2\neq0$. This solution may be obtained by three tricks: the Möbius transformation, the Cole–Hopf transformation or the relation

$$\frac{(\omega_1-\omega_2)(\omega_3-\omega_4)}{(\omega_1-\omega_3)(\omega_2-\omega_4)}=C=\text{const}.$$

of the solutions $\omega_i, 1\leq i\leq4$, beginning with three known solutions $1,-1,\tanh(R\xi)$.

(2) When $\mu_2=1$,

$$\omega=\omega(\xi)=\begin{cases}\tan(R\xi+\xi_0),\\-\cot(R\xi+\xi_0).\end{cases}\tag{9}$$

Thus according to Eqs. (4), (6), (8), (9) and the conclusions in step 4, we can obtain several travelling wave solutions of system (3).

## 3. Solitary wave solution and periodic wave solution for a new generalized Hirota–Satsuma coupled KdV system

To look for the solution of Eqs. (1), according to the above steps, we firstly make the following travelling wave transformation

$$u(x,t) = u(\xi), \quad v(x,t) = v(\xi),$$
$$w(x,t) = w(\xi), \quad \xi = x + \beta t, \tag{10}$$

where $\beta$ are constants to be determined.

Substituting (10) into (1) gives

$$\beta u' = \frac{1}{2}u''' - 3uu' + 3(vw)',$$
$$\beta v' = -v''' + 3uv',$$
$$\beta w' = -w''' + 3uw', \tag{11}$$

and integrating the first equation in (11) gives

$$\frac{1}{2}u'' - \frac{3}{2}u^2 + 3vw - \beta u = 0,$$
$$-v''' + 3uv' - \beta v' = 0,$$
$$-w''' + 3uw - \beta w' = 0. \tag{12}$$

According to step 1 in section 2, we suppose that (12) has the following two ansätze

$$\begin{aligned} u =& a_0 + a_1\omega + b_1\sqrt{\mu_1(1+\mu_2\omega^2)} \\ &+ a_2\omega^2 + b_2\omega\sqrt{\mu_1(1+\mu_2\omega^2)}, \end{aligned}$$
$$v = A_0 + A_1\omega + B_1\sqrt{\mu_1(1+\mu_2\omega^2)},$$
$$w = C_0 + C_1\omega + D_1\sqrt{\mu_1(1+\mu_2\omega^2)}, \tag{13}$$

and

$$\begin{aligned} u =& a_0 + a_1\omega + b_1\sqrt{\mu_1(1+\mu_2\omega^2)} \\ &+ a_2\omega^2 + b_2\omega\sqrt{\mu_1(1+\mu_2\omega^2)}, \\ v =& A_0 + A_1\omega + B_1\sqrt{\mu_1(1+\mu_2\omega^2)} \\ &+ A_2\omega^2 + B_2\omega\sqrt{\mu_1(1+\mu_2\omega^2)}, \\ w =& C_0 + C_1\omega + D_1\sqrt{\mu_1(1+\mu_2\omega^2)} \\ &+ C_2\omega^2 + D_2\omega\sqrt{\mu_1(1+\mu_2\omega^2)}, \end{aligned} \tag{14}$$

where $\omega = \omega(\xi)$ satisfies Eq.(7) and $a_0$, $a_1$, $a_2$, $b_1$, $b_2$, $A_0$, $A_1$, $A_2$, $B_1$, $B_2$, $C_0$, $C_1$, $C_2$, $D_1$, $D_2$ are constants to be determined later.

With the aid of *Mathematica*, substituting (13) into (12) along with (7) and collecting all terms with the same power in $\omega^i(\mu_1+\mu_1\mu_2\omega^2)^{j/2}$ $(j=0,1; i=0,1,2,3,4)$, yields a system of equations with respect to $\omega^i(\mu_1+\mu_1\mu_2\omega^2)^{j/2}$. Setting the coefficients of $\omega^i(\mu_1+\mu_1\mu_2\omega^2)^{j/2}$ $(j=0,1; i=0,1,2,3,4)$ in the obtained system of equations to zero, we can deduce the following set of over-determined algebraic polynomials with respect to the unknowns $a_0$, $a_1$, $a_2$, $b_1$, $b_2$, $A_0$, $A_1$, $A_2$, $B_1$, $B_2$, $C_0$, $C_1$, $C_2$, $D_1$, $D_2$, $R$, $\beta$, namely

$$\begin{aligned} &-3a_0^2 + 6A_0C_0 + 2a_2R^2 \\ &-2a_0\beta - 3b_1^2\mu_1 + 6B_1D_1\mu_1 = 0, \end{aligned}$$
$$\begin{aligned} &-6a_0a_1 + 6A_1C_0 + 6A_0C_1 \\ &-2a_1\beta - 6b_1b_2\mu_1 + 2a_1R^2\mu_2 = 0, \end{aligned}$$
$$\begin{aligned} &-3a_1^2 - 6a_0a_2 + 6A_1C_1 - 2a_2\beta - 3b_2^2\mu_1 \\ &+ 8a_2R^2\mu_2 - 3b_1^2\mu_1\mu_2 + 6B_1D_1\mu_1\mu_2 = 0, \end{aligned}$$
$$-6a_1a_2 + 2a_1R^2 - 6b_1b_2\mu_1\mu_2 = 0,$$
$$-3(a_2^2 - 2a_2R^2 + b_2^2\mu_1\mu_2) = 0,$$
$$-6a_0b_1 + 6B_1C_0 + 6A_0D_1 - 2b_1\beta + b_1R^2\mu_2 = 0,$$
$$-6a_1b_1 - 6a_0b_2 + 6B_1C_1 + 6A_1D_1 - 2b_2\beta + 5b_2R^2\mu_2 = 0,$$
$$-6a_2b_1 - 6a_1b_2 + 2b_1R^2 = 0,$$
$$b_2(-a_2 + R^2) = 0,$$
$$-A_1R(-3a_0 + \beta + 2R^2\mu_2) = 0,$$
$$3R(a_1A_1 + b_1B_1\mu_1\mu_2) = 0,$$
$$R(3B_1b_2\mu_1\mu_2 + A_1(3a_2 - 8R^2 + 3a_0\mu_2 - \beta\mu_2)) = 0,$$
$$3R(b_1B_1\mu_1 + a_1A_1\mu_2) = 0,$$
$$3R(B_1b_2\mu_1 + A_1\mu_2(a_2 - 2R^2\mu_2^2)) = 0,$$
$$3A_1b_1R = 0,$$
$$R(3A_1b_2 - B_1(5R^2 + (-3a_0 + \beta)\mu_2)) = 0,$$
$$3(A_1b_1 + a_1B_1)R\mu_2 = 0,$$
$$3R\mu_2(a_2B_1 + A_1b_2 - 2B_1R^2\mu_2^2) = 0,$$
$$-C_1R(-3a_0 + \beta + 2R^2\mu_2) = 0,$$
$$3R(a_1C_1 + b_1D_1\mu_1\mu_2) = 0,$$
$$R(3a_2C_1 + \mu_2(3a_0C_1 + 3b_2D_1\mu_1 - C_1(\beta + 8R^2\mu_2))) = 0,$$
$$3R\mu_2(a_1C_1 + b_1D_1\mu_1\mu_2) = 0,$$
$$3R\mu_2(a_2C_1 + \mu_2(b_2D_1\mu_1 - 2C_1R^2\mu_2)) = 0,$$
$$3b_1C_1R = 0,$$
$$R(3b_2C_1 - D_1\mu_2(-3a_0 + \beta + 5R^2\mu_2)) = 0,$$
$$3(b_1C_1 + a_1D_1)R\mu_2 = 0,$$
$$3R(b_2C_1 + D_1(a_2 - 2R^2))\mu_2 = 0.$$

From which, with the aid of *Mathematica*, we have

**Case 1**

$$a_0 = \frac{\beta + 2R^2\mu_2}{3}, \quad a_1 = 0, \quad a_2 = 2R^2,$$

$$A_0=\frac{-4R^2C_0(-\beta+R^2\mu_2)}{3C_1^2},$$
$$A_1=-\frac{4R^2(-\beta+R^2\mu_2)}{3C_1^2},$$
$$b_1=b_2=B_1=D_1=0,$$

where $C_0, R, \beta$ and $C_1 \neq 0$ are arbitrary constants, and we have the following identity

$$\frac{-24C_0^2R^2(R^2\mu_2-\beta)+C_1^2(8R^4-8R^2\beta\mu_2-3\beta^2)}{3C_1^2}=0.$$

**Case 2**

$$a_0=\frac{2R^2+\beta\mu_2}{3\mu_2},\quad a_1=0,\quad a_2=R^2,$$
$$b_1=0,\quad b_2=\frac{R^2}{\sqrt{\mu_1\mu_2}},$$
$$A_0=A_0,\quad A_1=-\frac{R^4-4R^2\beta\mu_2}{12C_1\mu_2},$$
$$B_1=-\frac{R^4-4R^2\beta\mu_2}{12C_1\mu_2\sqrt{\mu_1\mu_2}},$$
$$C_0=C_0,\quad C_1=\frac{\sqrt{C_0R^2(R^2-4\beta\mu_2)}}{2\sqrt{3A_0\mu_2}},$$
$$D_1=\frac{C_1}{\sqrt{\mu_1\mu_2}},$$

where $C_0, R, \beta$ and $A_0 \neq 0$ are arbitrary constants, and we have the following identity

$$36A_0C_0+R^4-6\beta^2-4R^2\beta\mu_2=0.$$

Therefore, according to step 5, eight families of explicit and exact travelling wave solutions, comprising solitary wave solutions, periodic wave solutions and new travelling wave solutions, are found as follows for (1).

**Case 1**–solutions:

$$u_{11}=\frac{\beta-2R^2}{3}+2R^2\tanh^2R(x+\beta t),$$
$$v_{11}=\frac{4R^2C_0(\beta+R^2)}{3C_1^2}+\frac{4R^2(\beta+R^2)}{3C_1^2}\tanh R(x+\beta t),$$
$$w_{11}=C_0+C_1\tanh R(x+\beta t),$$
$$u_{12}=\frac{\beta-2R^2}{3}+2R^2\coth^2R(x+\beta t),$$
$$v_{12}=\frac{4R^2C_0(\beta+R^2)}{3C_1^2}+\frac{4R^2(\beta+R^2)}{3C_1^2}\coth R(x+\beta t),$$
$$w_{12}=C_0+C_1\coth R(x+\beta t),$$

where $C_0, R, \beta$ and $C_1 \neq 0$ are arbitrary constants, and we have the following identity

$$\frac{24C_0^2R^2(R^2+\beta)+C_1^2(8R^4+8R^2\beta-3\beta^2)}{3C_1^2}=0.$$

$$u_{13}=\frac{\beta+2R^2}{3}+2R^2\tan^2R(x+\beta t),$$
$$v_{13}=\frac{4R^2C_0(\beta-R^2)}{3C_1^2}+\frac{4R^2(\beta-R^2)}{3C_1^2}\tan R(x+\beta t),$$
$$w_{13}=C_0+C_1\tan R(x+\beta t),$$
$$u_{14}=\frac{\beta+2R^2}{3}-2R^2\cot^2R(x+\beta t),$$
$$v_{14}=\frac{4R^2C_0(\beta-R^2)}{3C_1^2}-\frac{4R^2(\beta-R^2)}{3C_1^2}\cot R(x+\beta t),$$
$$w_{14}=C_0-C_1\cot R(x+\beta t),$$

where $C_0, R, \beta$ and $C_1 \neq 0$ are arbitrary constants, and we have the following identity

$$\frac{-24C_0^2R^2(R^2-\beta)+C_1^2(8R^4-8R^2\beta-3\beta^2)}{3C_1^2}=0.$$

**Case 2**–solutions:

$$u_{21}=-\frac{2R^2-\beta}{3}+R^2\tanh^2R(x+\beta t)\pm \mathrm{i}R^2\tanh R(x+\beta t)\mathrm{sech}R(x+\beta t),$$

$$v_{21}=A_0+\frac{R^4+4R^2\beta}{12C_1}\tanh R(x+\beta t)\pm \mathrm{i}\frac{R^4+4R^2\beta}{12C_1}\mathrm{sech}R(x+\beta t),$$

$$w_{21}=C_0+\frac{\sqrt{C_0R^2(R^2+4\beta)}}{2\sqrt{3A_0}}\tanh R(x+\beta t)\pm \mathrm{i}\frac{\sqrt{C_0R^2(R^2+4\beta)}}{2\sqrt{3A_0}}\mathrm{sech}R(x+\beta t),$$

$$u_{22}=-\frac{2R^2-\beta}{3}+R^2\coth^2R(x+\beta t)\pm R^2\coth R(x+\beta t)\mathrm{csch}R(x+\beta t),$$

$$v_{22}=A_0+\frac{R^4+4R^2\beta}{12C_1}\coth R(x+\beta t)\pm\frac{R^4+4R^2\beta}{12C_1}\mathrm{csch}R(x+\beta t),$$

$$w_{22}=C_0+\frac{\sqrt{C_0R^2(R^2+4\beta)}}{2\sqrt{3A_0}}\coth R(x+\beta t)\pm\frac{\sqrt{C_0R^2(R^2+4\beta)}}{2\sqrt{3A_0}}\mathrm{csch}R(x+\beta t),$$

where $C_0, R, \beta$ and $A_0 \neq 0$ are arbitrary constants, and we have the following identity

$$36A_0C_0 + R^4 - 6\beta^2 + 4R^2\beta = 0.$$

$$u_{23} = \frac{2R^2+\beta}{3} + R^2\tan^2 R(x+\beta t) \pm R^2 \tan R(x+\beta)\sec R(x+\beta t),$$

$$v_{23} = A_0 - \frac{R^4 - 4R^2\beta}{12C_1}\tan R(x+\beta t) \pm \frac{R^4-4R^2\beta}{12C_1}\sec R(x+\beta t),$$

$$w_{23} = C_0 + \frac{\sqrt{C_0R^2(R^2-4\beta)}}{2\sqrt{3A_0}}\tan R(x+\beta t) \pm C_1 \sec R(x+\beta t),$$

$$u_{24} = \frac{2R^2+\beta}{3} + R^2\cot^2 R(x+\beta t) \pm R^2 \cot R(x+\beta)\csc R(x+\beta t),$$

$$v_{24} = A_0 + \frac{R^4 - 4R^2\beta}{12C_1}\cot R(x+\beta t) \pm \frac{R^4-4R^2\beta}{12C_1}\csc R(x+\beta t),$$

$$w_{24} = C_0 - \frac{\sqrt{C_0R^2(R^2-4\beta)}}{2\sqrt{3A_0}}\cot R(x+\beta t) \pm C_1 \csc R(x+\beta t),$$

where $C_0, R, \beta$ and $A_0 \neq 0$ are arbitrary constants, and we have the following identity

$$36A_0C_0 + R^4 - 6\beta^2 - 4R^2\beta = 0.$$

Similarly, substituting ansatz (14) into (12) yields

$$-3a_0^2 + 6A_0C_0 + 2a_2R^2 - 2a_0\beta - 3b_1^2\mu_1 + 6B_1D_1\mu_1 = 0,$$

$$-6a_0a_1 + 6A_1C_0 + 6A_0C_1 - 2a_1\beta - 6b_1b_2\mu_1 + 6B_2D_1\mu_1 + 6B_1D_2\mu_1 + 2a_1R^2\mu_2 = 0,$$

$$-3a_1^2 - 6a_0a_2 + 6A_2C_0 + 6A_1C_1 + 6A_0C_2 - 2a_2\beta - 3b_2^2\mu_1 + 6B_2D_2\mu_1 + 8a_2R^2\mu_2 - 3b_1^2\mu_1\mu_2 + 6B_1D_1\mu_1\mu_2 = 0,$$

$$2(a_1(-3a_2+R^2)) + 3(A_2C_1 + A_1C_2 + (-b_1b_2 + B_2D_1 + B_1D_2)\mu_1\mu_2)) = 0,$$

$$-3(a_2^2 - 2A_2C_2 - 2a_2R^2 + (b_2^2 - 2B_2D_2)\mu_1\mu_2) = 0,$$

$$-6a_0b_1 + 6B_1C_0 + 6A_0D_1 - 2b_1\beta + b_1R^2\mu_2 = 0,$$

$$-6a_1b_1 - 6a_0b_2 + 6B_2C_0 + 6B_1C_1 + 6A_1D_1 + 6A_0D_2 - 2b_2\beta + 5b_2R^2\mu_2 = 0,$$

$$-6a_2b_1 - 6a_1b_2 + 6B_2C_1 + 6B_1C_2 + 6A_2D_1 + 6A_1D_2 + 2b_1R^2 = 0,$$

$$6(-a_2b_2 + B_2C_2 + A_2D_2 + b_2R^2) = 0,$$

$$R(3a_0C_1 + 3b_1D_2\mu_1 - C_1(\beta + 2R^2\mu_2)) = 0,$$

$$R(3a_1C_1 + 6a_0C_2 - 2C_2\beta + 3b_2D_2\mu_1 - 16C_2R^2\mu_2 + 3b_1D_1\mu_1\mu_2) = 0,$$

$$R(3a_2C_1 + 6a_1C_2 - 8C_1R^2 + 3a_0C_1\mu_2 - C_1\beta\mu_2 + 3b_2D_1\mu_1\mu_2 + 9b_1D_2\mu_1\mu_2) = 0,$$

$$R(6a_2C_2 - 2C_2(20R^2 - 3a_0\mu_2 + \beta\mu_2) + 3(b_1D_1\mu_1 + a_1C_1\mu_2 + 3b_2D_2\mu_1\mu_2)) = 0,$$

$$3R(b_2D_1\mu_1 + 2b_1D_2\mu_1 + a_2C_1\mu_2 + 2a_1C_2\mu_2 - 2C_1R^2\mu_2^3) = 0,$$

$$6R(b_2D_2\mu_1 + C_2\mu_2(a_2 - 4R^2\mu_2^2)) = 0,$$

$$R(3b_1C_1 + D_2(3a_0 - \beta - 5R^2\mu_2)) = 0,$$

$$R(3b_2C_1 + 6b_1C_2 + 3a_1D_2 - 5D_1R^2 + 3a_0D_1\mu_2 - D_1\beta\mu_2) = 0,$$

$$R(6b_2C_2 + 3a_2D_2 - 28D_2R^2 + 3b_1C_1\mu_2 + 3a_1D_1\mu_2 + 6a_0D_2\mu_2 - 2D_2\beta\mu_2) = 0,$$

$$3R\mu_2(b_2C_1 + 2b_1C_2 + a_2D_1 + 2a_1D_2 - 2D_1R^2\mu_2^2) = 0,$$

$$6R\mu_2(b_2C_2 + D_2(a_2 - 4R^2\mu_2^2)) = 0,$$

$$R(3a_0A_1 + 3b_1B_2\mu_1 - A_1(\beta + 2R^2\mu_2)) = 0,$$

$$R(3a_1A_1 + 6a_0A_2 - 2A_2\beta + 3b_2B_2\mu_1 - 16A_2R^2\mu_2 + 3b_1B_1\mu_1\mu_2) = 0,$$

$$R(A_1(3a_2 - 8R^2 + 3a_0\mu_2 - \beta\mu_2) + 3(2a_1A_2 + (B_1b_2 + 3b_1B_2)\mu_1\mu_2)) = 0,$$

$$R(6a_2A_2 - 2A_2(20R^2 - 3a_0\mu_2 + \beta\mu_2) + 3(b_1B_1\mu_1 + a_1A_1\mu_2 + 3b_2B_2\mu_1\mu_2)) = 0,$$

$$3R(B_1b_2\mu_1 + 2b_1B_2\mu_1 + A_1a_2\mu_2 + 2a_1A_2\mu_2 - 2A_1R^2\mu_2^3) = 0,$$

$$6R(b_2B_2\mu_1 + A_2\mu_2(a_2 - 4R^2\mu_2^2)) = 0,$$

$$R(3A_1b_1 + B_2(3a_0 - \beta - 5R^2\mu_2)) = 0,$$

$$R(6A_2b_1 + 3A_1b_2 + 3a_1B_2 - 5B_1R^2 + 3a_0B_1\mu_2 - B_1\beta\mu_2) = 0,$$

$$R(6A_2b_2 + 3a_2B_2 - 28B_2R^2 + 3A_1b_1\mu_2 + 3a_1B_1\mu_2 + 6a_0B_2\mu_2 - 2B_2\beta\mu_2) = 0,$$

$$3R\mu_2(2A_2b_1 + a_2B_1 + A_1b_2 + 2a_1B_2 - 2B_1R^2\mu_2^2) = 0,$$

$$6R\mu_2(A_2b_2 + B_2(a_2 - 4R^2\mu_2^2)) = 0.$$

From which we have

**Case 1**

$$a_0 = \frac{\beta\mu_2 + 8R^2}{3\mu_2}, \quad a_2 = 4R^2,$$

$$A_0 = \frac{4(-3R^4C_0\mu_2 + 2\beta R^2C_2\mu_2 + 4R^4C_2)}{3C_2^2\mu_2},$$

$$A_2 = \frac{4R^4}{C_2}, \quad b_1 = b_2 = B_1 = B_2 = D_1 = D_2 = 0,$$

where $C_0, R, \beta$ and $C_2 \neq 0$ are arbitrary constants, and holds the equation

$$C_0 = \frac{C_2(4\beta + (8R^2 \pm \sqrt{-16R^4 + 10\beta^2})\mu_2)}{12R^2}.$$

**Case 2**

$$a_0 = \frac{\beta + 5R^2\mu_2}{3}, \quad a_1 = 0, \quad a_2 = 2R^2,$$

$$b_1 = 0, \quad b_2 = \frac{2R^2\sqrt{\mu_2}}{\sqrt{\mu_1}},$$

$$A_0 = -\frac{3C_0R^4 - 4C_2R^2\beta - 5C_2R^4\mu_2}{3C_2^2},$$

$$A_1 = B_1 = 0, \quad A_2 = \frac{R^4}{C_2}, \quad B_2 = \frac{R^4\sqrt{\mu_2}}{C_2\sqrt{\mu_1}},$$

$$C_1 = D_1 = 0, \quad D_2 = \frac{C_2\sqrt{\mu_2}}{\sqrt{\mu_1}},$$

where $C_0, R, \beta$ and $C_2 \neq 0$ are arbitrary constants.

Therefore, according to step 5, eight families of explicit and exact travelling wave solution, comprising solitary wave solutions, periodic wave solutions and new travelling wave solutions, are found as follows for (1).

**Case 1**–solutions:

$$u_{11} = \frac{\beta - 8R^2}{3} + 4R^2\tanh^2 R(x + \beta t),$$

$$v_{11} = -\frac{4(3R^4C_0 - 2\beta R^2C_2 + 4R^4C_2)}{3C_2^2} + \frac{4R^4}{C_2}\tanh^2 R(x + \beta t),$$

$$w_{11} = C_0 + C_2\tanh^2 R(x + \beta t),$$

$$u_{12} = \frac{\beta - 8R^2}{3} + 4R^2\coth^2 R(x + \beta t),$$

$$v_{12} = -\frac{4(3R^4C_0 - 2\beta R^2C_2 + 4R^4C_2)}{3C_2^2} + \frac{4R^4}{C_2}\coth^2 R(x + \beta t),$$

$$w_{12} = C_0 + C_2\coth^2 R(x + \beta t),$$

where $C_0, R, \beta$ and $C_2 \neq 0$ are arbitrary constants, and we have the following identity

$$C_0 = \frac{C_2(4\beta - (8R^2 \pm \sqrt{-16R^4 + 10\beta^2}))}{12R^2}.$$

$$u_{13} = \frac{\beta + 8R^2}{3} + 4R^2\tan^2 R(x + \beta t),$$

$$v_{13} = -\frac{4(3R^4C_0 + 2\beta R^2C_2 + 4R^4C_2)}{3C_2^2} + \frac{4R^4}{C_2}\tan^2 R(x + \beta t),$$

$$w_{13} = C_0 + C_2\tan^2 R(x + \beta t),$$

$$u_{14} = \frac{\beta + 8R^2}{3} + 4R^2\cot^2 R(x + \beta t),$$

$$v_{14} = -\frac{4(3R^4C_0 + 2\beta R^2C_2 + 4R^4C_2)}{3C_2^2} + \frac{4R^4}{C_2}\cot^2 R(x + \beta t),$$

$$w_{14} = C_0 + C_2\cot^2 R(x + \beta t),$$

where $C_0, R, \beta$ and $C_2 \neq 0$ are arbitrary constants, and holds the equation

$$C_0 = \frac{C_2(4\beta + (8R^2 \pm \sqrt{-16R^4 + 10\beta^2}))}{12R^2}.$$

**Case 2**–solutions:

$$u_{21} = \frac{\beta - 5R^2}{3} + 2R^2\tanh^2 R(x + \beta t) \pm \mathrm{i}2R^2\tanh R(x + \beta t)\operatorname{sech} R(x + \beta t),$$

$$v_{21} = -\frac{3C_0R^4 - 4C_2R^2\beta + 5C_2R^4}{3C_2^2} + \frac{R^4}{C_2}\tanh^2 R(x+\beta t) \pm \frac{\mathrm{i}R^4}{C_2}\tanh R(x+\beta t)\mathrm{sech} R(x+\beta t),$$

$$w_{21} = C_0 + C_2\tanh^2 R(x+\beta t) \pm \mathrm{i}C_2\tanh R(x+\beta t)\mathrm{sech} R(x+\beta t),$$

$$u_{22} = \frac{\beta - 5R^2}{3} + 2R^2\coth^2 R(x+\beta t) \pm 2R^2\coth R(x+\beta t)\mathrm{csch} R(x+\beta t),$$

$$v_{22} = -\frac{3C_0R^4 - 4C_2R^2\beta + 5C_2R^4}{3C_2^2} + \frac{R^4}{C_2}\coth^2 R(x+\beta t) \pm \frac{R^4}{C_2}\coth R(x+\beta t)\mathrm{csch} R(x+\beta t),$$

$$w_{22} = C_0 + C_2\coth^2 R(x+\beta t) \pm C_2\coth R(x+\beta t)\mathrm{csch} R(x+\beta t),$$

$$u_{23} = \frac{\beta + 5R^2}{3} + 2R^2\tan^2 R(x+\beta t) \pm 2R^2\tan R(x+\beta t)\sec R(x+\beta t),$$

$$v_{23} = -\frac{3C_0R^4 - 4C_2R^2\beta - 5C_2R^4}{3C_2^2} + \frac{R^4}{C_2}\tan^2 R(x+\beta t) \pm \frac{R^4}{C_2}\tan R(x+\beta t)\sec R(x+\beta t),$$

$$w_{23} = C_0 + C_2\tan^2 R(x+\beta t) \pm C_2\tan R(x+\beta t)\sec R(x+\beta t),$$

$$u_{24} = \frac{\beta + 5R^2}{3} + 2R^2\cot^2 R(x+\beta t) \pm 2R^2\cot R(x+\beta t)\csc R(x+\beta t),$$

$$v_{24} = -\frac{3C_0R^4 - 4C_2R^2\beta - 5C_2R^4}{3C_2^2} + \frac{R^4}{C_2}\cot^2 R(x+\beta t) \pm \frac{R^4}{C_2}\cot R(x+\beta t)\csc R(x+\beta t),$$

$$w_{24} = C_0 + C_2\cot^2 R(x+\beta t) \pm C_2\cot R(x+\beta t)\csc R(x+\beta t),$$

where $C_0, R, \beta$ and $C_2 \neq 0$ are arbitrary constants.

**Remark 1**

1. It is easily seen that the above obtained solutions include the two kinds of solutions of Eqs. (1) obtained by Fan.[9] But to the best of our knowledge, the rest of solutions of Eqs. (1) have not been found before.

2. In Ref.[21], using a different method, Hu obtained five kinds of solutions for Eqs. (1), two kinds of the solutions are the same as the solutions by Fan and us. However, the other three kinds of solutions obtained by Hu are not covered by our improved method. So there is no general method for solving given nonlinear PDEs.

# 4. Solitary wave solution and periodic wave solution for the coupled MKdV equation

Let $u(x,t) = u(\xi)$, $v(x,t) = v(\xi)$, $\xi = x + \beta t$, then Eq.(2) becomes

$$\beta u' = \frac{1}{2}u''' - 3u^2u' + \frac{3}{2}v'' + 3(uv)' - 3\lambda u',$$

$$\beta v' = -v''' - 3vv' - 3u'v' + 3u^2v' + 3\lambda v'. \tag{15}$$

According to the same above-mentioned steps, we firstly make the following two ansätze

$$u = a_0 + a_1\omega + b_1\sqrt{\mu_1(1+\mu_2\omega^2)},$$

$$v = A_0 + A_1\omega + B_1\sqrt{\mu_1(1+\mu_2\omega^2)}, \tag{16}$$

and

$$u = a_0 + a_1\omega + b_1\sqrt{\mu_1(1+\mu_2\omega^2)},$$

$$v = A_0 + A_1\omega + B_1\sqrt{\mu_1(1+\mu_2\omega^2)} + A_2\omega^2 + B_2\omega\sqrt{\mu_1(1+\mu_2\omega^2)}. \tag{17}$$

With the aid of *Mathematica*, substituting (16) into (15), we can obtain the over-determined algebraic equations

$$R(-3a_0^2a_1 + 3a_0A_1 + a_1(3A_0 - \beta - 3\lambda - 3b_1^2\mu_1 + R^2\mu_2)) = 0,$$

$$3R(2a_1A_1 + A_1R\mu_2 + 2b_1B_1\mu_1\mu_2 - 2a_0(a_1^2 + b_1^2\mu_1\mu_2)) = 0,$$

$$R(-3a_1^3+3a_0A_1\mu_2-a_1\mu_2(3a_0^2-3A_0+\beta+3\lambda+12b_1^2\mu_1-4R^2\mu_2))=0,$$

$$3R\mu_2(2a_1A_1+A_1R\mu_2+2b_1B_1\mu_1\mu_2-2a_0(a_1^2+b_1^2\mu_1\mu_2))=0,$$

$$-3a_1R\mu_2(a_1^2+\mu_2(3b_1^2\mu_1-R^2\mu_2))=0,$$

$$\frac{3}{2}R(-4a_0a_1b_1+2A_1b_1+2a_1B_1+B_1R\mu_2)=0,$$

$$-\frac{1}{2}R(12a_1^2b_1+\mu_2(6a_0^2b_1-6a_0B_1+b_1(-6A_0+2\beta+6\lambda+6b_1^2\mu_1-5R^2\mu_2)))=0,$$

$$3R\mu_2(-4a_0a_1b_1+2A_1b_1+2a_1B_1+B_1R\mu_2)=0,$$

$$-3b_1R\mu_2(3a_1^2+\mu_2(b_1^2\mu_1-R^2\mu_2))=0,$$

$$-A_1R(-3a_0^2+3A_0+3a_1R+\beta-3\lambda-3b_1^2\mu_1+2R^2\mu_2)=0,$$

$$-3R(A_1^2+B_1^2\mu_1\mu_2-2a_0(a_1A_1+b_1B_1\mu_1\mu_2))=0,$$

$$R(3a_1^2A_1+a_1(-6A_1R\mu_2+6b_1B_1\mu_1\mu_2)-\mu_2(-3a_0^2A_1+3A_0A_1+A_1\beta-3A_1\lambda-6A_1b_1^2\mu_1+8A_1R^2\mu_2+3b_1B_1R\mu_1\mu_2))=0,$$

$$-3R\mu_2(A_1^2+B_1^2\mu_1\mu_2-2a_0(a_1A_1+b_1B_1\mu_1\mu_2))=0,$$

$$3R\mu_2(a_1^2A_1+a_1(-A_1R\mu_2+2b_1B_1\mu_1\mu_2)+\mu_2(A_1b_1^2\mu_1-2A_1R^2\mu_2-b_1B_1R\mu_1\mu_2))=0,$$

$$3A_1(2a_0b_1-B_1)R=0,$$

$$R(a_1(6A_1b_1-3B_1R\mu_2)-\mu_2(-3a_0^2B_1+3A_0B_1+3A_1b_1R+B_1\beta-3B_1\lambda-3b_1^2B_1\mu_1+5B_1R^2\mu_2))=0,$$

$$6(a_0A_1b_1+a_0a_1B_1-A_1B_1)R\mu_2=0,$$

$$3R\mu_2(a_1^2B_1+a_1(2A_1b_1-B_1R\mu_2)+\mu_2(-A_1b_1R+b_1^2B_1\mu_1-2B_1R^2\mu_2))=0.$$

From which, we find

**Case 1**

$$a_0=-\frac{A_1}{2R\mu_2},\quad a_1=-R\mu_2,\quad b_1=0,$$

$$A_0=\lambda,\quad \beta=\frac{3A_1^2+4R^4\mu_2}{4R^2},\quad B_1=0,$$

where $R\neq 0,\beta$ and $A_1$ are arbitrary constants.

**Case 2**

$$a_1=-\frac{R\mu_2}{2},\quad b_1=-\frac{R\sqrt{\mu_2}}{2\sqrt{\mu_1}},$$

$$A_0=\frac{-12a_0^2+4\beta+12\lambda-R^2\mu_2}{12},\quad A_1=2a_0a_1,$$

$$B_1=2a_0b_1,\quad \beta=\frac{12a_0^2+R^2\mu_2}{4},$$

where $a_0\neq 0,R,\beta$ is an arbitrary constant.

Therefore, according to step 5, eight families of explicit and exact travelling wave solution, comprising solitary wave solutions, periodic wave solutions and new travelling wave solution, are found as follows for Eqs. (2).

**Case 1**–solutions:

$$u_{11}=\frac{A_1}{2R}+R\tanh R\left(x+\left(\frac{3A_1^2}{4R^2}-R^2\right)t\right),$$

$$v_{11}=\lambda+A_1\tanh R\left(x+\left(\frac{3A_1^2}{4R^2}-R^2\right)t\right),$$

$$u_{12}=\frac{A_1}{2R}+R\coth R\left(x+\left(\frac{3A_1^2}{4R^2}-R^2\right)t\right),$$

$$v_{12}=\lambda+A_1\coth R\left(x+\left(\frac{3A_1^2}{4R^2}-R^2\right)t\right),$$

$$u_{13}=-\frac{A_1}{2R}-R\tan R\left(x+\left(\frac{3A_1^2}{4R^2}+R^2\right)t\right),$$

$$v_{13}=\lambda+A_1\tan R\left(x+\left(\frac{3A_1^2}{4R^2}+R^2\right)t\right),$$

$$u_{14}=-\frac{A_1}{2R}+R\cot R\left(x+\left(\frac{3A_1^2}{4R^2}+R^2\right)t\right),$$

$$v_{14}=\lambda-A_1\cot R\left(x+\left(\frac{3A_1^2}{4R^2}+R^2\right)t\right),$$

where $\lambda\neq 0,A_1\neq 0$ and $R\neq 0$ are arbitrary constants.

**Case 2**–solution:

$$u_{21}=a_0+\frac{R}{2}\tanh R\left(x+\frac{12a_0^2-R^2}{4}t\right)\pm \mathrm{i}\frac{R}{2}\mathrm{sech} R\left(x+\frac{12a_0^2-R^2}{4}t\right),$$

$$v_{21}=\lambda+Ra_0\tanh R\left(x+\frac{12a_0^2-R^2}{4}t\right)\pm \mathrm{i}Ra_0\mathrm{sech} R\left(x+\frac{12a_0^2-R^2}{4}t\right),$$

$$u_{22}=a_0+\frac{R}{2}\coth R\left(x+\frac{12a_0^2-R^2}{4}t\right)\pm \frac{R}{2}\mathrm{csch} R\left(x+\frac{12a_0^2-R^2}{4}t\right),$$

$$v_{22} = \lambda + Ra_0 \coth R\left(x + \frac{12a_0^2 - R^2}{4}t\right) + Ra_0 \mathrm{csch} R\left(x + \frac{12a_0^2 - R^2}{4}t\right),$$

where $a_0 \neq 0, R, \beta$ is an arbitrary constant.

$$u_{23} = a_0 - \frac{R}{2}\tan R\left(x + \frac{12a_0^2 + R^2}{4}t\right) \pm \frac{R}{2}\sec R\left(x + \frac{12a_0^2 + R^2}{4}t\right),$$

$$v_{23} = \lambda - Ra_0 \tan R\left(x + \frac{12a_0^2 + R^2}{4}t\right) \pm Ra_0 \sec R\left(x + \frac{12a_0^2 + R^2}{4}t\right),$$

$$u_{24} = a_0 + \frac{R}{2}\cot R\left(x + \frac{12a_0^2 + R^2}{4}t\right) \pm \frac{R}{2}\csc R\left(x + \frac{12a_0^2 + R^2}{4}t\right),$$

$$v_{24} = \lambda + Ra_0 \cot R\left(x + \frac{12a_0^2 + R^2}{4}t\right) - Ra_0 \csc R\left(x + \frac{12a_0^2 + R^2}{4}t\right),$$

where $a_0 \neq 0, R, \beta$ is an arbitrary constant.

For ansatz (17), similarly, we have

**Case 1**

$$a_0 = 0, \quad a_1 = -R\mu_2, \quad b_1 = 0, \quad \beta = R^2\mu_2,$$

$$A_0 = 2R^2\mu_2 + 4R, \quad A_1 = 0,$$

$$A_2 = -2R^2, \quad B_1 = 0, \quad B_2 = 0,$$

where $\lambda = 4(R^2\mu_2 + R)$.

Therefore, according to step 5, four families of explicit and exact travelling wave solutions, comprising solitary wave solutions, periodic wave solutions and new travelling wave solutions, are found as follows for Eq.(2).

**Case 2**

$$u_{11} = R\tanh R(x - R^2t),$$

$$v_{11} = 2R(-R+2) - 2R^2\tanh^2 R(x - R^2t),$$

$$u_{12} = R\coth R(x - R^2t),$$

$$v_{12} = 2R(-R+2) - 2R^2\coth^2 R(x - R^2t),$$

where $\lambda = 4(-R^2 + R)$.

$$u_{13} = R\tan R(x + R^2t),$$

$$v_{13} = 2R(R+2) - 2R^2\tan^2 R(x + R^2t),$$

$$u_{14} = -R\cot R(x + R^2t),$$

$$v_{14} = 2R(R+2) - 2R^2\cot^2 R(x + R^2t),$$

where $\lambda = 4(R^2 + R)$.

**Remark 2** It is easily seen that the above obtained solutions include the two kinds of solutions of Eqs. (2) by Fan.[9] But to the best of our knowledge, the rest of solutions of Eqs. (2) have not been found before.

## 5. Conclusions

In summary, based on the well-know Riccati equation, many new types of exact solutions for the new generalized Hirota–Satsuma coupled KdV system (1) and the new coupled KdV have been derived by a generalized transformation. Seven kinds of them are singular soliton solutions. Such solutions develop a singularity at a finite point, i.e. for any fixed $t = t_0$, there exists $x_0$ at which these solutions blow up. There is much current interest in the formation of so-called "hot spots" or "blow up" of solutions. It appears that these singular solutions will model this physical phenomena. The method can also be easily extended to other PDEs and is sufficient to seek more new solitary wave solutions of PDEs. These solutions contain the known ones.[9] Our method not only uses a more generalized transformation to produce an over-determined system of nonlinear algebraic equation, but also can find more solutions. With the aid of *Mathematica* (or *Maple*, *Reduce*) and Wu elimination method, the course of solving PDEs is computerizable, which allow us to perform complicated and tedious algebraic calculation on a computer.

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