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Applied Mathematics and Computation 181 (2006) 247–255
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Generalized algebraic method and new exact traveling wave solutions for (2 + 1)-dimensional dispersive long wave equation

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Abstract

With the help of the symbolic computation system *Maple*, a new generalized algebraic method to uniformly construct solutions in terms of special function of nonlinear partial differential equations is presented by means of a more general ansatz. As an application of the method, we choose a (2 + 1)-dimensional dispersive long wave equation to illustrate the method. As a result, we can successfully obtain the solutions found by the method proposed by Fan [E.G. Fan, Phys. Lett. A 300 (2002) 243] and find other new and more general solutions at the same time, which include polynomial solutions, exponential solutions, rational solutions, triangular periodic wave solutions, hyperbolic, and soliton solutions, Jacobi, and Weierstrass doubly periodic wave solutions.

Keywords: Symbolic computation; (2 + 1)-dimensional dispersive long wave equation; Weierstrass and Jacobi elliptic functions; Soliton solution; Periodic solution

1. Introduction

In recent years, nonlinear partial differential equation (PDE) system are widely used to describe many important phenomena and dynamic processes in physics, mechanics, chemistry, biology, etc. With the development of symbolic computation, there have been a great amount of activities aiming to find methods for exact solution of nonlinear PDE system. Among those, the tanh method [1–3] provides a straightforward and effective algorithm to obtain such particular solutions for a large number of nonlinear PDE system, in which the starting point is the ansatz that the solution sought is expressible as a finite series of tanh function. Recently, much research work has been concentrated on the various extensions and applications of tanh

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0096-3003/$ - see front matter © 2006 Published by Elsevier Inc.
doi:10.1016/j.amc.2006.01.038

method. Recently, Fan [4–6] has proposed an extended tanh-function method. More recently, Fan [7] and Yan [8] further developed this idea and made it much more lucid and straightforward for a class of nonlinear PDE system. Most recently, Elwakil et al. [9] modified extended tanh-function method and obtained some new exact solutions. Based on above work, Chen and Zheng [10,11] generalized extended tanh-function method. On the other hand, Gao and Tian [12,13] presented a generalized hyperbolic-function method by introducing coefficient functions. Li and Chen [14,15] presented the generalized Riccati equation expansion method to construct soliton-like solution of nonlinear PDE system. As we known, when applying above mentioned method, the choice of an appropriate ansatz is of great importance. Generally speaking, the more general ansatz proposed is, the more formals of solutions of nonlinear PDE system will be obtained.

In [16], Fan developed a new algebraic method which further exceeds the applicability of tanh method in obtaining a series of exact solutions of nonlinear PDE system. Compared with most of the existing tanh methods, the proposed method not only gives an unified formulation to construct various travelling wave solutions, but also provides a guideline to classify the various types of the travelling wave solutions according to the values of some parameters. The present work is motivated by the desire to generalize the work made in [16] by proposing a more general ansatz so that it can be used to obtained more types and general formal solutions which contain not only the results obtained by using the method [16] but also other types of solutions. For illustration, we apply the generalized method to solve $(2+1)$-dimensional dispersive long wave equation and successfully construct new and more general solutions including soliton wave solution, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions for $(2+1)$-dimensional dispersive long wave equation.

This paper is organized as follows. In Section 2, we summarize the generalized method. In Section 3, we apply the generalized method to $(2+1)$-dimensional dispersive long wave equation and bring out many solutions. Conclusions will be presented in finally.

2. Summary of the improved method: Generalized extended tanh-function method

In the following we would like to outline the main steps of our general method:

Step 1. For a given nonlinear PDE system with some physical fields $u_i(x,y,t)$ in three variables x, y, t,

$$F_i(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \ldots) = 0, \tag{2.1}$$

use the wave transformation

$$u_i(x,y,t) = u_i(\xi), \qquad \xi = k(x + ly - \lambda t), \tag{2.2}$$

where k, l and λ are constants to be determined later. Then the nonlinear partial differential (2.1) is reduced to a nonlinear ordinary differential equation (ODE) system:

$$G_i(u_i, u_i', u_i'', \ldots) = 0. \tag{2.3}$$

Step 2. We introduce a new and more general ansatz in the forms:

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \left\{ a_{ij}\phi^j + b_{ij}\phi^{-j} + f_{ij}\phi^{j-1}\sqrt{\sum_{p=0}^{4} c_p\phi^p} + k_{ij}\frac{\sqrt{\sum_{p=0}^{4} c_p\phi^p}}{\varphi^j} \right\}, \tag{2.4}$$

where the new variable $\phi = \phi(\xi)$ satisfies

$$\phi' = \frac{\mathrm{d}\phi}{\mathrm{d}\xi} = \sqrt{\sum_{p=0}^{4} c_p\phi^p}, \tag{2.5}$$

and $a_{i0}, a_{ij}, b_{ij}, f_{ij}, k_{ij}$ $(i = 1,2,\ldots;\ j = 1,2,\ldots,m_i)$ are constants to be determined later.Compared with the method proposed by Fan [16], our ansatz is more general than the ansatz presented in [16]. When $b_{ij} = f_{ij} = k_{ij} = 0$ in (2.4), (2.4) becomes the ansatz proposed by Fan.

Step 3. The underlying mechanism for a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions to occur is that differ effects that act to change wave forms in many nonlinear PDE, i.e. dispersion, dissipation and nonlinearity, either separately or various combination are able to balance out. We define the degree of $u_i(\xi)$ as $D[u_i(\xi)] = n_i$, which gives rise to the degrees of other expressions as

$$D\left[u_i^{(\alpha)}\right] = n_i + \alpha, \quad D\left[u_i^{\beta}\left(u_j^{(\alpha)}\right)^s\right] = n_i\beta + (\alpha + n_j)s. \tag{2.6}$$

Therefore we can get the value of m_i in (2.4). If n_i is a nonnegative integer, then we first make the transformation $u_i = \omega^{n_i}$.

Step 4. Substitute (2.4) into (2.2) along with (2.5) and then set all coefficients of $\omega^{\alpha}\left(\sqrt{\sum_{p=0}^{4} c_p\phi^p}\right)^{\beta}$ $(\beta = 0, 1;\ \alpha = 0, 1, 2, \ldots)$ to be zero to get an over-determined system of nonlinear algebraic equations with respect to λ, l, k, a_{i0}, a_{ij}, b_{ij}, f_{ij} and k_{ij} $(i = 1, 2, \ldots;\ j = 1, 2, \ldots, m_i)$.

Step 5. Solving the over-determined system of nonlinear algebraic equations by use of Maple, we would end up with the explicit expressions for λ, l, k, a_{i0}, a_{ij}, b_{ij}, f_{ij} and $k_{ij}(i = 1, 2, \ldots;\ j = 1, 2, \ldots, m_i)$.

Step 6. By using the results obtained in the above step, we can derive a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions. Because we are interested in solitary wave, Jacobi and Weierstrass doubly periodic solutions. On the other hand, tan and cot type solutions appear in pairs with tanh and coth type solutions. Respectively polynomial, rational triangular periodic solutions are omitted in this paper. By considering the different values of c_0, c_1, c_2, c_3 and c_4, (2.5) has many kinds of solitary wave, Jacobi and Weierstrass doubly periodic solutions which are listed as follows.

(i) Solitary wave solutions.

(a) Bell shaped solitary wave solutions

$$\phi = \sqrt{-\frac{c_2}{c_4}}\operatorname{sech}(\sqrt{c_2}\xi), \quad c_0 = c_1 = c_3 = 0, \quad c_2 > 0, \quad c_4 < 0, \tag{2.7}$$

$$\phi = -\frac{c_2}{c_3}\operatorname{sech}^2\left(\frac{\sqrt{c_2}}{2}\xi\right), \quad c_0 = c_1 = c_4 = 0, \quad c_2 > 0. \tag{2.8}$$

(b) Kink shaped solitary wave solutions

$$\phi = k\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right), \quad c_0 = \frac{c_2^2}{4c_4}, \quad c_1 = c_3 = 0, \quad c_2 < 0, \quad c_4 > 0. \tag{2.9}$$

(c) Solitary wave solutions

$$\phi = \frac{c_2\operatorname{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2k\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3}, \quad c_0 = c_1 = 0, \quad c_2 > 0. \tag{2.10}$$

(ii) Jacobi and Weierstrass doubly periodic solutions

$$\phi = \sqrt{\frac{-c_2m^2}{c_4(2m^2-1)}}\operatorname{cn}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right), \quad c_4 < 0, \quad c_2 > 0, \quad c_0 = \frac{c_2^2m^2(1-m^2)}{c_4(2m^2-1)^2}, \tag{2.11}$$

$$\phi = \sqrt{\frac{-m^2}{c_4(2-m^2)}}\operatorname{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right), \quad c_4 < 0, \quad c_2 > 0, \quad c_0 = \frac{c_2^2(1-m^2)}{c_4(2-m^2)^2}, \tag{2.12}$$

$$\phi = \sqrt{\frac{-c_2m^2}{c_4(m^2+1)}}\operatorname{sn}\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right), \quad c_4 > 0, \quad c_2 < 0, \quad c_0 = \frac{c_2^2m^2}{c_4(m^2+1)^2}, \tag{2.13}$$

where m is a modulus

$$\phi = \wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right), \quad c_2 = 0, \quad c_3 > 0, \tag{2.14}$$

where $g_2 = -4\frac{c_1}{c_3}$ and $g_3 = -4\frac{c_0}{c_3}$ are called invariants of Weierstrass elliptic function. The Jacobi elliptic functions are doubly periodic and possess properties of triangular functions:

$$\mathrm{sn}^2\xi + \mathrm{cn}^2\xi = 1, \quad \mathrm{dn}^2\xi = 1 - m^2\,\mathrm{sn}^2\xi,$$
$$(\mathrm{sn}\,\xi)' = \mathrm{cn}\,\xi\mathrm{dn}\,\xi, \quad (\mathrm{cn}\,\xi)' = -\mathrm{sn}\,\xi\mathrm{dn}\,\xi, \quad (\mathrm{dn}\,\xi)' = -m^2\,\mathrm{sn}\,\xi\mathrm{cn}\,\xi.$$

When $m \to 1$, the Jacobi functions degenerate to the hyperbolic functions, i.e.

$$\mathrm{sn}\,\xi \to \tanh\xi, \quad \mathrm{cn}\,\xi \to \mathrm{sech}\,\xi.$$

When $m \to 0$, the Jacobi functions degenerate to the triangular functions, i.e.

$$\mathrm{sn}\,\xi \to \sin\xi, \quad \mathrm{cn}\,\xi \to \cos\xi.$$

The more detailed notations for the Weierstrass and Jacobi elliptic functions can be found in Refs. [17–19].

Remark 1. The method can be extend to find soliton-like solutions and more types double periodic solutions of (2.1). Only will the restriction on $\xi(x,y,t)$ as merely a linear function x, y, t and the restrictions on the coefficients a_{i0}, a_{ij}, b_{ij}, f_{ij}, k_{ij} and c_i as constants be removed, i.e., the ansatz (2.2) and (2.4) generalize as

$$\begin{aligned} u_i(x,y,t) &= a_{i0}(x,y,t) + \sum_{j=1}^{m_i}\{a_{ij}(x,y,t)\phi^j + b_{ij}(x,y,t)\phi^{-j}\} \\ &\quad + \sum_{j=1}^{m_i}\left\{f_{ij}(x,y,t)\phi^{j-1}\sqrt{c_0+c_1\phi+c_2\phi^2+c_3\phi^3+c_4\phi^4} + k_{ij}(x,y,t)\frac{\sqrt{c_0+c_1\phi+c_2\phi^2+c_3\phi^3+c_4\phi^4}}{\phi^j}\right\}, \end{aligned}$$

where a_{i0}, a_{ij}, b_{ij}, f_{ij} and k_{ij} ($i = 1,2,\ldots; j = 1,2,\ldots,m_i$) are all functions of x, y, t; c_0, c_1, c_2, c_3 and c_4 are arbitrary constant to be determined later; and

$$\phi'(\xi(x,y,t)) = \sqrt{c_0 + c_1\phi(\xi(x,y,t)) + c_2\phi^2(\xi(x,y,t)) + c_3\phi^3(\xi(x,y,t)) + c_4\phi^4(\xi(x,y,t))},$$

where $\xi(x,y,t)$ is an arbitrary functions of x, y, t to be determined later.

Remark 2. Some kinds of solutions derived by the generalized transformation are singular soliton solution and Jacobi elliptic doubly periodic wave solution. There is much current interest in the formation of so-called "hot spots" or "blow up" of solutions. It appears that these singular solutions will model this physical phenomena.

Remark 3. The general solutions of sub-equation (2.5) are difficult to be listed, because it is a complicated nonlinear ODE. We just listed some particular solutions of (2.5) to illustrate the effectiveness of our algorithm. Other interested researchers can obtain much more solutions of (2.1) when they get more solutions of (2.5), but this is not the purpose of this paper.

3. Exact solutions of (2 + 1)-dimensional dispersive long wave equation

Let us consider the $(2+1)$-dimensional dispersive long wave equation (DLWE) [20–24],

$$\begin{aligned} &u_{yt} - \eta_{xx} + \alpha(uu_x)_y = 0, \\ &\eta_t + u_x + \beta(u\eta)_x + u_{xxy} = 0, \end{aligned} \tag{3.1}$$

where α, $\beta \neq 0$ are all constants. The DLWE were first obtained by Boiti et al. [20] as a compatibility condition for a "weak" Lax pair. A Kac–Moody–Virasoro Lie algebra for the DLWE was given by Paquin and

Winternitz [21]. Lou [22] has shown that the DLWE cannot pass the painleve test both in the ARS algorithm and in the WTC approach. The solitary wave solution of the DLWE were constructed by using a homogeneous method [23]. Recently, Fan used the method in [24] to DLWE and found new soliton solutions, which cannot be obtained by using the tanh method and the homogeneous balance method:

According to the above method, to seek the solutions of (3.1), we make the following transformation:

$$u(x,t)=\sigma(\xi),\quad \eta(\xi)=\tau(\xi),\quad \xi=x+ly-\lambda t, \tag{3.2}$$

where λ and l are constants to be determined later, and thus (3.1) becomes

$$\begin{aligned} &-\lambda l\sigma''-\tau''+\alpha\ell(\sigma\sigma')'=0,\\ &-\lambda\tau'+\sigma'+\beta(\sigma\tau)'+l\sigma'''=0. \end{aligned} \tag{3.3}$$

According to *Step* 1 in Section 2, if $a\neq 0$ and $\beta\neq 0$, by balancing $\sigma'''(\xi)$ and $(\sigma(\xi)\tau(\xi))'$ in (3.3) and by balancing $\tau''(\xi)$ and $(\sigma'(\xi))^2$ in (3.3), we suppose that (3.3) has the following formal solutions:

$$\begin{aligned} \sigma&=a_0+a_1\phi+\frac{b_1}{\phi}+f_1\sqrt{\sum_{i=0}^{4}c_i\phi^i}+k_1\frac{\sqrt{\sum_{i=0}^{4}c_i\phi^i}}{\phi},\\ \tau&=A_0+A_1\phi+\frac{B_1}{\phi}+F_1\sqrt{\sum_{i=0}^{4}c_i\phi^i}+K_1\frac{\sqrt{\sum_{i=0}^{4}c_i\phi^i}}{\phi}+A_2\phi^2+\frac{B_2}{\phi^2}+F_2\phi\sqrt{\sum_{i=0}^{4}c_i\phi^i}+K_2\frac{\sqrt{\sum_{i=0}^{4}c_i\phi^i}}{\phi^2}, \end{aligned} \tag{3.4}$$

where $\phi(\xi)$ satisfies (2.5), where $a_0, a_1, b_1, f_1, k_1, a_2, b_2, f_2, k_2, A_0, A_1, B_1, F_1, K_1, A_2, B_2, F_2$ and K_2 are constants to be determined later.

With the aid of *Maple*, substituting (3.4) along with (2.5) into (3.3), yields a set of algebraic equations for $\phi^p(\xi)\left(\sqrt{\sum_{i=0}^{4}c_i\phi^i}\right)^q$, $(p=0,1,\ldots;\ q=0,1)$. Setting the coefficients of these terms $\phi^p(\xi)\left(\sqrt{\sum_{i=0}^{4}c_i\phi^i}\right)^q$ to zero yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, f_1, k_1, a_2, b_2, f_2, k_2, A_0, A_1, B_1, F_1, K_1, A_2, B_2, F_2, K_2, l, and λ.

By use of the *Maple*, solving the over-determined algebraic equations, we get the following results.

Case 1.

$$\begin{aligned} &A_2=-2\frac{lc_4}{\beta},\quad K_1=\pm\frac{2}{3}\sqrt{\frac{1}{\beta\alpha}}l\lambda(-\beta+\alpha)\beta^{-1},\quad k_1=\pm 2\sqrt{\frac{1}{\beta\alpha}},\\ &a_0=\frac{\lambda(2\beta+\alpha)}{3\beta\alpha},\quad A_0=-\frac{-4l\lambda^2\beta\alpha+2\lambda^2l\alpha^2+9\beta\alpha+2l\lambda^2\beta^2}{9\beta^2\alpha},\\ &c_0=c_1=c_3=a_1=b_1=f_1=A_1=B_1=B_2=F_1=F_2=K_2=0. \end{aligned} \tag{3.5}$$

Case 2.

$$\begin{aligned} &B_2=-\alpha lb_1^2,\quad A_2=-\alpha la_1^2,\quad k_1=\pm\sqrt{\frac{1}{\beta\alpha}},\quad a_0=\frac{\lambda(2\beta+\alpha)}{3\beta\alpha},\quad f_1=F_2=0,\\ &c_0=\beta\alpha b_1^2,\quad K_2=\pm\alpha lb_1\sqrt{\frac{1}{\beta\alpha}},\quad B_1=-\frac{l(-2\beta\lambda b_1+2\lambda\alpha b_1+3c_1)}{6\beta},\\ &K_1=\pm\frac{1}{3}\sqrt{\frac{1}{\beta\alpha}}l\lambda(-\beta+\alpha)\beta^{-1},\quad A_0=-\frac{2l\lambda^2\beta^2-4l\lambda^2\beta\alpha+2\lambda^2l\alpha^2-18\beta^2b_1\alpha^2la_1+9\beta\alpha}{9\beta^2\alpha},\\ &F_1=\pm\alpha la_1\sqrt{\frac{1}{\beta\alpha}},\quad c_4=\beta\alpha a_1^2,\quad A_1=-\frac{l(-2\beta a_1\lambda+3c_3+2\lambda\alpha a_1)}{6\beta}. \end{aligned} \tag{3.6}$$

Case 3.

$$\begin{aligned} &A_2=-2\frac{lc_4}{\beta},\quad B_2=-2\frac{lc_0}{\beta},\quad a_0=\frac{\lambda(2\beta+\alpha)}{3\beta\alpha},\quad k_1=\pm 2\sqrt{\frac{1}{\beta\alpha}},\\ &K_1=\pm\frac{2}{3}\sqrt{\frac{1}{\beta\alpha}}l\lambda(-\beta+\alpha)\beta^{-1},\quad A_0=-\frac{-4l\lambda^2\beta\alpha+2\lambda^2l\alpha^2+9\beta\alpha+2l\lambda^2\beta^2}{9\beta^2\alpha},\\ &c_1=c_3=a_1=b_1=f_1=A_1=B_1=F_2=F_1=K_2=0. \end{aligned} \tag{3.7}$$

Case 4.

$$\begin{aligned} &A_2=-\alpha la_1^2,\quad B_1=-\frac{lc_1}{2\beta},\quad a_0=\frac{\lambda(2\beta+\alpha)}{3\beta\alpha},\quad F_1=\pm\alpha la_1\sqrt{\frac{1}{\beta\alpha}},\\ &K_1=\pm\frac{1}{3}\sqrt{\frac{1}{\beta\alpha}}l\lambda(-\beta+\alpha)\beta^{-1},\quad A_0=-\frac{-4l\lambda^2\beta\alpha+2\lambda^2l\alpha^2+9\beta\alpha+2l\lambda^2\beta^2}{9\beta^2\alpha},\\ &k_1=\pm\sqrt{\frac{1}{\beta\alpha}},\quad c_4=\beta\alpha a_1^2,\quad A_1=-\frac{l(-2\beta a_1\lambda+3c_3+2\lambda\alpha a_1)}{6\beta},\\ &c_0=b_1=f_1=B_2=F_2=K_2=0. \end{aligned} \tag{3.8}$$

Case 5.

$$\begin{aligned} &B_2=-\alpha lb_1^2,\quad a_0=\frac{\lambda(2\beta+\alpha)}{3\beta\alpha},\quad c_0=\beta\alpha b_1^2,\quad A_1=-\frac{lc_3}{2\beta},\\ &B_1=-\frac{l(-2\beta\lambda b_1+2\lambda\alpha b_1+3c_1)}{6\beta},\quad K_2=\pm\alpha lb_1\sqrt{\frac{1}{\beta\alpha}},\\ &A_0=-\frac{-4l\lambda^2\beta\alpha+2\lambda^2l\alpha^2+9\beta\alpha+2l\lambda^2\beta^2}{9\beta^2\alpha},\quad K_1=\pm\frac{1}{3}\sqrt{\frac{1}{\beta\alpha}}l\lambda(-\beta+\alpha)\beta^{-1},\\ &k_1=\pm\sqrt{\frac{1}{\beta\alpha}},\quad c_2=c_4=a_1=f_1=A_2=F_1=F_2=0. \end{aligned} \tag{3.9}$$

From (3.2), (3.4) and Case 1–5, we obtain the following solutions for (3.1):

Family 1. From (3.5), we obtain the following solutions for the DLWE, as follows:

$$u_1=a_0\pm k_1\sqrt{c_2-c_2\,\mathrm{sech}^2(\sqrt{c_2}\xi)}, \tag{3.10}$$

$$\eta_1=A_0-\frac{A_2c_2}{c_4}\mathrm{sech}^2(\sqrt{c_2}\xi)\pm K_1\sqrt{c_2-c_2\,\mathrm{sech}^2(\sqrt{c_2}\xi)}, \tag{3.11}$$

where $\xi=x+ly-\lambda t$, $a_0=\frac{\lambda(2\beta+\alpha)}{3\beta\alpha}$, $A_0=-\frac{-4l\lambda^2\beta\alpha+2\lambda^2l\alpha^2+9\beta\alpha+2l\lambda^2\beta^2}{9\beta^2\alpha}$, $A_2=-2\frac{lc_4}{\beta}$, $K_1=\pm\frac{2}{3}\sqrt{\frac{1}{\beta\alpha}}l\lambda(-\beta+\alpha)\beta^{-1}$, $k_1=\pm 2\sqrt{\frac{1}{\beta\alpha}}$, $c_2>0$, $c_4<0$, λ and l are arbitrary constants.

Family 2. From (3.6), when $b_1=0$, then we obtain the following solutions for the DLWE, as follows:

$$\begin{aligned} u_2=a_0+a_1&\frac{c_2\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)-c_3}\\ &+k_1\sqrt{c_2+c_3\frac{c_2\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)-c_3}+c_4\left(\frac{c_2\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)-c_3}\right)^2}, \end{aligned} \tag{3.12}$$

$$\eta_2 = A_0 + \frac{A_1 c_2 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3} + A_2\left(\frac{c_2 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3}\right) + K_1\sqrt{c_2 + c_3\frac{c_2 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3} + c_4\left(\frac{c_2 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3}\right)^2} + \frac{F_1 c_2 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3}\sqrt{c_2 + c_3\frac{c_2 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3} + c_4\left(\frac{c_2 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3}\right)^2}, \tag{3.13}$$

where $\xi = x + ly - \lambda t$, $a_0 = \frac{\lambda(2\beta+\alpha)}{3\beta\alpha}$, $A_2 = -\alpha l a_1^2$, $k_1 = \pm\sqrt{\frac{1}{\beta\alpha}}$, $F_1 = -\frac{lc_1}{2\beta}$, $K_1 = \pm\frac{1}{3}\sqrt{\frac{1}{\beta\alpha}}l\lambda(-\beta+\alpha)\beta^{-1}$, $A_0 = -\frac{2l\lambda^2\beta^2-4l\lambda^2\beta\alpha+2\lambda^2 l\alpha^2+9\beta\alpha}{9\beta^2\alpha}$, $F_1 = \pm\alpha l a_1\sqrt{\frac{1}{\beta\alpha}}$, $c_4 = \beta\alpha a_1^2$, $A_1 = -\frac{l(-2\beta a_1\lambda+3c_3+2\lambda\alpha a_1)}{6\beta}$, $c_2 > 0$, c_3, a_1, λ and l are arbitrary constants.

Family 3. From (3.7), then we obtain the following solutions for the DLWE, as follows:

$$u_3 = a_0 + k_1\frac{\sqrt{c_0 - \frac{c_2^2m^2\operatorname{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}{c_4(2m^2-1)} + \frac{c_2^2m^4\operatorname{cn}^4\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}{c_4(2m^2-1)^2}}}{\sqrt{\frac{-c_2m^2}{c_4(2m^2-1)}}\operatorname{cn}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}, \tag{3.14}$$

$$\eta_3 = -\frac{-4l\lambda^2\beta\alpha + 2\lambda^2 l\alpha^2 + 9\beta\alpha + 2l\lambda^2\beta^2}{9\beta^2\alpha} - \frac{A_2c_2m^2}{c_4(2m^2-1)}\operatorname{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right) - \frac{B_2c_4(2m^2-1)}{c_2m^2\operatorname{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)} + K_1\frac{\sqrt{c_0 - \frac{c_2^2m^2\operatorname{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}{c_4(2m^2-1)} + \frac{c_2^2m^4\operatorname{cn}^4\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}{c_4(2m^2-1)^2}}}{\sqrt{\frac{-c_2m^2}{c_4(2m^2-1)}}\operatorname{cn}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}, \tag{3.15}$$

where $\xi = x + ly - \lambda t$, $A_2 = -2\frac{lc_4}{\beta}$, $B_2 = -2\frac{lc_0}{\beta}$, $a_0 = \frac{\lambda(2\beta+\alpha)}{3\beta\alpha}$, $k_1 = \pm 2\sqrt{\frac{1}{\beta\alpha}}$, $K_1 = \pm\frac{2}{3}\sqrt{\frac{1}{\beta\alpha}}l\lambda(-\beta+\alpha)\beta^{-1}$, $c_0 = \frac{c_2^2m^2(1-m^2)}{c_4(2m^2-1)^2}$, $c_2 > 0$, $c_4 < 0$, λ and l are arbitrary constants.

Family 4. From (3.7), we can obtain the following solutions for the DLWE, as follows:

$$u_4 = a_0 + k_1\frac{\sqrt{c_0 - \frac{c_2m^2\operatorname{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}{c_4(2-m^2)} + \frac{m^4\operatorname{dn}^4\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}{c_4(2-m^2)^2}}}{\sqrt{\frac{-m^2}{c_4(2-m^2)}}\operatorname{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}, \tag{3.16}$$

$$\eta_4 = -\frac{-4l\lambda^2\beta\alpha + 2\lambda^2 l\alpha^2 + 9\beta\alpha + 2l\lambda^2\beta^2}{9\beta^2\alpha} - \frac{A_2m^2}{c_4(2-m^2)}\operatorname{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right) - \frac{B_2c_4(2-m^2)}{m^2\operatorname{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)} + K_1\frac{\sqrt{c_0 - \frac{c_2m^2\operatorname{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}{c_4(2-m^2)} + \frac{m^4\operatorname{dn}^4\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}{c_4(2-m^2)^2}}}{\sqrt{\frac{-m^2}{c_4(2-m^2)}}\operatorname{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}, \tag{3.17}$$

where $\xi = x + ly - \lambda t$, $A_2 = -2\frac{lc_4}{\beta}$, $B_2 = -2\frac{lc_0}{\beta}$, $a_0 = \frac{\lambda(2\beta+\alpha)}{3\beta\alpha}$, $k_1 = \pm 2\sqrt{\frac{1}{\beta\alpha}}$, $K_1 = \pm\frac{2}{3}\sqrt{\frac{1}{\beta\alpha}}l\lambda(-\beta+\alpha)\beta^{-1}$, $c_0 = \frac{c_2^2m^2}{c_4(m^2+1)^2}$, $c_2 > 0$, $c_4 < 0$, λ and l are arbitrary constants.

Family 5. From (3.7), we can obtain the following solutions for the DLWE, as follows:

$$u_5 = a_0 + k_1\frac{\sqrt{c_0 - \frac{c_2^2m^2\operatorname{sn}^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}{c_4(m^2+1)} + \frac{c_2^2m^4\operatorname{sn}^4\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}{c_4(m^2+1)^2}}}{\sqrt{\frac{-c_2m^2}{c_4(m^2+1)}}\operatorname{sn}\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}, \tag{3.18}$$

$$\eta_5 = -\frac{-4l\lambda^2\beta\alpha + 2\lambda^2 l\alpha^2 + 9\beta\alpha + 2l\lambda^2\beta^2}{9\beta^2\alpha} - \frac{A_2 c_2 m^2}{c_4(m^2+1)}\mathrm{sn}^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)$$
$$-\frac{B_2 c_4(m^2+1)}{c_2 m^2 \mathrm{sn}^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)} + K_1 \frac{\sqrt{c_0 - \frac{c_2^2 m^2\,\mathrm{sn}^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}{c_4(m^2+1)} + \frac{c_2^2 m^4 \mathrm{sn}^4\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}{c_4(m^2+1)^2}}}{\sqrt{\frac{-c_2 m^2}{c_4(m^2+1)}}\mathrm{sn}\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}, \quad (3.19)$$

where $\xi = x + ly - \lambda t$, $A_2 = -2\frac{lc_4}{\beta}$, $B_2 = -2\frac{lc_0}{\beta}$, $a_0 = \frac{\lambda(2\beta+\alpha)}{3\beta\alpha}$, $k_1 = \pm 2\sqrt{\frac{1}{\beta\alpha}}$, $K_1 = \pm\frac{2}{3}\sqrt{\frac{1}{\beta\alpha}}l\lambda(-\beta+\alpha)\beta^{-1}$, $c_0 = \frac{c_2^2(1-m^2)}{c_4(2-m^2)^2}$, $c_2 < 0$, $c_4 > 0$, λ and l are arbitrary constants.

Family 6. From (3.8), when $c_3 = c_4 = 0$, we can obtain the following solutions for the DLWE, as follows.

$$u_6 = a_0 - \frac{a_1 c_1}{2c_2}(1 - \sinh(2\sqrt{c_2}\xi)) - \frac{2c_2 k_1}{c_1}\frac{\sqrt{\frac{c_1^2}{2c_2}\left(-1 + \sinh\left(2\sqrt{c_2}\xi\right)\right) + \frac{c_1^2}{4c_2}\left(1 - \sinh\left(2\sqrt{c_2}\xi\right)\right)^2}}{\left(1 - \sinh\left(2\sqrt{c_2}\xi\right)\right)}, \quad (3.20)$$

$$\eta_6 = A_0 - \frac{A_1 c_1}{2c_2}(1 - \sinh(2\sqrt{c_2}\xi)) + \frac{A_2 c_1^2}{4c_2^2}(1 - \sinh(2\sqrt{c_2}\xi))^2$$
$$+ F_1\sqrt{\frac{c_1^2}{2c_2}(-1 + \sinh(2\sqrt{c_2}\xi)) + \frac{c_1^2}{4c_2}(1 - \sinh(2\sqrt{c_2}\xi))^2}$$
$$-\frac{2c_2 K_1}{c_1}\frac{\sqrt{\frac{c_1^2}{2c_2}\left(-1 + \sinh\left(2\sqrt{c_2}\xi\right)\right) + \frac{c_1^2}{4c_2}\left(1 - \sinh\left(2\sqrt{c_2}\xi\right)\right)^2}}{\left(1 - \sinh\left(2\sqrt{c_2}\xi\right)\right)}, \quad (3.21)$$

where $\xi = x + ly - \lambda t$, $A_2 = -\alpha l a_1^2$, $F_1 = -\frac{lc_1}{2\beta}$, $a_0 = \frac{\lambda(2\beta+\alpha)}{3\beta\alpha}$, $F_1 = \pm\alpha l a_1\sqrt{\frac{1}{\beta\alpha}}$, $K_1 = \pm\frac{1}{3}\sqrt{\frac{1}{\beta\alpha}}l\lambda(-\beta+\alpha)\beta^{-1}$, $A_0 = -\frac{-4l\lambda^2\beta\alpha+2\lambda^2 l\alpha^2+9\beta\alpha+2l\lambda^2\beta^2}{9\beta^2\alpha}$, $k_1 = \pm\sqrt{\frac{1}{\beta\alpha}}$, $c_4 = \beta\alpha a_1^2$, $A_1 = -\frac{l(-2\beta a_1\lambda+2\lambda\alpha a_1)}{6\beta}$, $c_2 > 0$, a_1, c_1, λ and l are arbitrary constants.

Family 7. From (3.9), then we can obtain the following solutions for the DLWE, as follows:

$$u_7 = \frac{\lambda(2\beta+\alpha)}{3\beta\alpha} + \frac{b_1}{\wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)} \pm \sqrt{\frac{1}{\beta\alpha}}\frac{\sqrt{c_0 + c_1\wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right) + c_3\wp^3(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3)}}{\wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)}, \quad (3.22)$$

$$\eta_7 = -\frac{-4l\lambda^2\beta\alpha + 2\lambda^2 l\alpha^2 + 9\beta\alpha + 2l\lambda^2\beta^2}{9\beta^2\alpha} - \frac{lc_3}{2\beta}\wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right) - \frac{\alpha l 1_1^2}{\wp^2\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)}$$
$$-\frac{l(-2\beta\lambda b_1 + 2\lambda\alpha b_1 + 3c_1)}{6\beta\wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)} \pm \alpha l b_1\sqrt{\frac{1}{\beta\alpha}}\frac{\sqrt{c_0 + c_1\wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right) + c_3\wp^3\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)}}{\wp^2\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)}$$
$$\pm\frac{1}{3}\sqrt{\frac{1}{\beta\alpha}}l\lambda(-\beta+\alpha)\beta^{-1}\frac{\sqrt{c_0 + c_1\wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right) + c_3\wp^3\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)}}{\wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)}, \quad (3.23)$$

where $\xi = x + ly - \lambda t$, $g_2 = -4\frac{c_1}{c_3}$, $g_3 = -4\frac{c_0}{c_3}$, $c_0 = \beta\alpha b_1^2$, $c_3 > 0$, c_1, b_1, λ and l are arbitrary constants.

Remark 4. The solutions (3.10) and (3.11) become the solutions (4.16) in [24], when $\alpha = \beta = 1$, $\lambda = e$ and $l = d$. The solutions (3.10) and (3.11) become the solutions (4.25) and (4.26) in [23], when $\alpha = \beta = 1$, $\lambda = \alpha^*$, $l = \beta^*$, $\delta^* = 0$ and $c_2 = (\frac{\alpha^*}{2})^2$.

Remark 5. The other solutions obtained here, to our knowledge, are all new families of exact solutions of the DLWE.

4. Summary and conclusions

Generally speaking, to construct the more general and more formal solutions of nonlinear PDE system, the various extensions and improvement of tanh method have developed and can be classified into two class: One is called the direct method, which represents the solutions of given nonlinear PDE system as the sum of a polynomial in exponential solutions. It requires solving the recurrent coefficient relation or derivative relation for the terms of polynomial for computation closed. For example, tanh-function method [1–3], generalized hyperbolic-function method [12,13], Jacobi elliptic function expansion method [25] and extended Jacobi elliptic function expansion algorithm [26]. The second is called the indirect method, which consists of looking for the solutions of given nonlinear PDE system as a polynomial in a variable which satisfies a equation or equations (subequation). For example, Riccati equation [7–11] and Projective Riccati equations [27–29]. Further work about various extensions and improvement of tanh method need us to find the more general ansatz and the more general subequation.

Acknowledgement

The work is partially supported by the National Key Basic Research Project of China under the Grant No. 2004CB318000.

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