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Generalized Q–S (lag, anticipated and complete) synchronization in modified Chua's circuit and Hindmarsh–Rose systems

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Abstract

Based on symbolic computation system *Maple* and Lyapunov stability theory, a systematic, powerful and concrete scheme is extended to study the generalized Q–S (lag, anticipated and complete) synchronization between two identical modified Chua's circuit with different initial values and between two different chaotic systems: Hindmarsh–Rose system and modified Chua's circuit. Numerical simulations are used to verify the effectiveness of the obtained controller.

Keywords: Generalized Q–S (lag, anticipated and complete) synchronization; Modified Chua's circuit; Hindmarsh–Rose system

1. Introduction

The origin of the word synchronization is a Greek root (which means to share the common time). The original meaning of synchronization has been maintained up to now in the colloquial use of this word, as agreement or correlation in time of different processes [1]. Historically, the analysis of synchronization phenomena in the evolution of dynamical systems has been a subject of active investigation since the earlier days of physics. It started in the 17th century with the finding of Huygens that two very weakly coupled pendulum clocks (hanging at the same beam) become synchronized in phase [2,3]. Recently, the search for synchronization has moved to chaotic systems. In 1990, Pecora and Carroll [4] presented the chaos synchronization method to synchronize two identical chaotic systems with different initial values. Chaos synchronization has become an active research subject in nonlinear science and has attracted more attention in many fields. Many powerful methods have been reported to investigate chaos synchronization of some types of chaotic (hyperchaotic) attractors [5–10]. More recently, Yan [11] first defined the generalized (lag, anticipated and complete) synchronization between the drive system and the response system which includes generalized lag synchronization,

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generalized anticipated synchronization. Then a systematic, powerful and concrete scheme was developed to investigate the generalized (lag, anticipated, and complete) synchronization between the drive system and response system based on the active control idea. In Ref. [12], based on a backstepping design with one controller, Yan presented a systematic, concrete, and automatic scheme to investigate the generalized Q–S (lag, anticipated, and complete) synchronization between the drive system and response system with a strict-feedback form [6].

In this paper, we extend Yan's [12] scheme to study the generalized Q–S (lag, anticipated and complete) synchronization between two identical modified Chua's circuit [13] with different initial values and between two different chaotic systems: Hindmarsh–Rose system [14] and modified Chua's circuit. Numerical simulations are used to verify the effectiveness of the obtained controller.

2. Concept of Q–S (lag, anticipated and complete) synchronization

In this section, we introduce the concept of generalized Q–S (lag, anticipated and complete) synchronization defined by Yan [12].

For two continuous-time dynamical systems

$$\dot{x} = F(x,t), \tag{2.1a}$$

$$\dot{y} = G(y,t) + u(x,y,t), \quad (x,y) \in R^m \times R^m. \tag{2.1b}$$

Let $Q(y,t) = [Q_1(y,t), Q_2(y,t), \ldots, Q_h(y,t)]^{\mathrm{T}}$ and $S(x,t) = [S_1(x,t), S_2(x,t), \ldots, S_h(x,t)]^{\mathrm{T}}$ be two smooth, boundary vector functions and error states be $E(t) = Q(y(t),t) - S(x(t-\tau), t-\tau) = [Q_1(y(t),t) - S_1(x(t-\tau), t-\tau), \ldots, Q_h(y(t),t) - S_h(x(t-\tau), t-\tau)]^{\mathrm{T}}$. Thus, the Q–S error dynamical system between driver system (2.1a) and response system (2.1b) is

$$\begin{aligned}\dot{E}(t) &= \dot{Q}(y(t),t) - \dot{S}(x(t-\tau), t-\tau) \\ &= DQ(y(t),t)(G(y(t),t) + u(x(t-\tau), y(t), t)) + Q_t(y(t),t) + DS(x(t-\tau), t-\tau) \\ &\quad \times F(x(t-\tau), t-\tau) - S_t(x(t-\tau), t-\tau),\end{aligned} \tag{2.2}$$

where $DQ(y(t),t)$ and $DS(x(t-\tau), t-\tau)$ are Jacobian matrixes of the vector functions $Q(y(t),t)$ w.r.t. $y(t)$ and $S(x(t-\tau), t-\tau)$ w.r.t. $x(t-\tau)$, respectively. It is said that the drive system (2.1a) and response system (2.1b) are globally (i) *Q–S lag synchronized* ($\tau > 0, \tau$ is called the *Q–S synchronization lag*); (ii) *Q–S synchronized* ($\tau = 0$); (iii) *Q–S anticipated synchronized* ($\tau < 0, -\tau > 0$ is called the *Q–S synchronization anticipation*) with respect to the $Q(y(t),t)$ and $S(x(t-\tau), t-\tau)$, if there exists a controller $u(x,y,t)$ such that all trajectories $(x(t-\tau), y(t))$ in (2.1a) and (2.1b) with any initial conditions $(x(0), y(0))$ in $P = R_x^m \times R_y^n \subset R^m \times R_n$ approach the manifold $M = \{(x(t-\tau), y(t)) | Q_i(y(t),t) = S_i(x(t-\tau), t-\tau), i = 1, 2, \ldots, h\}$ with $M \subset P$ as time t goes to infinity, that is to say, $\lim_{t\to+\infty} = [Q_i(y(t),t) - S_i(x(t-\tau), t-\tau)] = 0$, which implies that the error dynamical system (2.2) between the drive system (2.1a) and (2.1b) is globally asymptotical stable.

According to the conditions of $Q_1(y(t),t)$ and $S_1(x(t-\tau), t-\tau)$, there exist many types of choices. Generally speaking, one can choose $Q_1(y(t),t)$ as one of the forms $\sinh(y_1(t))$, $\tanh(y_1(t))$, $e^{y_1(t)}$, $\sum_{i=0}^{m} |a_{2j+1}| y_1^{2j+1}(t)$ $(a_1 \neq 0), b_1 y_1(t) + b_2 \cos(y_1(t))$ $(|b_1| > |b_2|)$, etc., which are smooth, boundary functions and $\frac{dQ_1(y(t))}{dy_1(t)} \neq 0$, and $S_1(x(t-\tau), t-\tau)$ as one of the forms $\mathrm{sech}(x(t-\tau))$, $\sin(x(t-\tau))$, $\sum_{i=0}^{l} |a_j| x^j(t-\tau)$.

3. Two identical modified Chua's circuit

Unlike the classic Chua's circuit, the nonlinear function of this modified Chua's circuit is governed by a trigonometric function [13], which is a continuous function. It is reported that n-scroll attractors can be obtained, as shown in Fig. 1. The dimensionless state equation is given by

$$\begin{cases} \dot{x_1} = -\beta x_2, \\ \dot{x_2} = x_3 - x_2 + x_1, \\ \dot{x_3} = \alpha(x_2 - f(x_3)), \end{cases} \tag{3.1}$$

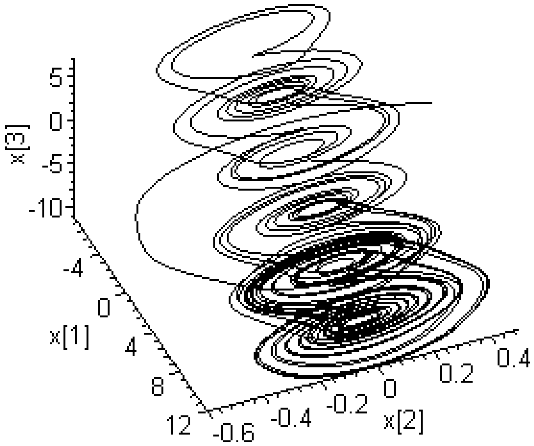

Fig. 1. Five-scroll attractors of the modified Chua's circuit with a sine function.

where

$$f(x_3) = \begin{cases} \frac{b\pi}{2a}(x_3 - 2ac) & \text{if } x_3 \geqslant 2ac, \\ -b\sin\left(\frac{\pi x_3}{2a} + d\right) & \text{if } -2ac < x_3 < 2ac, \\ \frac{b\pi}{2a}(x_3 + 2ac) & \text{if } x_3 \leqslant -2ac. \end{cases} \tag{3.2}$$

Here, in Eqs. (3.1) and (3.2), α, β, a, b, c, d are suitable constants, and $\alpha > 0$, $\beta > 0$, $a > 0$, $b > 0$.

An n-scroll attractor is generated with the following relationship:

$$n = c + 1, \tag{3.3}$$

and

$$d = \begin{cases} \pi & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}. \end{cases} \tag{3.4}$$

Here we choose (3.1) as the drive system and the controlled modified Chua's circuit system

$$\begin{cases} \dot{y_1} = -\beta y_2, \\ \dot{y_2} = y_3 - y_2 + y_1, \\ \dot{y_3} = \alpha(y_2 - f(y_3)) + u(x, y, t) \end{cases} \tag{3.5}$$

is the response system. $L(t) = \frac{1}{2}(E_1^2 + E_2^2 + E_3^2 + E_4^2), Q_1 = \sinh(y_1(t)), S_1 = \operatorname{sech}(x_1(t-\tau)), \frac{dQ_1(y(t))}{dy_1(t)} = \cosh(y_1(t)) \neq 0$. With the help of symbolic computation, by using the proposed scheme [12], we can obtain

$$E_1 = \sinh(y_1(t)) - \operatorname{sech}(x_1(t-\tau)), \tag{3.6}$$

$$E_2(t) = Q_2(y(t)) - S_2(x(t-\tau)) = h_1E_1(t) + \dot{E}_1(t), \tag{3.7}$$

$$E_3(t) = Q_3(y(t)) - S_3(x(t-\tau)) = E_1(t) + h_2E_2(t) + \dot{E}_2(t), \tag{3.8}$$

where $h_j \in R^+$ $(j = 1, 2, 3)$, $Q_i(y(t))$ and $S_i(x(t-\tau))$ $(i = 2, 3)$ are given in Appendix A.

Finally, from the equation $E_2(t) + h_3E_3(t) + \dot{E}_3(t) = 0$, we can determine the scalar controller $u(x(t-\tau), y(t))$ given in Appendix A. The substitution of (3.6)–(3.8) and u into the above-mentioned Lyapunov function $L(t)$ yields

$$\dot{L}(t) = -h_1E_1^2(t) - h_2E_2^2(t) - h_3E_3^2(t) \leqslant -\min\{h_1, h_2, h_3\}\sum_{i=1}^{3}E_i^2(t) = -2\min\{h_1, h_2, h_3\}L(t),$$

which denotes that

$$\lim_{t\to\infty} E_i(t) = \lim_{t\to\infty}[Qi(y(t)) - S_i(x(t-\tau))] = 0 \quad (i = 1, 2, 3).$$

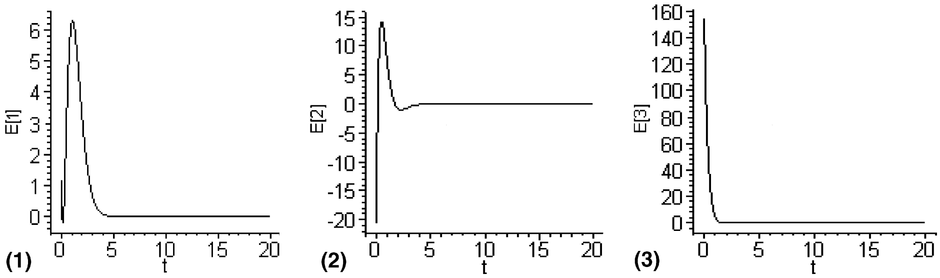

Fig. 2. Generalized lag synchronization errors: (1) $E_1(t) = \sinh(y_1(t)) - \text{sech}(x_1(t-0.5))$; (2) $E_2(t) = Q_2(y(t)) - S_2(x(t-0.5))$; (3) $E_3(t) = Q_3(y(t)) - S_3(x(t-0.5))$.

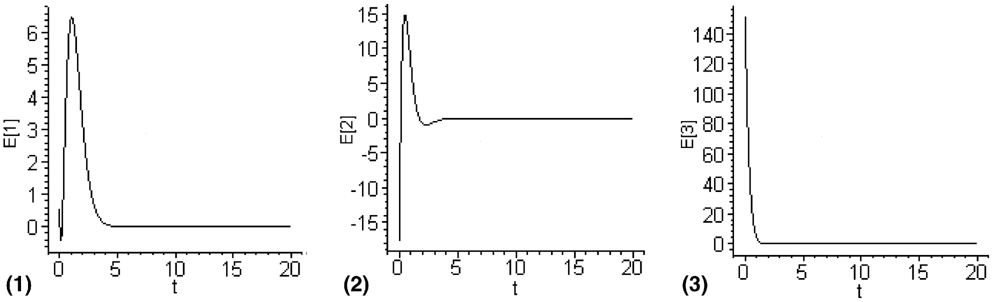

Fig. 3. Generalized (complete) synchronization errors: (1) $E_1(t) = \sinh(y_1(t)) - \text{sech}(x_1(t))$; (2) $E_2(t) = Q_2(y(t)) - S_2(x(t))$; (3) $E_3(t) = Q_3(y(t)) - S_3(x(t))$.

In what follows we would like to use numerical simulations to verify the effectiveness of the obtained scalar controller $u(x(t-\tau), y(t), t)$ defined in Appendix A. Let the initial values of system (3.1) and (3.5) as $\alpha = 10.814$, $\beta = 14$, $a = 1.3$, $b = 0.11$, $c = 4$, $d = \pi$, $h_1 = 1$, $h_2 = 2$, $[x_1(0) = -1, x_2(0) = 0.5, x_3(0) = 0]$ and $[y_1(0) = 1, y_2(0) = 1, y_3(0) = 0.5]$.

Case 1a. *Generalized lag synchronization*. In the case $\tau > 0$, without less of generality, we set $\tau = 0.5$. Thus by calculation, the initial values of the error dynamical system (3.6)–(3.8) is $E_1(0) = \sinh(y_1(0)) - \text{sech}(x_1(-0.5)) = 1.162284035$, $E_2(0) = -20.65236026$ and $E_3(0) = 154.2806069$. The dynamical of generalized lag synchronization errors for the drive system (3.1) and the response system (3.5) are shown in Fig. 2 (1)–(3).

Case 1b. *Generalized (complete) synchronization*. In the case $\tau = 0$. Thus the initial values of the error dynamical system (3.6)–(3.8) are $E_1(0) = \sinh(y_1(0)) - \text{sech}(x_1(0)) = 0.5271469203$, $E_2(0) = -17.62110154$ and $E_3(0) = 151.2275486$. Similarly, we also display the dynamical of generalized synchronization errors for the drive system (3.1) and the response system (3.5) (see Fig. 3(1)–(3)).

Case 1c. *Generalized anticipated synchronization*. In the case $\tau < 0$. Without less of generality, we set $\tau = -0.5$. Thus the initial values of the error dynamical system (3.6)–(3.8) are $E_1(0) = \sinh(y_1(0)) - \text{sech}(x_1(0.5)) = 0.8563056744$, $E_2(0) = -21.85450616$ and $E_3(0) = 144.7692987$. The dynamical of generalized anticipated synchronization errors for the drive system (3.1) and the response system (3.5) are shown in Fig. 4(1)–(3).

4. Chaos synchronization between two different Hindmarsh–Rose system and modified Chua's circuit

In the following, based on the backstepping design method [5,6], with help of symbolic computation, we will study the generalized *Q*–*S* synchronization between Hindmarsh–Rose system [14] (see Fig. 5):

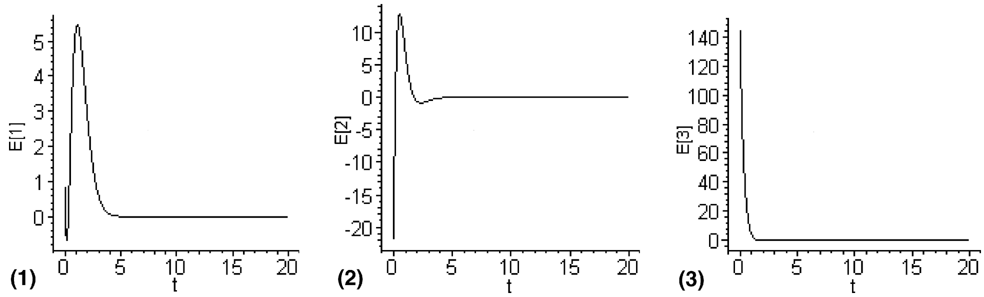

Fig. 4. Generalized anticipated synchronization errors: (1) $E_1(t) = \sinh(y_1(t)) - \mathrm{sech}(x_i(t + 0.5))$; (2) $E_2(t) = Q_2(y(t)) - S_2(x(t + 0.5))$; (3) $E_3(t) = Q_3(y(t)) - S_3(x(t + 0.5))$.

$$\begin{cases} \dot{x_1} = 0.006(4(x_2 + 1.56) - x_1), \\ \dot{x_2} = x_3 - x_2^3 + 3x_2^2 - x_1 + 3, \\ \dot{x_3} = 1 - 5x_2^2 - x_3, \end{cases} \tag{4.1}$$

and the controlled modified Chua's circuit (3.5).

Let the Lyapunov function be $L(t) = \frac{1}{2}(E_1^2 + E_2^2 + E_3^2 + E_4^2)$, Q_1=tanh$(y_1(t))$, $S_1 = \mathrm{e}^{x_1(t-\tau)}, \frac{\mathrm{d}Q_1(y(t))}{\mathrm{d}y_1(t)} = \mathrm{sech}^2(y_1(t)) \neq 0$. With the help of symbolic computation, by using the proposed scheme [12], we can obtain

$$E_1 = \tanh(y_1(t)) - \mathrm{e}^{x_1(t-\tau)}, \tag{4.2}$$

$$E_2(t) = Q_2(y(t)) - S_2(x(t-\tau)) = h_1E_1(t) + \dot{E}_1(t), \tag{4.3}$$

$$E_3(t) = Q_3(y(t)) - S_3(x(t-\tau)) = E_1(t) + h_2E_2(t) + \dot{E}_2(t), \tag{4.4}$$

where $h_j \in R^+$ $(j = 1, 2, 3)$, $Q_i(y(t))$ and $S_i(x(t-\tau))(i = 2, 3)$ are given in Appendix B.

Finally, from the equation $E_2(t) + h_3E_3(t) + \dot{E}_3(t) = 0$, we can determine the scalar controller $u(x(t-\tau), y(t))$ given in Appendix B. The substitution of (4.2)–(4.4) and u into the above-mentioned Lyapunov function $L(t)$ yields

$$\dot{L}(t) = -h_1E_1^2(t) - h_2E_2^2(t) - h_3E_3^2(t) \leqslant -\min\{h_1, h_2, h_3\}\sum_{i=1}^{3} E_i^2(t) = -2\min\{h_1, h_2, h_3\}L(t),$$

which denotes that

$$\lim_{t\to\infty} E_i(t) = \lim_{t\to\infty}[Qi(y(t)) - S_i(x(t-\tau))] = 0 \quad (i = 1, 2, 3).$$

In what follows we would like to use numerical simulations to verify the effectiveness of the obtained scalar controller $u(x(t-\tau), y(t))$ defined in Appendix B. Let the initial values of system (4.1) and (3.5) as $\alpha = 10.814$,

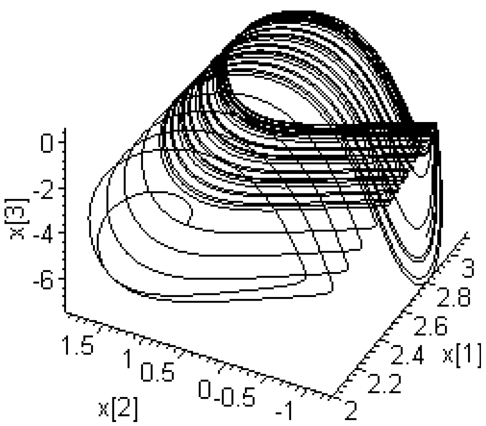

Fig. 5. Hindmarsh–Rose chaotic attractors.

$\beta = 14$, $a = 1.3$, $b = 0.11$, $c = 4$, $d = \pi$, $h_1 = 1$, $h_2 = 2$, $[x_1(0) = 2, x_2(0) = 0.5, x_3(0) = -2]$ and $[y_1(0) = -1, y_2(0) = 0.5, y_3(0) = 0]$.

Case 2a. *Generalized lag synchronization*. In the case $\tau > 0$, without less of generality, we set $\tau = 0.5$. Thus by calculation, the initial values of the error dynamical system (4.2)–(4.4) is $E_1(0) = \tanh(y_1(0)) - e^{x_1(-0.5)} = -8.003305015$, $E_2(0) = -11.25695559$ and $E_3(0) = 6.46556770$. The dynamical of generalized lag synchronization errors for the drive system (4.1) and the response system (3.5) are shown in Fig. 6(1)–(3).

Case 2b. *Generalized (complete) synchronization*. In the case $\tau = 0$. Thus the initial values of the error dynamical system (4.2)–(4.4) is $E_1(0) = \tanh(y_1(0)) - e^{x_1(0)} = -8.150650255$, $E_2(0) = -11.36711691$ and $E_3(0) = 6.12121461$. Similarly, we also display the dynamical of generalized synchronization errors for the drive system (4.1) and the response system (3.5) (see Fig. 7(1)–(3)).

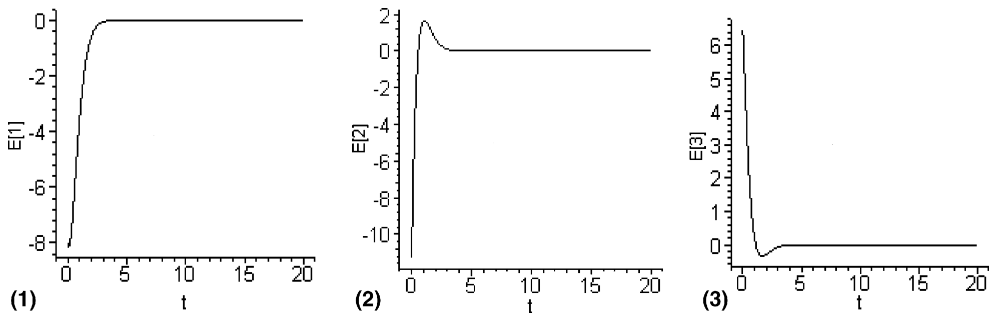

Fig. 6. Generalized lag synchronization errors: (1) $E_1(t) = \tanh(y_1(t)) - e^{x_1(t-0.5)}$; (2) $E_2(t) = Q_2(y(t)) - S_2(x(t-0.5))$; (3) $E_3(t) = Q_3(y(t)) - S_3(x(t-0.5))$.

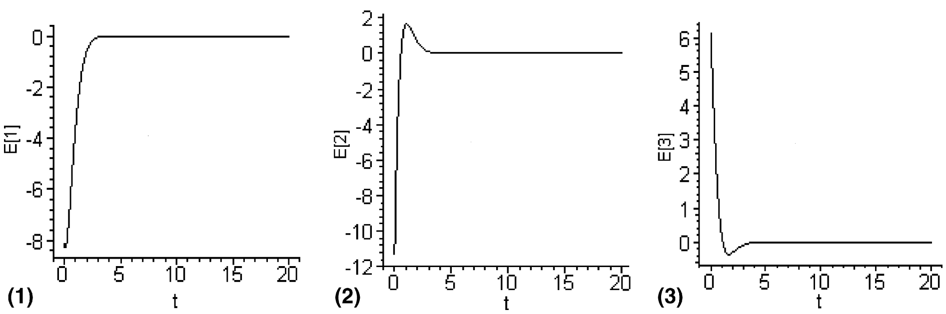

Fig. 7. Generalized (complete) synchronization errors: (1) $E_1(t) = \tanh(y_1(t)) - \sin(x_1(t))$; (2) $E_2(t) = Q_2(y(t)) - S_2(x(t))$; (3) $E_3(t) = Q_3(y(t)) - S_3(x(t))$m.

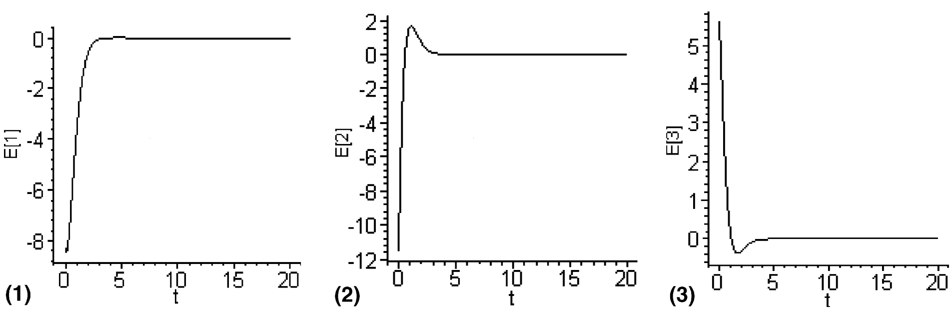

Fig. 8. Generalized anticipated synchronization errors: (1) $E_1(t) = \sinh(y_1(t)) - \mathrm{sech}(x_t(t+0.5))$; (2) $E_2(t) = Q_2(y(t)) - S_2(x(t+0.5))$; (3) $E_3(t) = Q_3(y(t)) - S_3(x(t+0.5))$m.

Case 2c. *Generalized anticipated synchronization*. In the case $\tau < 0$. Without less of generality, we set $\tau = -0.5$. Thus the initial values of the error dynamical system (4.2)–(4.4) is $E_1(0) = \tanh(y_1(0)) - e^{x_1(0.5)} = -8.285898362$, $E_2(0) = -11.50026851$ and $E_3(0) = 5.60012040$. The dynamical of generalized anticipated synchronization errors for the drive system (4.1) and the response system (3.5) are shown in Fig. 8(1)–(3).

5. Summary and conclusions

Based on the symbolic computation system *Maple*, we extend a systematic, powerful and concrete scheme to study the generalized *Q–S* (lag, anticipated and complete) synchronization between two identical modified Chua's circuit with different initial values and between two different chaotic systems: Hindmarsh–Rose system and modified Chua's circuit. Then, by means of numerical simulations, the simulation results demonstrate the effectiveness of the proposed controller. The combination of symbolic computation system and numerical simulations is a very powerful tool to investigate nonlinear science. We believe that the combination tool will be used to explore the interesting dynamical properties found in nonlinear systems.

Acknowledgements

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Appendix A

$$Q_2 = \sinh(y_1(t)) - \cosh(y_1(t))\beta y_2(t), \tag{A.1}$$

$$S_2 = -\mathrm{sech}(x_1(t-\tau)) - \mathrm{sech}(x_1(t-\tau))\tanh(x_1(t-\tau))\beta x_2(t-\tau), \tag{A.2}$$

$$\begin{aligned} Q_3 = {} & 3\sinh(y_1(t)) - 2\cosh(y_1(t))\beta y_2(t) + \sinh(y_1(t))\beta^2 y_2^2(t) \\ & - \cosh(y_1(t))\beta y_3(t) - \cosh(y_1(t))\beta y_1(t), \end{aligned} \tag{A.3}$$

$$\begin{aligned} S_3 = {} & -\mathrm{sech}(x_1(t-\tau))\tanh(x_1(t-\tau))\beta x_3(t-\tau) - 2\mathrm{sech}(x_1(t-\tau))\tanh(x_1(t-\tau))\beta x_2(t-\tau) \\ & - 2\mathrm{sech}(x_1(t-\tau))\tanh^2(x_1(t-\tau))\beta^2 x_2^2(t-\tau) + \mathrm{sech}(x_1(t-\tau))\beta^2 x_2^2(t-\tau) + 3\mathrm{sech}(x_1(t-\tau)) \\ & - \mathrm{sech}(x_1(t-\tau))\tanh(x_1(t-\tau))\beta x_1(t-\tau), \end{aligned} \tag{A.4}$$

$$\begin{aligned} u = {} & \beta y_2(t) - 5y_1(t) - 8y_2(t) - 5y_3(t) + \alpha f_2 - \alpha y_2(t) + 3\frac{\beta\sinh(y_1(t))y_2^2(t)}{\cosh(y_1(t))} + \frac{10\sinh(y_1(t))}{\cosh(y_1(t))\beta} \\ & - \beta^2 y_2^3(t) - \frac{3\beta x_2^2(t-\tau)}{\cosh(x_1(t-\tau))\cosh(y_1(t))} + \frac{6\beta x_2^2(t-\tau)}{\cosh^3(x_1(t-\tau))\cosh(y_1(t))} + \frac{6\beta x_2(t-\tau)x_1(t-\tau)}{\cosh^3(x_1(t-\tau))\cosh(y_1(t))} \\ & + \frac{6\beta x_2(t-\tau)x_3(t-\tau)}{\cosh^3(x_1(t-\tau))\cosh(y_1(t))} + \frac{6\beta^2\sinh(x_1(t-\tau))x_2^3(t-\tau)}{\cosh^4(x_1(t-\tau))\cosh(y_1(t))} + \frac{\sinh(x_1(t-\tau))\alpha f_1}{\cosh^2(x_1(t-\tau))\cosh(y_1(t))} \\ & + \frac{3\beta\sinh(y_1(t))y_2(t)y_3(t)}{\cosh(y_1(t))} + \frac{3\beta\sinh(y_1(t))y_2(t)y_1(t)}{\cosh(y_1(t))} + \frac{\beta\sinh(x_1(t-\tau))x_2(t-\tau)}{\cosh^2(x_1(t-\tau))\cosh(y_1(t))} \\ & - \frac{5\sinh(x_1(t-\tau))x_3(t-\tau)}{\cosh^2(x_1(t-\tau))\cosh(y_1(t))} - \frac{5\sinh(x_1(t-\tau))x_1(t-\tau)}{\cosh^2(x_1(t-\tau))\cosh(y_1(t))} - \frac{\beta^2\sinh(x_1(t-\tau))x_2^3(t-\tau)}{\cosh^2(x_1(t-\tau))\cosh(y_1(t))} \\ & - \frac{3\beta x_2(t-\tau)x_1(t-\tau)}{\cosh(x_1(t-\tau))\cosh(y_1(t))} - \frac{8\sinh(x_1(t-\tau))x_2(t-\tau)}{\cosh^2(x_1(t-\tau))\cosh(y_1(t))} - \frac{3\beta x_2(t-\tau)x_3(t-\tau)}{\cosh(x_1(t-\tau))\cosh(y_1(t))} \\ & - \frac{\sinh(x_1(t-\tau))\alpha x_2(t-\tau)}{\cosh^2(x_1(t-\tau))\cosh(y_1(t))} - \frac{10}{\cosh(x_1(t-\tau))\cosh(y_1(t))\beta}, \end{aligned} \tag{A.5}$$

where

$$f_1 = \begin{cases} \frac{b\pi}{2a}(x_3(t-\tau)-2ac) & \text{if } x_3(t-\tau) \geqslant 2ac, \\ -b\sin(\frac{\pi x_3(t-\tau)}{2a}+d) & \text{if } -2ac < x_3(t-\tau) < 2ac, \\ \frac{b\pi}{2a}(x_3(t-\tau)+2ac) & \text{if } x_3(t-\tau) \leqslant -2ac, \end{cases} \quad f_2 = \begin{cases} \frac{b\pi}{2a}(y_3(t)-2ac) & \text{if } y_3(t) \geqslant 2ac, \\ -b\sin(\frac{\pi y_3(t)}{2a}+d) & \text{if } -2ac < y_3(t) < 2ac, \\ \frac{b\pi}{2a}(y_3(t)+2ac) & \text{if } y_3(t) \leqslant -2ac. \end{cases}$$

Appendix B

$$Q_2 = \tanh(y_1(t)) - \beta y_2(t) + \beta y_2(t)\tanh^2(y_1(t)), \tag{B.1}$$

$$S_2 = -\frac{3242}{3125}e^{x_1(t-\tau)} - \frac{3}{125}e^{x_1(t-\tau)}x_2(t-\tau) + \frac{3}{500}e^{x_1(t-\tau)}x_1(t-\tau), \tag{B.2}$$

$$\begin{aligned} Q_3 &= 3\tanh(y_1(t)) - 2\beta y_2(t) + 2\beta y_2(t)\tanh^2(y_1(t)) - \beta y_3(t) + \beta\tanh^2(y_1(t))y_3(t) - \beta y_1(t) \\ &\quad + \beta\tanh^2(y_1(t))y_1(t) - 2\beta^2 y_2^2(t)\tanh(y_1(t)) + 2\beta^2 y_2^2(t)\tanh^3(y_1(t)), \end{aligned} \tag{B.3}$$

$$\begin{aligned} S_3 &= -\frac{124433481}{39062500}e^{x_1(t-\tau)} - \frac{115083}{1562500}e^{x_1(t-\tau)}x_2(t-\tau) + \frac{265083}{6250000}e^{x_1(t-\tau)}x_1(t-\tau) - \frac{1134}{15625}e^{x_1(t-\tau)}x_2^2(t-\tau) \\ &\quad + \frac{9}{31250}e^{x_1(t-\tau)}x_2(t-\tau)x_1(t-\tau) - \frac{3}{125}e^{x_1(t-\tau)}x_3(t-\tau) + \frac{3}{125}e^{x_1(t-\tau)}x_2^3(t-\tau) - \frac{9}{250000}e^{x_1(t-\tau)}x_1^2(t-\tau), \end{aligned} \tag{B.4}$$

$$\begin{aligned} u &= \frac{5652}{15625\beta}\cosh^2(y_1(t))e^{x_1(t-\tau)}x_2^4(t-\tau) + \frac{27}{125000000\beta}\cosh^2(y_1(t))e^{x_1(t-\tau)}x_1^3(t-\tau) - 5\cosh^2(y_1(t))y_1(t) \\ &\quad - \frac{9}{125\beta}\cosh^2(y_1(t))e^{x_1(t-\tau)}x_2^5(t-\tau) - \frac{14816331291}{19531250000\beta}\cosh^2(y_1(t))e^{x_1(t-\tau)}x_2(t-\tau) + 2\beta^2\cosh^2(y_1(t))y_2^3(t) \\ &\quad + \frac{17828181291}{78125000000\beta}\cosh^2(y_1(t))e^{x_1(t-\tau)}x_1(t-\tau) - \frac{2035611}{3125000000\beta}\cosh^2(y_1(t))e^{x_1(t-\tau)}x_1^2(t-\tau) \\ &\quad - 8\cosh^2(y_1(t))y_2(t) - \frac{2270673}{7812500\beta}\cosh^2(y_1(t))e^{x_1(t-\tau)}x_2^3(t-\tau) - \frac{191487}{1562500\beta}\cosh^2(y_1(t))e^{x_1(t-\tau)}x_3(t-\tau) \\ &\quad - \cosh^2(y_1(t))\tanh^2(y_1(t))\alpha f - \frac{81}{31250000\beta}\cosh^2(y_1(t))e^{x_1(t-\tau)}x_2(t-\tau)x_1^2(t-\tau) \\ &\quad + \frac{27}{62500\beta}\cosh^2(y_1(t))e^{x_1(t-\tau)}x_3(t-\tau)x_1(t-\tau) - \frac{27}{62500\beta}\cosh^2(y_1(t))e^{x_1(t-\tau)}x_2^3(t-\tau)x_1(t-\tau) \\ &\quad + \frac{9}{125\beta}\cosh^2(y_1(t))e^{x_1(t-\tau)}x_2^2(t-\tau)x_3(t-\tau) - 6\beta\cosh^2(y_1(t))y_2^2(t)\tanh(y_1(t)) \\ &\quad + 6\beta\cosh^2(y_1(t))y_2^2(t)\tanh^3(y_1(t)) + 5\cosh^2(y_1(t))\tanh^2(y_1(t))y_3(t) + 5\cosh^2(y_1(t))\tanh^2(y_1(t))y_1(t) \\ &\quad + 6\beta^2\cosh^2(y_1(t))y_2^3(t)\tanh^4(y_1(t)) + 6\beta\cosh^2(y_1(t))\tanh^3(y_1(t))y_1(t)y_2(t) \\ &\quad - 6\beta\cosh^2(y_1(t))\tanh(y_1(t))y_3(t)y_2(t) - \cosh^2(y_1(t))\alpha y_2(t) \\ &\quad + \frac{57610611}{390625000\beta}\cosh^2(y_1(t))e^{x_1(t-\tau)}x_2(t-\tau)x_1(t-\tau) + 6\beta\cosh^2(y_1(t))\tanh(y_1(t))y_1(t)y_2(t) \\ &\quad - \frac{5232684}{48828125\beta}\cosh^2(y_1(t))e^{x_1(t-\tau)}x_2^2(t-\tau) + \frac{10}{\beta}\cosh^2(y_1(t))\tanh(y_1(t)) \\ &\quad - \frac{2277}{15625\beta}\cosh^2(y_1(t))e^{x_1(t-\tau)}x_3(t-\tau)x_2(t-\tau) - \frac{276147}{3906250\beta}\cosh^2(y_1(t))e^{x_1(t-\tau)}x_2^2(t-\tau)x_1(t-\tau) \\ &\quad + 6\beta\cosh^2(y_1(t))\tanh^3(y_1(t))y_3(t)y_2(t) + \cosh^2(y_1(t))\tanh^2(y_1(t))\alpha y_2(t) - 5\cosh y_1^2(t)y_3(t) \\ &\quad - \frac{5349886601333}{488281250000\beta}\cosh^2(y_1(t))e^{x_1(t-\tau)} - \beta\cosh^2(y_1(t))\tanh^2(y_1(t))y_2(t), \end{aligned} \tag{B.5}$$

where

$$f = \begin{cases} \frac{b\pi}{2a}(y_3(t) - 2ac) & \text{if } y_3(t) \geqslant 2ac, \\ -b\sin(\frac{\pi y_3(t)}{2a} + d) & \text{if } -2ac < y_3(t) < 2ac, \\ \frac{b\pi}{2a}(y_3(t) + 2ac) & \text{if } y_3(t) \leqslant -2ac. \end{cases}$$

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