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New Exact Travelling Wave Solutions to Hirota Equation and (1+1)-Dimensional Dispersive Long Wave Equation*

WANG Qi,[1,4] CHEN Yong,[2,3,4,†] LI Biao,[1,4,‡] and ZHANG Hong-Qing[1,4]

[1]Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China

[2]Department of Physics, Ningbo University, Ningbo 315211, China

[3]Department of Physics, Shanghai Jiao Tong University, Shanghai 200030, China

[4]Key Laboratory of Mathematics and Mechanization, the Chinese Academy of Sciences, Beijing 100080, China

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Abstract *Based on the computerized symbolic Maple, we study two important nonlinear evolution equations, i.e., the Hirota equation and the (1+1)-dimensional dispersive long wave equation by use of a direct and unified algebraic method named the general projective Riccati equation method to find more exact solutions to nonlinear differential equations. The method is more powerful than most of the existing tanh method. New and more general form solutions are obtained. The properties of the new formal solitary wave solutions are shown by some figures.*

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1 Introduction

The nonlinear evolution equations (NEEs) are widely used to describe many important phenomena and dynamic processes in physics, mechanics, chemistry, biology, etc. In recent year, directly finding exact solutions to NEEs has become more and more attractive.[1−12] The success of the symbolic mathematical computation discipline is striking. The availability of computer systems like ***Maple*** and ***Mathematica*** helps us perform some complicated and tedious algebraic calculation on a computer to realize our algorithms.

Hirota equation[13,14]

$$\mathrm{i}u_t + u_{xx} + 2|u|^2u + \mathrm{i}\alpha\, u_{xxx} + 6\,\mathrm{i}\alpha\,|u|^2u_x = 0\,, \qquad (1)$$

which is the standard Schrödinger equation in the case when $\alpha = 0$, is a famous mathematical and physical equation.

The (1+1)-dimensional dispersive long wave equation (DLWE) is[15−18]

$$v_t + vv_x + w_x = 0\,, \quad w_t + (wv)_x + \frac{1}{3}v_{xxx} = 0\,. \qquad (2)$$

In Ref. [15], Wu and Zhang derived three sets of model equations of the Boussinesq class for modelling nonlinear and dispersive long gravity waves travelling in two horizontal directions in shallow water of uniform depth. Their comparative study of these models is directed to exploring the intrinsic properties in physical and mathematical terms that these models possess (see Refs. [15] ∼ [18] for details). In Ref. [16], some special types of soliton solutions to (2+1) WZ equations are derived by the standard and nonstandard truncation of the WTC's approach and the modified Conte's invariant Painleve expansion for the (3+1)-dimensional WZ equations, which can be reduced to the (1+1)-dimensional dispersive long wave equation (2) by scaling transformation and symmetry reduction. Chen *et al.*[17] used elliptic function method to obtain three families of soliton solutions to (2+1)-dimensional WZ equations. Zheng *et al.*[18] used the generalized extended tanh-function method to construct new explicit exact solutions of the (1+1)-dimensional dispersive long wave equations.

In Ref. [19], Conte and Musette presented a direct and unified algebraic method for building new solitary wave solutions to nonlinear PDEs that can be expressed as a polynomial in two elementary functions, which satisfy the coupled projective Riccati equations.[20] Recently, Yan[21] improved Conte's method and presented the general projective Riccati equation method to find more exact solutions to nonlinear differential equations based upon a more general form of the projective Riccati equations.

The present work is motivated by the desire to improve the generally projective Riccati equation method to study Hirota equation (1) and the (1+1)-dimensional dispersive long wave equation (DLWE) (2). Hirota equation (1) is not made of the NLPDE $E(u) = 0$ polynomial in u and its derivatives. However, by the proper transformation, Hirota equation (1) can be changed into polynomial. In

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†E-mail: chenyong@dlut.edu.cn
‡E-mail: E-mail: libiao@dlut.edu.cn

Ref. [21], Yan did not apply the method to the system of differential equations. As a result, for the two equations, we can successfully recover the previously known solitary wave solutions that have been found by the extended tanh-function method and other more sophisticated method. More importantly, we also obtain other new and more general solutions at the same time, which include kink-profile solitary-wave solutions, bell-shaped solitary-wave solutions, periodic wave solutions, singular solutions, and new formal solutions. The properties of new formal soliton solutions for the system are shown by some figures.

This paper is organized as follows. In Sec. 2, we summarize the improved method. In Sec. 3, exact solutions to Hirota equation (1) are obtained. In Sec. 4, exact solutions for the (1+1)-dimensional dispersive long wave equation (DLWE) (2) are obtained. Conclusions will be presented finally.

2 Summary of Improved Method

Now we will improve the generalize projective Riccati equation method and apply it to the system of differential equations.

Consider a given system of nonlinear PDEs, say, in two variables:

$$\begin{aligned} &F(u,H,u_t,H_t,u_x,H_x,u_{xt},H_{x,t},u_{tt},H_{tt},u_{xx},H_{xx},\ldots)=0\,,\\ &G(u,H,u_t,H_t,u_x,H_x,u_{xt},H_{x,t},u_{tt},H_{tt},u_{xx},H_{xx},\ldots)=0\,. \end{aligned} \tag{3}$$

Under the transformation $u(x,t)=u(\xi)$, $H(x,t)=H(\xi)$, $\xi=x-\lambda t$, equations (3) reduce to

$$F(u,H,u',H',u'',H'',u''',H''',\cdots)=0\,,\qquad G(u,h',u',H',u'',H'',u''',H''',\cdots)=0\,, \tag{4}$$

where "$'$" denotes $\mathrm{d}/\mathrm{d}\xi$.

In order to seek for the travelling wave solutions to Eqs. (3), we need the following steps:

Step 1

Balancing the highest order derivative terms and the nonlinear terms in Eqs. (4), we get balance constant m and n (m and n are usually positive integers). If m or n is s fraction or a negative integer, we make the following transformation

$$u(\xi)=v^m(\xi)\,,\qquad H(\xi)=w^n(\xi)\,. \tag{5}$$

then return to determine balance constants m and n again.

Step 2

We express the solutions to Eqs. (4) as the following forms.

Type 1

$$u(\xi)=a_0+\sum_{i=1}^{m}\sigma^{i-1}[a_i\sigma(\xi)+b_i\tau(\xi)]\,,\qquad H(\xi)=A_0+\sum_{i=1}^{n}\sigma^{i-1}[A_i\sigma(\xi)+B_i\tau(\xi)]\,, \tag{6}$$

where $\sigma(\xi)$ and $\tau(\xi)$ satisfy the following projective Riccati equations

$$\sigma'(\xi)=\epsilon\sigma(\xi)\tau(\xi)\,,\quad \tau'(\xi)=R+\epsilon\tau^2(\xi)-\mu\sigma(\xi)\,,\quad \epsilon=\pm1\,,\quad R,\mu=\text{constant}\,, \tag{7}$$

which admits the first integral with $R\neq0$,

$$\tau^2(\xi)=-\epsilon\Big[R-2\mu\sigma(\xi)+\frac{\mu^2-1}{R}\sigma^2(\xi)\Big]\,,\quad (R\neq0)\,, \tag{8}$$

where "$'$" $=\mathrm{d}/\mathrm{d}\xi$. When $\epsilon=-1$, $R=1$, $\mu\to\mu/K$, equation (7) becomes a projective Riccati equation.[13]

Type 2

When $R=\mu=0$ in Eqs. (7),

$$u(\xi)=\sum_{i=0}^{m}a_i\tau^i(\xi)\,,\qquad H(\xi)=\sum_{i=0}^{n}A_i\tau^i(\xi)\,, \tag{9}$$

where $\tau(\xi)$ satisfies

$$\tau'(\xi)=\tau^2(\xi)\,. \tag{10}$$

Step 3

When $R\neq0$, substituting Eqs. (6) along with the conditions (7) and (8) into Eqs. (4), and when $R=\mu=0$, substituting Eqs. (9) along with $\tau'(\xi)=\tau^2(\xi)$ into Eqs. (4), yields a set of algebraic equations for $\sigma^j(\xi)\tau^i(\xi),(j=0,1,\ldots;i=0,1)(\tau^l(\xi),\ l=0,1,\ldots)$. Setting the coefficients of these terms $\sigma^j\tau^i$ (or $\tau^l(\xi)$) to zero yields a set of over-determined algebraic equations in λ, a_i, b_i, A_i, B_i, R, and μ.

Step 4

With the aid of ***Maple***, solving the above set of equations obtained in step 3, yields the values of a_i, b_i, A_i, B_i, R, λ, and μ.

Step 5

We know that equations (7) admit the following solutions

Case 1

When $\epsilon = -1$, $R \neq 0$,

$$\sigma_1(\xi) = \frac{R\,\mathrm{sech}\,(\sqrt{R}\,\xi)}{\mu\,\mathrm{sech}\,(\sqrt{R}\,\xi) + 1}\,, \qquad \tau_1(\xi) = \frac{\sqrt{R}\ \tanh(\sqrt{R}\,\xi)}{\mu\,\mathrm{sech}\,(\sqrt{R}\,\xi) + 1}\,,$$
$$\sigma_2(\xi) = \frac{R\,\mathrm{csch}(\sqrt{R}\,\xi)}{\mu\,\mathrm{csch}(\sqrt{R}\,\xi) + 1}\,, \qquad \tau_2(\xi) = \frac{\sqrt{R}\ \coth(\sqrt{R}\,\xi)}{\mu \mathrm{csch}(\sqrt{R}\,\xi) + 1}\,. \tag{11}$$

Case 2

When $\epsilon = 1$, $R \neq 0$,

$$\sigma_3(\xi) = \frac{R\sec(\sqrt{R}\,\xi)}{\mu\sec(\sqrt{R}\,\xi) + 1}\,, \qquad \tau_3(\xi) = \frac{\sqrt{R}\ \tan(\sqrt{R}\,\xi)}{\mu\sec(\sqrt{R}\,\xi) + 1}\,,$$
$$\sigma_4(\xi) = \frac{R\csc(\sqrt{R}\,\xi)}{\mu\csc(\sqrt{R}\,\xi) + 1}\,, \qquad \tau_4(\xi) = -\frac{\sqrt{R}\ \cot(\sqrt{R}\,\xi)}{\mu\csc(\sqrt{R}\,\xi) + 1}\,. \tag{12}$$

Case 3

When $R = \mu = 0$,

$$\sigma_5(\xi) = \frac{C}{\xi} = C\epsilon\tau_5(\xi)\,, \qquad \tau_5(\xi) = \frac{1}{\epsilon\xi}\,, \tag{13}$$

where C is a constant.

Thus according to Eqs. (6) and (11) $\sim$ (13) and the conclusions in ***Step 4***, we can obtain many solutions to Eqs. (3).

3 Exact Solutions to Hirota Equation

Let us consider the Hirota equations, i.e., Eqs. (1). According to the above method, to seek travelling wave solutions to Eqs. (1), we make the transformation

$$u(x,t) = \mathrm{e}^{\mathrm{i}\theta}\phi(\xi)\,, \qquad \xi = x + \lambda t\,, \qquad \theta = px + qt\,, \tag{14}$$

where λ, p, and q are constants to be determined later, and thus equations (1) become

$$(\lambda + 2\,p - 3\alpha\,p^2)\phi' + \alpha\phi''' + 6\alpha\phi^2\phi' = 0\,, \qquad (-q - p^2 + \alpha\,p^3)\phi + (1 - 3\alpha)\phi'' + (2 - 6\alpha\,p)\phi^3 = 0\,. \tag{15}$$

According to ***Step 1*** in Sec. 2, by balancing $\phi'''(\xi)$ and $\phi^2(\xi)\phi'(\xi)$ in Eqs. (15), we get $n = 1$. Therefore we suppose that equations (15) have the following formal solutions

$$\phi(\xi) = a_0 + a_1\sigma(\xi) + b_1\tau(\xi)\,, \tag{16}$$

where $\sigma(\xi)$ and $\tau(\xi)$ satisfy Eqs. (7) and (8), and a_0, a_1, b_1 are constants to be determined later.

With the aid of ***Maple***, substituting Eq. (16) along with Eqs. (7) and (8) into Eqs. (15) yields a set of algebraic equations for $\sigma^j(\xi)\tau^i(\xi)$ ($j = 0, 1, \ldots;\ \ i = 0, 1$). Setting the coefficients of these terms $\sigma^j\tau^i$ to zero yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, R, p, q, μ, and λ (note that here we take $\epsilon = -1$).

$$-6\alpha {b_1}^3 + 18 b_1 R\alpha {a_1}^2 - 6\,b_1\alpha\mu^4 + 12\alpha\mu^2{b_1}^3 - 6\alpha\mu^4{b_1}^3 + 12\,b_1\alpha\,\mu^2 - 18\,b_1 R\alpha\,{a_1}^2\mu^2 - 6\,b_1\alpha = 0\,, \tag{17}$$

$$24\,R^2\alpha\,a_1{b_1}^2\mu + 12\,R\alpha\,a_0{b_1}^2 + 6\,R^2\alpha\,a_1\mu - 12\,R\alpha\,a_0{b_1}^2\mu^2 - 12\,R^2\alpha\,a_0{a_1}^2 = 0\,, \tag{18}$$

$$-R^3\alpha\,a_1 + 3\,R^2\alpha\,p^2a_1 - 2\,R^2pa_1 - 6\,R^2\alpha\,{a_0}^2a_1 - R^2\lambda\,a_1 + 12\,R^2\alpha\,a_0{b_1}^2\mu - 6\,R^3\alpha\,a_1{b_1}^2 = 0\,, \tag{19}$$

$$30\,b_1\alpha\,{a_1}^2\mu\,R^2 + 24\,b_1R\alpha\,a_0a_1 - 24\,b_1R\alpha\,a_0a_1\mu^2 + 12\,b_1\alpha\,\mu^3R - 12\,b_1\alpha\,\mu\,R + 18\,\alpha\,\mu^3Rb_1{}^3 - 18\,\alpha\,\mu\,Rb_1{}^3 = 0\,, \tag{20}$$

$$18\,R\alpha\,{b_1}^2a_1 + 6\,R\alpha\,a_1 - 18\,R\alpha\,{b_1}^2a_1\mu^2 - 6\,R^2\alpha\,{a_1}^3 - 6\,R\alpha\,a_1\mu^2 = 0\,, \tag{21}$$

$$-3\,b_1\alpha\,p^2\mu\,R^2 - 12\,b_1\alpha\,a_0a_1R^3 + 6\,\alpha\,R^3\mu\,{b_1}^3 + b_1\alpha\,R^3\mu + b_1\lambda\,\mu\,R^2 + 2\,b_1p\mu\,R^2 + 6\,b_1\alpha\,{a_0}^2\mu\,R^2 = 0\,, \tag{22}$$

$$\begin{aligned}&-3\,\alpha\,p^2b_1R - 7\,b_1\alpha\,\mu^2R^2 - b_1R\lambda\,\mu^2 + 6\,\alpha\,{a_0}^2b_1R - 12\,b_1\alpha\,{a_1}^2R^3 - 2\,b_1Rp\mu^2 + 3\,b_1R\alpha\,p^2\mu^2 + 6\,\alpha\,{b_1}^3R^2\\ &\quad + 36\,b_1\alpha\,a_0a_1\mu\,R^2 - 6\,b_1R\alpha\,{a_0}^2\mu^2 + 4\,b_1\alpha\,R^2 + \lambda\,b_1R + 2\,pb_1R - 18\,\alpha\,R^2\mu^2{b_1}^3 = 0\,,\end{aligned} \tag{23}$$

$$2\,{a_0}^3R - 6\,\alpha\,p{a_0}^3R + \alpha\,p^3a_0R - qa_0R - p^2a_0R + 6\,a_0{b_1}^2R^2 - 18\,\alpha\,pa_0{b_1}^2R^2 = 0\,, \tag{24}$$

$$2\,{b_1}^3\mu^2 - 18\,b_1\alpha\,p{a_1}^2R + 6\,b_1{a_1}^2R - 2\,b_1 - 6\,{b_1}^3\alpha\,p\mu^2 + 6\,b_1\alpha\,p - 6\,b_1\alpha\,p\mu^2 - 2\,{b_1}^3 + 6\,{b_1}^3\alpha\,p + 2\,b_1\mu^2 = 0\,, \tag{25}$$

$$6\,\alpha\,pa_1 + 18\,\alpha\,pa_1{b_1}^2 - 6\,a_1{b_1}^2 + 2\,{a_1}^3R - 6\,\alpha\,p{a_1}^3R - 18\,\alpha\,pa_1{b_1}^2\mu^2 - 2\,a_1 + 2\,a_1\mu^2 - 6\,\alpha\,pa_1\mu^2 + 6\,a_1{b_1}^2\mu^2 = 0\,, \tag{26}$$

$$6\,a_0{a_1}^2R-12\,a_1{b_1}^2\mu\,R-18\,\alpha\,pa_0{b_1}^2\mu^2+9\,\alpha\,pa_1\mu\,R+18\,\alpha\,pa_0{b_1}^2+6\,a_0{b_1}^2\mu^2-6\,a_0{b_1}^2-3\,a_1\mu\,R-18\,\alpha\,pa_0{a_1}^2R+36\,\alpha\,pa_1{b_1}^2\mu\,R=0\,,\tag{27}$$

$$-4\,{b_1}^3\mu\,R-36\,b_1\alpha\,pa_0a_1R+12\,{b_1}^3\alpha\,p\mu\,R-b_1\mu\,R+12\,b_1a_0a_1R+3\,b_1\alpha\,p\mu\,R=0\,,\tag{28}$$

$$6\,a_1{b_1}^2R^2+36\,\alpha\,pa_0{b_1}^2\mu\,R-12\,a_0{b_1}^2\mu\,R-18\,\alpha\,p{a_0}^2a_1R+a_1R^2-3\,\alpha\,pa_1R^2-qa_1R-18\,\alpha\,pa_1{b_1}^2R^2-p^2a_1R+6\,{a_0}^2a_1R+\alpha\,p^3a_1R=0\,,\tag{29}$$

$$b_1\alpha\,p^3R-6\,{b_1}^3\alpha\,pR^2+2\,{b_1}^3R^2-b_1p^2R+6\,b_1{a_0}^2R-b_1qR-18\,b_1\alpha\,p{a_0}^2R=0\,.\tag{30}$$

By use of the ***Maple*** soft package "charsets" by Dongming Wang, which is based on the Wu-elimination method,[22] solving Eqs. (17) ∼ (30), we get the following results.

Case 1

$$\lambda=3\,\alpha\,p^2-\alpha\,R-2\,p\,,\qquad a_1=-\frac{1}{\sqrt{R}}\,,\qquad q=R-3\,Rp\alpha-p^2+\alpha\,p^3\,,\qquad b_1=a_0=\mu=0\,.\tag{31}$$

Case 2

$$q=\alpha\,p^3+6\,Rp\alpha-2\,R-p^2\,,\qquad \lambda=3\,\alpha\,p^2+2\,\alpha\,R-2\,p\,,\qquad a_0=a_1=\mu=0\,,\quad b_1=\pm\mathrm{i}\,.\tag{32}$$

Case 3

$$\lambda=\frac{1}{2}\,\alpha\,R+3\,\alpha\,p^2-2\,p\,,\qquad q=\alpha\,p^3+\frac{3}{2}\,Rp\alpha-\frac{1}{2}\,R-p^2\,,\qquad a_0=\mu=0\,,\qquad b_1=\mp\frac{\mathrm{i}}{2}\,,\qquad a_1=\pm\frac{\sqrt{R}}{2}\,.\tag{33}$$

Case 4

$$b_1=0\,,\qquad p=\frac{1}{3R}\,,\qquad q=-\frac{2}{27\alpha^2}\,,\qquad \lambda=-\frac{1+18\,\alpha^2{a_0}^2+3\alpha^2R}{3\alpha}\,,\qquad a_1=\pm\frac{1}{\sqrt{R+4\,{a_0}^2}}\,,\qquad \mu=\pm2\,\frac{a_0}{\sqrt{R+4\,{a_0}^2}}\,.\tag{34}$$

Case 5

$$\lambda=\frac{1}{2}\,\alpha\,R+3\,\alpha\,p^2-2\,p\,,\qquad \mu=\pm\sqrt{-4\,R{a_1}^2+1}\,,\qquad q=\alpha\,p^3+\frac{3}{2}\,Rp\alpha-\frac{1}{2}\,R-p^2\,,\qquad a_0=0\,,\qquad b_1=\frac{1}{2}\,\mathrm{i}\,.\tag{35}$$

From Eqs. (14) and (16) and *Cases* 1 ∼ 5, we obtain the following solutions to Eq. (1).

Family 1

From Eq. (31), we obtain the following solutions to the Hirota equation,

$$u_{11}=-\,\mathrm{e}^{\mathrm{i}\theta}\sqrt{R}\,\mathrm{sech}\,(\sqrt{R}\,\xi)\,,\tag{36}$$

$$u_{12}=-\,\mathrm{e}^{\mathrm{i}\theta}\sqrt{R}\,\mathrm{csch}(\sqrt{R}\,\xi)\,,\tag{37}$$

where $\xi=x+\lambda t$, $\theta=px+qt$, $\lambda=3\,\alpha\,p^2-\alpha\,R-2\,p$, $q=R-3\,Rp\alpha-p^2+\alpha\,p^3$, p, and R are arbitrary constants.

Family 2

From Eq. (32), we obtain the following solutions to the Hirota equation,

$$u_{21}=\pm\,\mathrm{e}^{\mathrm{i}\theta}\mathrm{i}\sqrt{R}\,\tanh(\sqrt{R}\,\xi)\,,\tag{38}$$

$$u_{22}=\pm\,\mathrm{e}^{\mathrm{i}\theta}\mathrm{i}\sqrt{R}\,\coth(\sqrt{R}\,\xi)\,,\tag{39}$$

where $\xi=x+\lambda t$, $\theta=px+qt$, $q=\alpha\,p^3+6\,Rp\alpha-2\,R-p^2$, $\lambda=3\,\alpha\,p^2+2\,\alpha\,R-2\,p$, and p and R are arbitrary constants.

Family 3

From Eq. (33), we obtain the following solutions for the Hirota equation,

$$u_{31}=\,\mathrm{e}^{\mathrm{i}\theta}\Big(\pm\frac{1}{2}\,R^{3/2}\mathrm{sech}(\sqrt{R}\,\xi)\mp\frac{\sqrt{R}}{2}\,\mathrm{i}\tanh(\sqrt{R}\,\xi)\Big)\,,\tag{40}$$

$$u_{32}=\,\mathrm{e}^{\mathrm{i}\theta}\Big(\pm\frac{1}{2}\,R^{3/2}\mathrm{csch}(\sqrt{R}\,\xi)\mp\frac{\sqrt{R}}{2}\,\mathrm{i}\coth(\sqrt{R}\,\xi)\Big)\,,\tag{41}$$

where $\xi=x+\lambda t$, $\theta=px+qt$, $\lambda=\alpha R/2+3\alpha\,p^2-2\,p$, $q=\alpha\,p^3+3Rp\alpha/2-R/2-p^2$, p, and R are arbitrary constants.

Family 4

From Eq. (34), we obtain the following solutions to the Hirota equation,

$$u_{41}=\,\mathrm{e}^{\mathrm{i}\theta}\Big(a_0\pm\frac{R\,\mathrm{sech}(\sqrt{R}\,\xi)}{\sqrt{R+4\,{a_0}^2}(\mu\,\mathrm{sech}(\sqrt{R}\,\xi)+1)}\Big),\tag{42}$$

$$u_{42}=\,\mathrm{e}^{\mathrm{i}\theta}\Big(a_0\pm\frac{R\,\mathrm{csch}(\sqrt{R}\,\xi)}{\sqrt{R+4\,{a_0}^2}(\mu\,\mathrm{csch}(\sqrt{R}\,\xi)+1)}\Big),\tag{43}$$

where $\xi=x+\lambda t$, $\theta=px+qt$, $p=1/3R$, $q=-227\alpha^2$, $\lambda=-(1+18\,\alpha^2{a_0}^2+3\,\alpha^2R)/3\alpha$, $\mu=\pm2a_0/\sqrt{R+4\,{a_0}^2}$, and R is an arbitrary constant.

Family 5

From Eq. (35), we obtain the following solutions to the Hirota equation,

$$u_{51}=\,\mathrm{e}^{\mathrm{i}\theta}\Big(\frac{a_1R\,\mathrm{sech}(\sqrt{R}\,\xi)}{\mu\,\mathrm{sech}(\sqrt{R}\,\xi)+1}+\frac{\mathrm{i}\sqrt{R}\tanh(\sqrt{R}\,\xi)}{2\mu\,\mathrm{sech}(\sqrt{R}\,\xi)+2}\Big),\tag{44}$$

$$u_{52}=\,\mathrm{e}^{\mathrm{i}\theta}\Big(\frac{a_1R\,\mathrm{csch}(\sqrt{R}\,\xi)}{\mu\,\mathrm{csch}(\sqrt{R}\,\xi)+1}+\frac{\mathrm{i}\sqrt{R}\coth(\sqrt{R}\,\xi)}{2\mu\,\mathrm{csch}(\sqrt{R}\,\xi)+2}\Big),\tag{45}$$

where $\xi = x+\lambda t$, $\theta = px+qt$, $q = \alpha p^3+3Rp\alpha/3-R/2-p^2$ $\lambda = \alpha R/2 + 3\alpha p^2 - 2p$, $\mu = \pm\sqrt{-4Ra_1{}^2+1}$, a_1, p, and R are arbitrary constants.

Remark 1 The properties of new formal solitary wave solutions u_{41} and u_{51} are shown in Figs. 1 and 2.

4 Exact Solutions to (1+1)-Dimensional Dispersive Long Wave Equation

Let us consider the (1+1)-dimensional dispersive long wave equation (DLWE), i.e. Eqs. (2). According to the above method, to seek travelling wave solutions of Eqs. (2), we make the transformation

$$u(x,t) = \phi(\xi)\,, \quad v(\xi) = \theta(\xi)\,, \quad \xi = x - \lambda t\,, \tag{46}$$

where λ is constant to be determined later, and thus equations (2) becomes

$$-\lambda\phi' + \phi\phi' + \theta' = 0\,,$$
$$-\lambda\theta' + (\theta\phi)' + \frac{1}{3}\phi''' = 0\,. \tag{47}$$

Integrating the second equation of Eqs. (47) once with regard to ξ, we obtain

$$-\lambda\phi' + \phi\phi' + \theta' = 0\,,$$
$$-\lambda\theta + \theta\phi + \frac{1}{3}\phi'' = 0 \tag{48}$$

with the integration constants taken to be zero. According to ***Step 1*** in Sec. 2, by balancing $\theta'(\xi)$ and $\phi(\xi)\phi'(\xi)$ in Eqs. (48), we get $m=1$ and by balancing $\phi''(\xi)$ and $\phi(\xi)\theta(\xi)$ in Eqs. (48), we get $n=2$.

Therefore we suppose that equations (48) have the following formal solutions

$$\phi(\xi) = a_0 + a_1\sigma(\xi) + b_1\tau(\xi)\,, \qquad \tau(\xi) = A_0 + A_1\sigma(\xi) + B_1\tau(\xi) + A_2\sigma^2(\xi) + B_2\sigma(\xi)\tau(\xi)\,, \tag{49}$$

where $\sigma(\xi)$, $\tau(\xi)$ satisfy Eqs. (7) and (8), a_0, a_1, b_1, A_0, A_1, B_1, A_2, and B_2 are constants to be determined later.

With the aid of ***Maple***, substituting Eqs. (49) along with Eqs. (7) and (8) into Eqs. (48) yields a set of algebraic equations for $\sigma^j(\xi)\tau^i(\xi)$ $(j=0,1,\ldots;i=0,1)$. Setting the coefficients of terms $\sigma^j\tau^i$ to zero yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, A_0, A_1, B_1, A_2, B_2, R, μ, and λ (note that here we take $\epsilon = -1$).

$$-a_0a_1R + b_1{}^2\mu R - A_1R + \lambda a_1R = 0\,, \tag{50}$$
$$2B_2 + 2b_1a_1 - 2B_2\mu^2 - 2b_1a_1\mu^2 = 0\,, \tag{51}$$
$$3a_1b_1\mu R - a_0b_1\mu^2 - B_1\mu^2 + \lambda b_1\mu^2 + a_0b_1 + B_1 - \lambda b_1 + 3B_2\mu R = 0\,, \tag{52}$$
$$-2A_2R - a_1{}^2R - b_1{}^2\mu^2 + b_1{}^2 = 0\,, \tag{53}$$
$$-\lambda b_1\mu R + a_0b_1\mu R + B_1\mu R - a_1b_1R^2 - B_2R^2 = 0\,, \tag{54}$$
$$-3\lambda A_0R + 3b_1B_1R^2 + 3a_0A_0R = 0\,, \tag{55}$$
$$a_1R^2 - 6b_1B_1\mu R + 3a_0A_1R - 3\lambda A_1R + 3b_1B_2R^2 + 3a_1A_0R = 0\,, \tag{56}$$
$$3b_1A_0R + 3a_0B_1R - 3\lambda B_1R = 0\,, \tag{57}$$
$$-3b_1B_2 + 3a_1A_2R + 3b_1B_2\mu^2 - 2a_1 + 2a_1\mu^2 = 0\,, \tag{58}$$
$$3a_1A_1R + 3b_1B_1\mu^2 - 3\lambda A_2R - 3b_1B_1 - 3a_1\mu R + 3a_0A_2R - 6b_1B_2\mu R = 0\,, \tag{59}$$
$$3b_1A_1R - 3\lambda B_2R - b_1\mu R + 3a_0B_2R + 3a_1B_1R = 0\,, \tag{60}$$
$$2b_1\mu^2 + 3b_1A_2R + 3a_1B_2R - 2b_1 = 0\,. \tag{61}$$

By use of the ***Maple*** soft package "charsets" by Dongming Wang, which is based on the Wu-elimination method,[22] solving Eqs. (50) $\sim$ (61), we get the following results.

Case 1

$$A_0 = A_1 = B_1 = \mu = 0\,, \qquad a_1 = \pm\frac{1}{\sqrt{-3R}}\,, \qquad b_1 = \pm\frac{\sqrt{3}}{3}\,, \qquad B_2 = \frac{\sqrt{-R}}{3R}\,, \qquad \lambda = a_0\,, \qquad A_2 = \frac{1}{3R}\,. \tag{62}$$

Case 2

$$A_0 = A_1 = B_1 = \mu = 0\,, \qquad a_1 = \mp\frac{1}{\sqrt{-3R}}\,, \qquad b_1 = \pm\frac{\sqrt{3}}{3}\,, \qquad B_2 = \frac{\sqrt{-R}}{3R}\,, \qquad \lambda = a_0\,, \qquad A_2 = \frac{1}{3R}\,. \tag{63}$$

Case 3

$$b_1 = A_1 = B_2 = \mu = 0\,, \qquad a_1 = \pm\frac{2\sqrt{-3R}}{3R}\,, \qquad A_0 = -\frac{R}{3}\,, \qquad A_2 = \frac{2}{3R}\,, \qquad \lambda = a_0\,. \tag{64}$$

Case 4

$$a_1 = A_0 = A_1 = B_1 = B_2 = \mu = 0\,, \qquad b_1 = \pm\frac{2\sqrt{R}}{3}\,, \qquad A_2 = \frac{2}{3R}\,. \tag{65}$$

Case 5

$$\begin{aligned} &A_0 = B_1 = 0\,, \qquad b_1 = \pm\frac{\sqrt{3}}{3}\,, \qquad A_2 = -a_1^2\,, \\ &A_1 = \pm\frac{\sqrt{3a_1^2R+1}}{3}\,, \qquad \mu = \pm\sqrt{3a_1^2R+1}\,, \qquad B_2 = \mp\frac{\sqrt{3}}{3}a_1\,. \end{aligned} \tag{66}$$

From Eqs. (46) and (49) and Cases 1 ∼ 5, we obtain the following solutions to Eqs. (2).

Family 1

From Eq. (62), we obtain the following solutions to the DLWE,

$$u_{11} = a_0 \mp \frac{\sqrt{-3R}}{3}\,\mathrm{sech}(\sqrt{R}\,\xi) \pm \frac{\sqrt{3R}}{3}\tanh(\sqrt{R}\,\xi)\,, \qquad v_{11} = \frac{R}{3}\,\mathrm{sech}^2(\sqrt{R}\,\xi) + \frac{R}{3}\,\mathrm{i}\,\mathrm{sech}(\sqrt{R}\,\xi)\tanh(\sqrt{R}\,\xi)\,, \tag{67}$$

$$u_{12} = a_0 \mp \frac{\sqrt{-3R}}{3}\,\mathrm{csch}(\sqrt{R}\,\xi) \pm \frac{\sqrt{3R}}{3}\coth(\sqrt{R}\,\xi)\,, \qquad v_{12} = \frac{R}{3}\,\mathrm{csch}^2(\sqrt{R}\,\xi) + \frac{R}{3}\,\mathrm{i}\,\mathrm{csch}(\sqrt{R}\,\xi)\coth(\sqrt{R}\xi)\,, \tag{68}$$

where R and a_0 are arbitrary constants and $\xi = x - \lambda t$, $\lambda = a_0$.

Family 2

From Eq. (63), the DLWEs (2) have the following solutions,

$$u_{21} = a_0 \pm \frac{\sqrt{-3R}}{3}\,\mathrm{sech}\,(\sqrt{R}\,\xi) \pm \frac{\sqrt{3R}}{3}\tanh(\sqrt{R}\,\xi)\,, \qquad v_{21} = \frac{R}{3}\,\mathrm{sech}^2(\sqrt{R}\,\xi) - \frac{R}{3}\,\mathrm{i}\,\mathrm{sech}(\sqrt{R}\,\xi)\tanh(\sqrt{R}\,\xi)\,, \tag{69}$$

$$u_{22} = a_0 \pm \frac{\sqrt{-3R}}{3}\mathrm{csch}(\sqrt{R}\,\xi) \pm \frac{\sqrt{3R}}{3}\coth(\sqrt{R}\,\xi)\,, \qquad v_{22} = \frac{R}{3}\mathrm{csch}^2(\sqrt{R}\,\xi) - \frac{R}{3}\,\mathrm{i}\,\mathrm{csch}(\sqrt{R}\,\xi)\coth(\sqrt{R}\xi)\,. \tag{70}$$

where R and a_0 are arbitrary constants and $\xi = x - \lambda t$, $\lambda = a_0$.

Family 3

From Eq. (64), the DLWEs (2) have the following solutions,

$$u_{31} = a_0 \pm \frac{2\sqrt{-3\,R}}{3}\,\mathrm{sech}(\sqrt{R}\,\xi)\,, \qquad v_{31} = -\frac{R}{3} + \frac{2}{3}R\,\mathrm{sech}^2(\sqrt{R}\,\xi)\,, \tag{71}$$

$$u_{32} = a_0 \pm \frac{2\sqrt{-3R}}{3}\mathrm{csch}(\sqrt{R}\,\xi)\,, \qquad v_{32} = -\frac{R}{3} + \frac{2}{3}R\,\mathrm{csch}^2(\sqrt{R}\,\xi)\,, \tag{72}$$

where R and a_0 are arbitrary constants and $\xi = x - \lambda t$, $\lambda = a_0$.

Family 4

From Eq. (65), we obtain the following solutions to the DLWE,

$$u_{41} = a_0 \pm \frac{2\sqrt{3R}}{3}\tanh(\sqrt{R}\,\xi)\,, \qquad v_{41} = \frac{2R}{3}\,\mathrm{sech}^2(\sqrt{R}\,\xi)\,, \tag{73}$$

$$u_{42} = a_0 \pm \frac{2\sqrt{3R}}{3}\coth(\sqrt{R}\,\xi)\,, \qquad v_{42} = \frac{2R}{3}\,\mathrm{csch}^2(\sqrt{R}\,\xi)\,, \tag{74}$$

where R and a_0 are arbitrary constants and $\xi = x - \lambda t$, $\lambda = a_0$.

Family 5

From Eq. (66), we obtain the following solutions to the DLWE,

$$\begin{aligned} u_{51} &= a_0 + \frac{a_1R\,\mathrm{sech}(\sqrt{R}\,\xi)}{\mu\,\mathrm{sech}(\sqrt{R}\,\xi)+1} \pm \frac{\sqrt{3R}}{3}\frac{\tanh(\sqrt{R}\,\xi)}{\mu\,\mathrm{sech}(\sqrt{R}\,\xi)+1}\,, \\ v_{51} &= \pm\frac{\sqrt{3\,a_1{}^2R+1}}{3}\frac{R\,\mathrm{sech}(\sqrt{R}\xi)}{\mu\,\mathrm{sech}(\sqrt{R}\,\xi)+1} - \frac{a_1{}^2R^2\mathrm{sech}(\sqrt{R}\,\xi)}{(\mu\,\mathrm{sech}(\sqrt{R}\,\xi)+1)^2} \mp \frac{\sqrt{3}a_1R^{3/2}\mathrm{sech}(\sqrt{R}\,\xi)\tanh(\sqrt{R}\,\xi)}{3(\mu\,\mathrm{sech}(\sqrt{R}\,\xi)+1)^2}\,, \end{aligned} \tag{75}$$

$$\begin{aligned} u_{52} &= a_0 + \frac{a_1R\,\mathrm{csch}(\sqrt{R}\,\xi)}{\mu\,\mathrm{csch}(\sqrt{R}\,\xi)+1} \pm \frac{\sqrt{3R}}{3}\frac{\coth(\sqrt{R}\,\xi)}{\mu\,\mathrm{csch}(\sqrt{R}\,\xi)+1}\,, \\ v_{52} &= \pm\frac{\sqrt{3a_1{}^2R+1}}{3}\frac{R\,\mathrm{csch}(\sqrt{R}\,\xi)}{\mu\,\mathrm{csch}(\sqrt{R}\,\xi)+1} - \frac{a_1{}^2R^2\,\mathrm{csch}(\sqrt{R}\,\xi)}{(\mu\,\mathrm{csch}(\sqrt{R}\,\xi)+1)^2} \mp \frac{\sqrt{3}a_1R^{3/2}\,\mathrm{csch}(\sqrt{R}\,\xi)\coth(\sqrt{R}\,\xi)}{3(\mu\,\mathrm{csch}(\sqrt{R}\,\xi)+1)^2}\,, \end{aligned} \tag{76}$$

where R, a_0, and a_1 are arbitrary constants and $\xi = x - \lambda t$, $\lambda = a_0$, $\mu = \pm\sqrt{3a_1{}^2R+1}$.

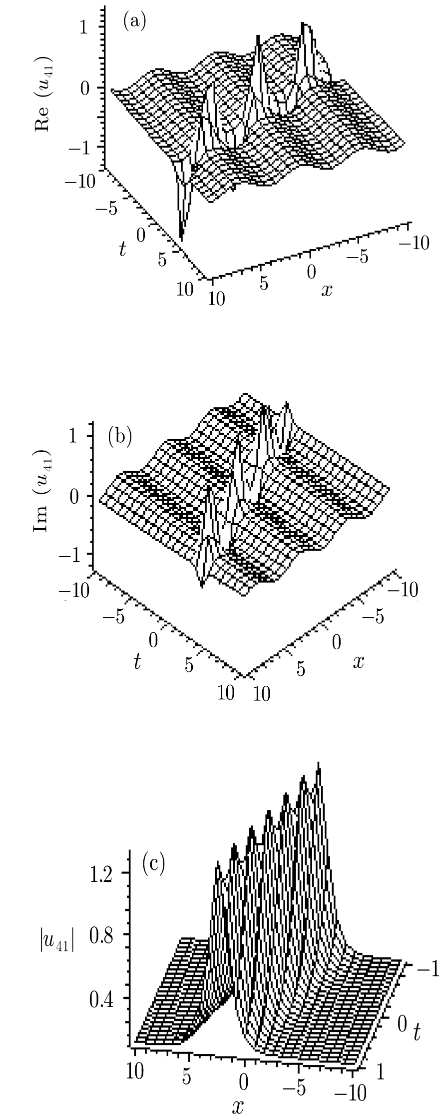

Fig. 1 The solitary solution to Hirota equation, u_{41}. The real part (a), imaginary part (b), and the modulus (c), where $R = 2$, $p = 1$, $\alpha = 1$, and $a_0 = 0.1$.

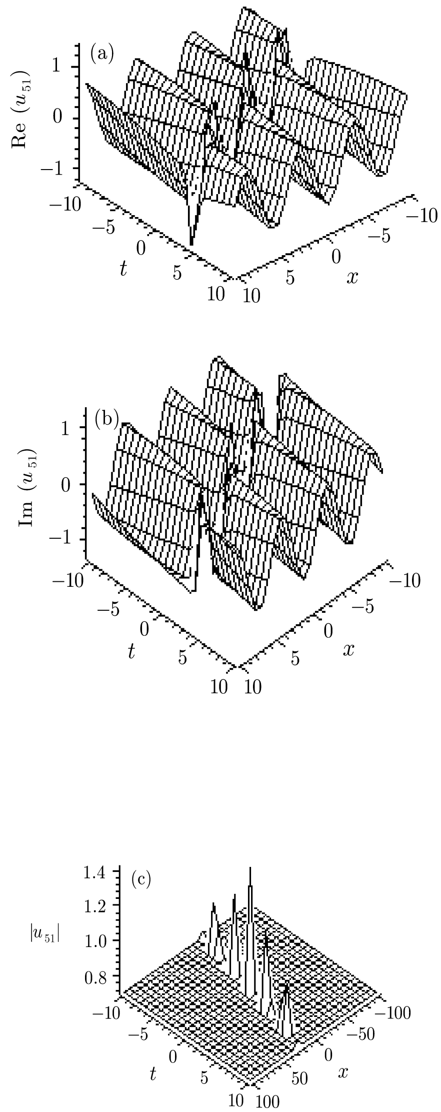

Fig. 2 The solitary solution to Hirota equation, u_{51}. The real part (a), imaginary part (b), and the modulus (c), where $R = 2$, $p = 1$, $\alpha = 1$, and $a_0 = 0$.

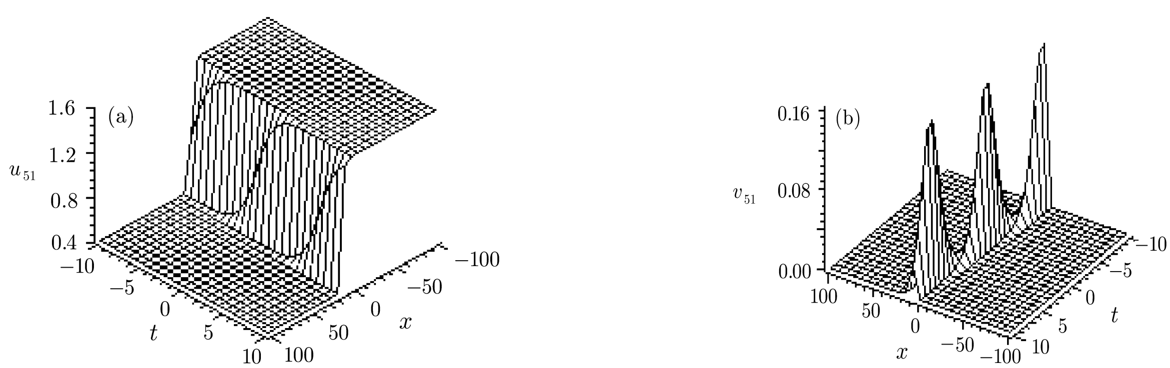

Fig. 3 The solitary solutions to DLWE, u_{51}, and v_{51}, where $a_0 = 1$, $a_1 = 1$, and $R = 1$.

Remark 2

(i) The solutions (73) and (74) reproduce the solutions of cases 1, 3, 4, and 5 in Ref. [18], when $a_0 = \pm 2\sqrt{-3R}/3$ and $R = -R'$; the solutions (67) $\sim$ (70) reproduce the solutions of cases 6, 7, 9, and 10 in Ref. [18], when $a_0 = -\sqrt{-3R}/3$ and $R = -R'$; the solutions (71) and (72) reproduce the solutions of cases 2 and 8 in Ref. [18], when $a_0 = \pm\sqrt{6R}/3$ and $R = -R'$; the solutions (73) and (74) reproduce the solutions of case 15 in Ref. [18] , when $a_0 = \pm\sqrt{R}\, d_{11}/2$ and $R = -3d_{11}/4$; The solutions (75) and (76) reproduce the solutions of cases 16 and 17, in Ref. [18], when $a_0 = \pm\, \mathrm{i}d_{11}$, $a_1 = \mathrm{i}/3d_{11}$, $\mu = 0$, and $R = -3d_{11}^2$.

(ii) The other solutions obtained here, to our knowledge, are all new families of exact solutions of the DLWE. The properties of the new formal solitary wave solutions obtained are shown in Fig. 3. Here periodic wave solutions and rational wave solutions are omitted.

(iii) We corrected some errors Eqs. (11) and (12) in Ref. [21].

5 Summary and Conclusions

In summary, by use of the general projective Riccati equation method, more general forms of solutions for Hirota equation and the system DLWE are obtained. The method of introducing the general projective Riccati system as subequations of a nonlinear ODE prevents the drawback of having to sum the entire series in exponential solutions of the linearized equation. Of course, this method can be extended to other systems of nonlinear differential equations. On the other hand, we will extend this method to seek soliton-like solutions for some PDEs in the forthcoming works.

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