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New Complexiton Solutions of (1+1)-Dimensional Dispersive Long Wave Equation*

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Abstract *By means of two different Riccati equations with different parameters as subequation in the components of finite rational expansion method, new complexiton solutions for the (1+1)-dimensional dispersive long wave equation are successfully constructed, which include various combination of trigonometric periodic and hyperbolic function solutions, various combination of trigonometric periodic and rational function solutions, and various combination of hyperbolic and rational function solutions.*

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1 Introduction

In this paper, the (1+1)-dimensional dispersive long wave equation is investigated by use of the multiple Riccati equations rational expansion method presented with two different Riccati equations with different parameters and a new general transformation. The presented method is directly extended from the Riccati equation rational expansion method by us.[1] The key idea is that the solutions of two different Riccati equations with different parameters are used to replace the variables in the traditional finite rational expansion method[1] in the following forms:

$$U_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} \phi^{r_{j1}}(\xi)\psi^{r_{j2}}(\xi)}{(\mu_1\phi(\xi) + \mu_2\psi(\xi) + 1)^j}\,, \tag{1}$$

where $a^j_{r_{j1}r_{j2}}$, μ_1, and μ_2 ($r_{jn} = 1, 2, \ldots, j$; $n = 1, 2$; $i = 1, 2, \ldots$) are constants to be determined later and the new variables $\phi = \phi(\xi)$ and $\psi = \psi(\xi)$ satisfy two different Riccati equations, i.e.

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = h_1 + h_2\phi^2\,, \quad \frac{\mathrm{d}\psi}{\mathrm{d}\xi} = h_3 + h_4\psi^2\,, \tag{2}$$

where h_1, h_2, h_3 and h_4 are constants. Due to using two different Riccati equations with different parameters as subequation, we can easily see that when $h_1 \neq h_3$ or $h_2 \neq h_4$, ϕ and ψ satisfy the different Riccati equation, so hyperbolic functions and triangular functions can appear in one solution at the same time. We can successfully construct, if the given nonlinear evolution equation has this style of solutions, various combination of trigonometric periodic and hyperbolic function solutions, various combination of trigonometric periodic and rational function solutions, various combination of hyperbolic and rational function solutions for the given nonlinear evolution equations. These solutions have not been obtained by any other Riccati equation expansion methods or projective Riccati equation expansion methods.[2−5]

In Ref. [6], Lou presented and answered the problem "Are there any exact explicit multiple periodic wave solutions and periodic-solitary wave solutions for the nSG equation?" with help of the mapping relations among the sine-Gordon field equation and the cubic nonlinear Klein–Gordon fields. In Ref. [7], Ma found a novel class of explicit exact solutions to the Korteweg-de Vries equation through its bilinear form and defined the solutions as complexiton solutions. Such solutions possess singularities of combinations of trigonometric function waves and exponential function waves, which have different travelling speeds of new type. The above mentioned solutions of nonlinear evolution equations have a common character: combination of trigonometric function waves and exponential function waves. For unification and conciseness, so we call the solutions obtained by Lou, the solutions obtained by Ma and the solutions obtained by us in this paper as complexiton solutions.

Illustrative example is an application to the (1+1)-dimensional dispersive long wave equation, of the resulting complexiton solutions are exhibited, in the hope that they will lead to a deeper and more comprehensive understanding of the complex structures resulted from the nonlinearity of evolution equation, for example, the (1+1)-dimensional dispersive long wave equation.

2 Summary of Multiple Riccati Equations Rational Expansion Method

In the following we would like to outline the main steps of our method.

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Step 1 Given a system of polynomial nonlinear evolution equation with constant coefficients and some physical fields $u_i(x,y,t)$ in three variables x, y, t,

$$\Delta(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \ldots) = 0\,, \tag{3}$$

use the wave transformation

$$u_i(x,y,t) = U_i(\xi)\,, \quad \xi = k(x + ly + \lambda t)\,, \tag{4}$$

where k, l, and λ are constants to be determined later. Then the system (3) is reduced to a nonlinear ordinary differential system:

$$\Theta(U_i, U_i', U_i'', \ldots) = 0\,. \tag{5}$$

Step 2 We introduce a new ansätz in terms of finite rational formal expansion in the following forms:

$$U_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} \phi^{r_{j1}}(\xi)\psi^{r_{j2}}(\xi)}{(\mu_1\phi(\xi) + \mu_2\psi(\xi) + 1)^j}\,, \tag{6}$$

where $a^j_{r_{j1}r_{j2}}$, μ_1, and μ_2 ($r_{jn} = 1,2,\ldots,j$; $n = 1,2$; $i = 1,2,\ldots$) are constants to be determined later and the new variables $\phi = \phi(\xi)$ and $\psi = \psi(\xi)$ satisfy the Riccati equation, i.e.,

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = h_1 + h_2\phi^2\,, \quad \frac{\mathrm{d}\psi}{\mathrm{d}\xi} = h_3 + h_4\psi^2\,, \tag{7}$$

where h_1, h_2, h_3, and h_4 are constants.

Step 3 Determine the m_i in the rational formal polynomial solutions (6) by respectively balancing the highest nonlinear terms and the highest-order partial derivative terms in the given system equations (see Refs. [3] $\sim$ [5] for details), and then give the formal solutions.

Step 4 Substitute Eq. (6) into Eq. (5) along with Eq. (7) and then set all coefficients of $\phi^p(\xi)\psi^q(\xi)$, ($p = 0,1,2,\ldots; q = 0,1,2,\ldots$) of the resulting system's numerator to be zero to get an over-determined system of nonlinear algebraic equations with respect to k, μ_1, μ_2 and $a^j_{r_{j1}r_{j2}}$ ($r_{jn} = 1,2,\ldots,j$; $j = 1,2,\ldots,m_i$; $n = 1,2$; $i = 1,2,\ldots$).

Step 5 By solving the over-determined system of nonlinear algebraic equations by use of symbolic computation system *Maple*, we end up with the explicit expressions for k, μ_1, μ_2, and $a^j_{r_{j1}r_{j2}}$ ($r_{jn} = 1,2,\ldots,j$; $n = 1,2$; $i = 1,2,\ldots$).

Step 6 The general solutions of the Riccati equation (7)

$$\frac{\mathrm{d}F}{\mathrm{d}\xi} = R_1 + R_2F^2$$

are

1) when $R_1 = 1/2$ and $R_2 = -1/2$,

$$F(\xi) = \tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}\,(\xi)\,, \quad F(\xi) = \coth(\xi) \pm \mathrm{csch}\,(\xi)\,. \tag{8}$$

2) when $R_1 = R_2 = \pm 1/2$,

$$F(\xi) = \sec(\xi) \pm \tan(\xi)\,, \quad F(\xi) = \csc(\xi) \pm \cot(\xi)\,. \tag{9}$$

3) when $R_1 = 1$ and $R_2 = -1$,

$$F(\xi) = \tanh(\xi)\,, \quad F(\xi) = \coth(\xi)\,. \tag{10}$$

4) when $R_1 = R_2 = 1$,

$$F(\xi) = \tan(\xi)\,. \tag{11}$$

5) when $R_1 = R_2 = -1$,

$$F(\xi) = \cot(\xi)\,. \tag{12}$$

6) when $R_1 = 0$ and $R_2 \neq 0$,

$$F(\xi) = -\frac{1}{R_2\xi + c_0}\,. \tag{13}$$

According to system (4) and (6), the conclusions in *Step 5* and the general solutions (8) $\sim$ (13), we can obtain many types of complexiton solution of system (3) as following.

(i) When $h_1 = 1$, $h_2 = -1$, and $h_3 = h_4 = \pm 1$, we can get combining tanh (coth) and tan function solutions and combining tanh (coth) and sec function solutions:

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} \tanh^{r_{j1}}(\xi)\tan^{r_{j2}}(\xi)}{(\mu_1\tanh(\xi) + \mu_2\tan(\xi) + 1)^j}\,, \tag{14}$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} \tanh^{r_{j1}}(\xi)\sec^{r_{j2}}(\xi)}{(\mu_1\tanh(\xi) + \mu_2\sec(\xi) + 1)^j}\,, \tag{15}$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} \coth^{r_{j1}}(\xi)\tan^{r_{j2}}(\xi)}{(\mu_1\coth(\xi) + \mu_2\tan(\xi) + 1)^j}\,, \tag{16}$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} \coth^{r_{j1}}(\xi)\sec^{r_{j2}}(\xi)}{(\mu_1\coth(\xi) + \mu_2\sec(\xi) + 1)^j}\,. \tag{17}$$

(ii) When $h_1 = 1$, $h_2 = -1$, and $h_3 = h_4 = \pm 1/2$, we can get combining tanh (coth), sec, and tan function solutions and combining tanh (coth), csc, sec, cot function solutions:

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} \tanh^{r_{j1}}(\xi)(\sec(\xi) \pm \tan(\xi))^{r_{j2}}}{(\mu_1\tanh(\xi) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^j}\,, \tag{18}$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} \tanh^{r_{j1}}(\xi)(\csc(\xi) \pm \cot(\xi))^{r_{j2}}}{(\mu_1\tanh(\xi) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^j}\,, \tag{19}$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} \coth^{r_{j1}}(\xi)(\sec(\xi) \pm \tan(\xi))^{r_{j2}}}{(\mu_1\coth(\xi) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^j}\,, \tag{20}$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} \coth^{r_{j1}}(\xi)(\csc(\xi) \pm \cot(\xi))^{r_{j2}}}{(\mu_1 \coth(\xi) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^j}. \quad (21)$$

(iii) When $h_1 = 1$, $h_2 = -1$, $h_3 = 0$, and $h_4 \neq 0$, we can get combining tanh (coth) and rational function solutions:

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} \tanh^{r_{j1}}(\xi)\,[-1/(h_4\xi + c_0)]^{r_{j2}}}{(\mu_1 \tanh(\xi) - \mu_2/(h_4\xi + c_0) + 1)^j}, \quad (22)$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} \coth^{r_{j1}}(\xi)\,[-1/(h_4\xi + c_0)]^{r_{j2}}}{(\mu_1 \coth(\xi) - \mu_2/(h_4\xi + c_0) + 1)^j}. \quad (23)$$

(iv) When $h_1 = -1/2$, $h_2 = 1/2$, and $h_3 = h_4 = \pm 1$, we can get combining tanh, sech and tan (cot) function solutions and combining coth, csch and tan (cot) function solutions:

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} (\tanh(\xi) \mathrm{i}\,\mathrm{sech}(\xi))^{r_{j1}} \tan^{r_{j2}}(\xi)}{(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2 \tan(\xi) + 1)^j}, \quad (24)$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} (\coth(\xi) \pm \mathrm{csch}(\xi))^{r_{j1}} \tan^{r_{j2}}(\xi)}{(\mu_1(\coth(\xi) \pm \mathrm{csch}(\xi)) + \mu_2 \tan(\xi) + 1)^j}, \quad (25)$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} (\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))^{r_{j1}} \cot^{r_{j2}}(\xi)}{(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2 \cot(\xi) + 1)^j}, \quad (26)$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} (\coth(\xi) \pm \mathrm{csch}(\xi))^{r_{j1}} \cot^{r_{j2}}(\xi)}{(\mu_1(\coth(\xi) \pm \mathrm{csch}(\xi)) + \mu_2 \cot(\xi) + 1)^j}. \quad (27)$$

(v) When $h_1 = -1/2$, $h_2 = 1/2$, and $h_3 = h_4 = \pm 1/2$, we can get combining tanh, sech (coth, csch), sec and tan function solutions and combining tanh, sech (coth, csch), csc and cot function solutions:

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} (\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))^{r_{j1}} (\sec(\xi) \pm \tan(\xi))^{r_{j2}}}{(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^j}, \quad (28)$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} (\coth(\xi) \pm \mathrm{csch}(\xi))^{r_{j1}} (\sec(\xi) \pm \tan(\xi))^{r_{j2}}}{(\mu_1(\coth(\xi) \pm \mathrm{csch}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^j}, \quad (29)$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} (\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))^{r_{j1}} (\csc(\xi) \pm \cot(\xi))^{r_{j2}}}{(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)^j}, \quad (30)$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} (\coth(\xi) \pm \mathrm{csch}(\xi))^{r_{j1}} (\csc(\xi) \pm \cot(\xi))^{r_{j2}}}{(\mu_1(\coth(\xi) \pm \mathrm{csch}(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)^j}. \quad (31)$$

(vi) When $h_1 = -1/2$, $h_2 = 1/2$, $h_3 = 0$ and $h_4 \neq 0$, we can get combining tanh, sech (coth, csch), rational function solutions:

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} (\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))^{r_{j1}} [-1/(h_4\xi + c_0)]^{r_{j2}}}{(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) - \mu_2/(h_4\xi + c_0) + 1)^j}, \quad (32)$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} (\coth(\xi) \pm \mathrm{csch}(\xi))^{r_{j1}} [-1/(h_4\xi + c_0)]^{r_{j2}}}{(\mu_1(\coth(\xi) \pm \mathrm{csch}(\xi)) - \mu_2/(h_4\xi + c_0) + 1)^j}. \quad (33)$$

(vii) When $h_1 = h_2 = \pm 1$, $h_3 = 0$, and $h_4 \neq 0$, we can get combining tan (cot), rational function solutions:

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} \tan^{r_{j1}}(\xi)\,[-1/(h_4\xi + c_0)]^{r_{j2}}}{(\mu_1 \tan(\xi) - \mu_2/(h_4\xi + c_0) + 1)^j}, \quad (34)$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} \cot^{r_{j1}}(\xi)\,[-1/(h_4\xi + c_0)]^{r_{j2}}}{(\mu_1 \cot(\xi) - \mu_2/(h_4\xi + c_0) + 1)^j}\,, \tag{35}$$

(viii) When $h_1 = h_2 = \pm 1/2$, $h_3 = 0$, and $h_4 \neq 0$, we can get combining tan (cot), sec (csc), rational function solutions:

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} (\sec(\xi) \pm \tan(\xi))^{r_{j1}}\,[-1/(h_4\xi + c_0)]^{r_{j2}}}{(\mu_1(\sec(\xi) \pm \tan(\xi)) - \mu_2/(h_4\xi + c_0) + 1)^j}\,, \tag{36}$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} (\csc(\xi) \pm \cot(\xi))^{r_{j1}}\,[-1/(h_4\xi + c_0)]^{r_{j2}}}{(\mu_1(\csc(\xi) \pm \cot(\xi)) - \mu_2/(h_4\xi + c_0) + 1)^j}\,. \tag{37}$$

Remark 1 We can easily see that when $h_1 \neq h_3$ or $h_2 \neq h_4$, ϕ and ψ satisfy the different Riccati equations, so hyperbolic functions and triangular functions can appear in one solution at the same time. These solutions have not been obtained by any other Riccati equation expansion methods or projective Riccati equation expansion methods.

3 Complexiton Solutions of (1+1)-Dimensional Dispersive Long Wave Equation

Let us consider the (1+1)-dimensional dispersive long wave equation (DLWE),

$$v_t + vv_x + w_x = 0\,, \quad w_t + (wv)_x + \frac{1}{3}v_{xxx} = 0\,, \tag{38}$$

where w is the elevation of the water wave, v is the surface velocity of water along x-direction. The equation system (38) can be traced back to the works of Broer,[8] Kaup,[9] Jaulent–Miodek,[10] Martinez,[11] Kupershmidt,[12] etc. A good understanding of all the solutions of Eq. (38) is very helpful for coastal and civil engineers to apply the nonlinear water wave model in a harbor and coastal design. Therefore, finding more types of exact solutions of Eq. (38) is of fundamental interest in fluid dynamics. There are an amount of papers devoted to this equation.[13−17]

In order to get some families of rational form wave solutions to the (1 + 1)-dimensional DLWE, by considering the wave transformations $w(x,t) = W(\xi)$, $v(x,t) = V(\xi)$, and $\xi = k(x + \lambda t)$, we change Eq. (38) to the form

$$\lambda V' + VV' + W' = 0\,, \quad \lambda W' + (WV)' + \frac{1}{3}k^2 V''' = 0\,. \tag{39}$$

For the (1+1)-dimensional DLWE, by balancing the highest nonlinear terms and the highest order partial derivative terms in Eq. (39), We suppose that equation (39) has the following formal travelling wave solution:

$$V(\xi) = a_0 + \frac{a_1\phi + b_1\psi}{\mu_1\phi + \mu_2\psi + 1}\,, \quad W(\xi) = A_0 + \frac{A_1\phi + B_1\psi}{\mu_1\phi + \mu_2\psi + 1} + \frac{A_2\phi^2 + B_2\phi\psi + C_1\psi^2}{(\mu_1\phi + \mu_2\psi + 1)^2}\,, \tag{40}$$

where μ_1, μ_2, a_0, a_i, b_i, A_0, A_i, B_i, and C_1 $(i = 1, 2)$ are constants to be determined later and the new variables ϕ and ψ satisfy Eq. (7).

With the aid of *Maple*, substituting Eq. (40) along with Eq. (7) into Eq. (39) and setting the coefficients of these terms $\phi^i\psi^j$ to be zero yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, A_0, A_1, B_1 A_2, B_2, C_1, μ_1, μ_2, k, and λ.

By use of the *Maple* soft package "Charsets" by Dongming Wang, which is based on the Wu-elimination method, solving the over-determined algebraic equations, we get the following results. Note that since the solutions obtained here are so many, we just list some new solutions for the (1 + 1)-dimensional DLWE to illustrate the efficiency of our method.

$$B_1 = \frac{b_1 A_1}{a_1}\,, \quad A_2 = -\frac{a_1^2}{2}\,, \quad B_2 = -a_1 b_1\,, \quad \lambda = -\frac{A_1 + a_0 a_1}{a_1}\,, \tag{41}$$

where a_0, A_0, a_1, b_1, A_1, C_1, and k are arbitrary constants.

According to Eqs. (40) and (41) and the general solutions of Eq. (7) listed in Sec. 2, we will obtain the following wave solutions for (1+1)-dimensional DLWE.

Family 1 When $h_1 = 1$, $h_2 = -1$, and $h_2 = h_4 = \pm 1$, then we can get combining tanh and tan function solution:

$$\begin{aligned} v_1 &= a_0 + \frac{a_1 \tanh(\xi)}{\mu_1 \tanh(\xi) + \mu_2 \tan(\xi) + 1} + \frac{b_1 \tan(\xi)}{\mu_1 \tanh(\xi) + \mu_2 \tan(\xi) + 1}\,, \\ w_1 &= A_0 + \frac{A_1 \tanh(\xi)}{\mu_1 \tanh(\xi) + \mu_2 \tan(\xi) + 1} + \frac{b_1 A_1 \tan(\xi)}{a_1(\mu_1 \tanh(\xi) + \mu_2 \tan(\xi) + 1)} - \frac{a_1^2 \tanh^2(\xi)}{2(\mu_1 \tanh(\xi) + \mu_2 \tan(\xi) + 1)^2} \\ &\quad - \frac{a_1 b_1 \tanh(\xi)\tan(\xi)}{(\mu_1 \tanh(\xi) + \mu_2 \tan(\xi) + 1)^2} - \frac{b_1^2 \tan^2(\xi)}{2(\mu_1 \tanh(\xi) + \mu_2 \tan(\xi) + 1)^2}\,, \end{aligned} \tag{42}$$

where $\xi = k(x + \lambda t)$, $\lambda = -(A_1 + a_0 a_1)/a_1$, a_0, A_0, a_1, b_1, A_1, C_1, and k are arbitrary constants.

Family 2 When $h_1 = 1$, $h_2 = -1$, and $h_3 = h_4 = \pm 1/2$, then we can get combining tanh, sec and tan function solution:

$$\begin{aligned} v_2 &= a_0 + \frac{a_1 \tanh(\xi)}{\mu_1 \tanh(\xi) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1} + \frac{b_1(\sec(\xi) \pm \tan(\xi))}{\mu_1 \tanh(\xi) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1}\,, \\ w_2 &= A_0 + \frac{A_1 \tanh(\xi)}{\mu_1 \tanh(\xi) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1} + \frac{b_1 A_1(\sec(\xi) \pm \tan(\xi))}{a_1(\mu_1 \tanh(\xi) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)} \\ &\quad - \frac{a_1^2 \tanh^2(\xi)}{2(\mu_1 \tanh(\xi) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^2} - \frac{a_1 b_1 \tanh(\xi)(\sec(\xi) \pm \tan(\xi))}{(\mu_1 \tanh(\xi) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^2} \\ &\quad - \frac{b_1^2(\sec(\xi) \pm \tan(\xi))^2}{2(\mu_1 \tanh(\xi) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^2}\,, \end{aligned} \tag{43}$$

where $\xi = k(x + \lambda t)$, $\lambda = -(A_1 + a_0 a_1)/a_1$, a_0, A_0, a_1, b_1, A_1, C_1, and k are arbitrary constants.

Family 3 When $h_1 = 1$, $h_2 = -1$, $h_3 = 0$, and $h_4 \neq 0$, then we can get combining tanh and rational function solution:

$$\begin{aligned} v_3 &= a_0 + \frac{a_1 \tanh(\xi)(h_4\xi + c_0) - b_1}{\mu_1 \tanh(\xi)(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0}\,, \\ w_3 &= A_0 + \frac{A_1 a_1 \tanh(\xi)(h_4\xi + c_0) + b_1 A_1}{a_1(\mu_1 \tanh(\xi)(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0)} - \frac{a_1^2 \tanh^2(\xi)(h_4\xi + c_0)^2}{2(\mu_1 \tanh(\xi)(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0)^2} \\ &\quad + \frac{a_1 b_1 \tanh(\xi)(h_4\xi + c_0)}{(\mu_1 \tanh(\xi)(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0)^2} - \frac{b_1^2}{2(\mu_1 \tanh(\xi)(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0)^2}\,, \end{aligned} \tag{44}$$

where $\xi = k(x + \lambda t)$, $\lambda = -(A_1 + a_0 a_1)/a_1$, $h_4 \neq 0$, c_0, a_0, A_0, a_1, b_1, A_1, C_1, and k are arbitrary constants.

Family 4 When $h_1 = -1/2$, $h_2 = 1/2$ and $h_3 = h_4 = \pm 1$, then we can get combining tanh, sech and tan function solution:

$$\begin{aligned} v_4 &= a_0 + \frac{a_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))}{\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2 \tan(\xi) + 1} + \frac{b_1 \tan(\xi)}{\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2 \tan(\xi) + 1}\,, \\ w_4 &= A_0 + \frac{A_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))}{\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2 \tan(\xi) + 1} + \frac{b_1 A_1 \tan(\xi)}{a_1(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2 \tan(\xi) + 1)} \\ &\quad - \frac{{a_1}^2(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))^2}{2(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2 \tan(\xi) + 1)^2} - \frac{a_1 b_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) \tan(\xi)}{(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2 \tan(\xi) + 1)^2} \\ &\quad - \frac{{b_1}^2 \tan^2(\xi)}{2(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2 \tan(\xi) + 1)^2}\,, \end{aligned} \tag{45}$$

where $\xi = k(x + \lambda t)$, $\lambda = -(A_1 + a_0 a_1)/a_1$, a_0, A_0, a_1, b_1, A_1, C_1, and k are arbitrary constants.

Family 5 When $h_1 = -1/2$, $h_2 = 1/2$, and $h_3 = h_4 = \pm 1/2$, then we combining tanh, sech, sec and tan function solution:

$$\begin{aligned} v_5 &= a_0 + \frac{a_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))}{\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1} + \frac{b_1(\sec(\xi) \pm \tan(\xi))}{\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1}\,, \\ w_5 &= A_0 + \frac{A_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))}{\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1} + \frac{b_1 A_1(\sec(\xi) \pm \tan(\xi))}{a_1(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)} \\ &\quad - \frac{a_1^2(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))^2}{2(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^2} - \frac{a_1 b_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))(\sec(\xi) \pm \tan(\xi))}{(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^2} \\ &\quad - \frac{b_1^2(\sec(\xi) \pm \tan(\xi))^2}{2(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^2}\,, \end{aligned} \tag{46}$$

where $\xi = k(x + \lambda t)$, $\lambda = -(A_1 + a_0 a_1)/a_1$, a_0, A_0, a_1, b_1, A_1, C_1, and k are arbitrary constants.

Family 6 When $h_1 = -1/2$, $h_2 = 1/2$, $h_3 = 0$, and $h_4 \neq 0$, then we can get combining tanh, sech, rational function solution:

$$\begin{aligned} v_6 &= a_0 + \frac{a_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))(h_4\xi + c_0) - b_1}{(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0)}\,, \\ w_6 &= A_0 + \frac{A_1 a_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))(h_4\xi + c_0) - b_1 A_1}{a_1(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0)} \end{aligned}$$

$$- \frac{{a_1}^2 (\tanh(\xi) \pm \mathrm{isech}(\xi))^2 (h_4\xi + c_0)^2}{2(\mu_1 (\tanh(\xi) \pm \mathrm{isech}(\xi))(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0)^2}$$
$$+ \frac{a_1 b_1 (\tanh(\xi) \pm \mathrm{isech}(\xi))(h_4\xi + c_0)}{(\mu_1 (\tanh(\xi) \pm \mathrm{isech}(\xi))(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0)^2}$$
$$- \frac{b_1^2}{2(\mu_1 (\tanh(\xi) \pm \mathrm{isech}(\xi))(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0)^2}, \tag{47}$$

where $\xi = k(x + \lambda t)$, $\lambda = -(A_1 + a_0 a_1)/a_1$, $h_4 \neq 0$, c_0, a_0, A_0, a_1, b_1, A_1, C_1, and k are arbitrary constants.

Family 7 When $h_1 = h_2 = \pm 1$, $h_3 = 0$, and $h_4 \neq 0$, then we can get combining tan and rational function solution:

$$v_7 = a_0 + \frac{a_1 \tan(\xi) - b_1}{\mu_1 \tan(\xi)(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0},$$
$$w_7 = A_0 + \frac{A_1 a_1 \tan(\xi)(h_4\xi + c_0) - b_1 A_1}{a_1(\mu_1 \tan(\xi)(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0)} - \frac{a_1^2 \tan^2(\xi)(h_4\xi + c_0)^2}{2(\mu_1 \tan(\xi)(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0)^2}$$
$$+ \frac{a_1 b_1 \tan(\xi)(h_4\xi + c_0)}{(\mu_1 \tan(\xi)(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0)^2} - \frac{b_1^2}{2(\mu_1 \tan(\xi)(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0)^2}, \tag{48}$$

where $\xi = k(x + \lambda t)$, $\lambda = -(A_1 + a_0 a_1)/a_1$, $h_4 \neq 0$, c_0, a_0, A_0, a_1, b_1, A_1, C_1, and k are arbitrary constants.

Family 8 When $h_1 = h_2 = \pm 1/2$, $h_3 = 0$, and $h_4 \neq 0$, then we can get combining tan, sec, rational function solution:

$$v_8 = a_0 + \frac{a_1(\sec(\xi) \pm \tan(\xi)) - b_1}{\mu_1(\sec(\xi) \pm \tan(\xi))(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0},$$
$$w_8 = A_0 + \frac{A_1 a_1(\sec(\xi) \pm \tan(\xi))(h_4\xi + c_0) - b_1 A_1}{a_1(\mu_1(\sec(\xi) \pm \tan(\xi))(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0)}$$
$$- \frac{a_1^2(\sec(\xi) \pm \tan(\xi))^2(h_4\xi + c_0)^2}{2(\mu_1(\sec(\xi) \pm \tan(\xi))(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0)^2}$$
$$+ \frac{a_1 b_1(\sec(\xi) \pm \tan(\xi))(h_4\xi + c_0)}{(\mu_1(\sec(\xi) \pm \tan(\xi))(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0)^2}$$
$$- \frac{b_1^2}{2(\mu_1(\sec(\xi) \pm \tan(\xi))(h_4\xi + c_0) - \mu_2 + h_4\xi + c_0)^2}, \tag{49}$$

where $\xi = k(x + \lambda t)$, $\lambda = -(A_1 + a_0 a_1)/a_1$, $h_4 \neq 0$, c_0, a_0, A_0, a_1, b_1, A_1, C_1, and k are arbitrary constants.

At the same time, we can also get some new solutions which are not the complexiton solutions and cannot be obtained by other tanh method, such as

Family 9 When $h_1 = h_3 = 1/2$ and $h_2 = h_4 = -1/2$, then we obtain the following solutions

$$v_9 = a_0 + \frac{a_1(\tanh(\xi) \pm \mathrm{isech}(\xi)) + b_1(\coth(\xi) \pm \mathrm{csch}(\xi))}{\mu_1(\tanh(\xi) \pm \mathrm{isech}(\xi)) + \mu_2(\coth(\xi) \pm \mathrm{csch}(\xi)) + 1},$$
$$w_9 = A_0 + \frac{A_1 a_1(\tanh(\xi) \pm \mathrm{isech}(\xi)) + b_1 A_1(\coth(\xi) \pm \mathrm{csch}(\xi))}{a_1(\mu_1(\tanh(\xi) \pm \mathrm{isech}(\xi)) + \mu_2(\coth(\xi) \pm \mathrm{csch}(\xi)) + 1)}$$
$$- \frac{a_1^2(\tanh(\xi) \pm \mathrm{isech}(\xi))^2}{2(\mu_1(\tanh(\xi) \pm \mathrm{isech}(\xi)) + \mu_2(\coth(\xi) \pm \mathrm{csch}(\xi)) + 1)^2}$$
$$- \frac{a_1 b_1(\tanh(\xi) \pm \mathrm{isech}(\xi))(\coth(\xi) \pm \mathrm{csch}(\xi))}{(\mu_1(\tanh(\xi) \pm \mathrm{isech}(\xi)) + \mu_2(\coth(\xi) \pm \mathrm{csch}(\xi)) + 1)^2}$$
$$- \frac{b_1^2(\coth(\xi) \pm \mathrm{csch}(\xi))^2}{(\mu_1(\tanh(\xi) \pm \mathrm{isech}(\xi)) + \mu_2(\coth(\xi) \pm \mathrm{csch}(\xi)) + 1)^2}, \tag{50}$$

where $\xi = k(x + \lambda t)$, $\lambda = -(A_1 + a_0 a_1)/a_1$, a_0, A_0, a_1, b_1, A_1, C_1, and k are arbitrary constants.

Family 10 When $h_1 = h_2 = \pm 1/2$ and $h_3 = h_4 = \pm 1/2$, then we obtain the following solutions

$$v_{10} = a_0 + \frac{a_1(\sec(\xi) \pm \tan(\xi)) + b_1(\csc(\xi) \pm \cot(\xi))}{\mu_1(\sec(\xi) \pm \tan(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1}$$
$$w_{10} = A_0 + \frac{A_1 a_1(\sec(\xi) \pm \tan(\xi)) + b_1 A_1(\csc(\xi) \pm \cot(\xi))}{a_1(\mu_1(\sec(\xi) \pm \tan(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)}$$

$$
- \frac{a_1^2(\sec(\xi) \pm \tan(\xi))^2}{2(\mu_1(\sec(\xi) \pm \tan(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)^2}
- \frac{a_1 b_1(\sec(\xi) \pm \tan(\xi))(\csc(\xi) \pm \cot(\xi))}{(\mu_1(\sec(\xi) \pm \tan(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)^2}
- \frac{b_1^2(\csc(\xi) \pm \cot(\xi))^2}{2(\mu_1(\sec(\xi) \pm \tan(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)^2}, \quad (51)
$$

where $\xi = k(x + \lambda t)$, $\lambda = -(A_1 + a_0 a_1)/a_1$, a_0, A_0, a_1, b_1, A_1, C_1, and k are arbitrary constants.

4 Discussions and Conclusion

Although our method cannot recover all complexiton solutions obtained by Ma's method and Lou's method, other new types of complexiton solutions cannot be found by Ma's method and Lou's method. In particular, our method is a unified straightforward and pure algebraic algorithm to integrable equations and nonintegrable equations, which is implemented in a computer algebraic system. The complexiton solutions presented above do not satisfy the definition by Ma,[7] but one of the basic character of the solutions by Ma[7] and the solutions by us are the same, i.e. possessing various combination of trigonometric periodic and hyperbolic function solutions, various combination of trigonometric periodic and rational function solutions, various combination of hyperbolic and rational function solutions. Of course our method can also be extended to other integrable systems and nonintegrable systems. Our further work is finding different travelling wave complexiton solutions by this direct and unified method for nonlinear evolution equations.

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