New Complexiton Solutions of (1+1)-Dimensional Dispersive Long Wave Equation∗

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Abstract By means of two different Riccati equations with different parameters as subequation in the components of finite rational expansion method, new complexiton solutions for the (1+1)-dimensional dispersive long wave equation are successfully constructed, which include various combination of trigonometric periodic and hyperbolic function solutions, various combination of trigonometric periodic and rational function solutions, and various combination of hyperbolic and rational function solutions.

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1 Introduction

In this paper, the (1+1)-dimensional dispersive long wave equation is investigated by use of the multiple Riccati equations rational expansion method presented with two different Riccati equations with different parameters and a new general transformation. The presented method is directly extended from the Riccati equation rational expansion method by us.[1] The key idea is that the solutions of two different Riccati equations with different parameters are used to replace the variables in the traditional finite rational expansion method[1] in the following forms:

\[ U_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \frac{a^i_{j,r_1,r_2}}{(\mu_1 \phi(\xi) + \mu_2 \psi(\xi) + 1)^j} , \tag{1} \]

where \(a^i_{j,r_1,r_2}, \mu_1, \) and \(\mu_2\) (\(r_1, r_2 = 1, 2, \ldots, j; n = 1, 2; i = 1, 2, \ldots\)) are constants to be determined later and the new variables \(\phi = \phi(\xi)\) and \(\psi = \psi(\xi)\) satisfy two different Riccati equations, i.e.

\[ \frac{d\phi}{d\xi} = h_1 + h_2 \phi^2, \quad \frac{d\psi}{d\xi} = h_3 + h_4 \psi^2, \tag{2} \]

where \(h_1, h_2, h_3 \) and \(h_4\) are constants. Due to using two different Riccati equations with different parameters as subequation, we can easily see that when \(h_1 \neq h_3\) or \(h_2 \neq h_4\), \(\phi\) and \(\psi\) satisfy the different Riccati equation, so hyperbolic functions and triangular functions can appear in one solution at the same time. We can successfully construct, if the given nonlinear evolution equation has this style of solutions, various combination of trigonometric periodic and hyperbolic function solutions, various combination of trigonometric periodic and rational function solutions, various combination of hyperbolic and rational function solutions for the given nonlinear evolution equations. These solutions have not been obtained by any other Riccati equation expansion methods or projective Riccati equation expansion methods.[2−5]

In Ref. [6], Lou presented and answered the problem “Are there any exact explicit multiple periodic wave solutions and periodic-solitary wave solutions for the nSG equation?” with help of the mapping relations among the sine-Gordon field equation and the cubic nonlinear Klein−Gordon fields. In Ref. [7], Ma found a novel class of explicit exact solutions to the Korteweg−de Vries equation through its bilinear form and defined the solutions as complexiton solutions. Such solutions possess singularities of combinations of trigonometric function waves and exponential function waves, which have different travelling speeds of new type. The above mentioned solutions of nonlinear evolution equations have a common character: combination of trigonometric function waves and exponential function waves. For unification and conciseness, so we call the solutions obtained by Lou, the solutions obtained by Ma and the solutions obtained by us in this paper as complexiton solutions.

Illustrative example is an application to the (1+1)-dimensional dispersive long wave equation, of the resulting complexiton solutions are exhibited, in the hope that they will lead to a deeper and more comprehensive understanding of the complex structures resulted from the nonlinearity of evolution equation, for example, the (1+1)-dimensional dispersive long wave equation.

2 Summary of Multiple Riccati Equations Rational Expansion Method

In the following we would like to outline the main steps of our method.
Step 1 Given a system of polynomial nonlinear evolution equations with constant coefficients and some physical fields \( u_i(x, y, t) \) in three variables \( x, y, t \),

\[
\Delta(u_i, u_{ix}, u_{iyy}, u_{iyt}, u_{ixy}, u_{iyy}, \ldots) = 0,
\]

(3)

use the wave transformation

\[
u_i(x, y, t) = U_i(\xi), \quad \xi = k(x + ly + \lambda t),
\]

(4)

where \( k, l, \) and \( \lambda \) are constants to be determined later. Then the system (3) is reduced to a nonlinear ordinary differential system:

\[
\Theta(U_i, U'_i, U''_i, \ldots) = 0.
\]

(5)

Step 2 We introduce a new ansatz in terms of finite rational formal expansion in the following forms:

\[
U_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} a_{i1r_1, i2r_2 = j} (\phi(\xi) + r_1 \psi(\xi)) (\xi)^{r_1, r_2}(\xi)
\]

(6)

where \( a_{i1r_1, i2r_2 = j} \), \( \mu_1 \), and \( \mu_2 \) \((r_n = 1, 2, \ldots; j = 1, 2; i = 1, 2, \ldots)\) are constants to be determined later and the new variables \( \phi = \phi(\xi) \) and \( \psi = \psi(\xi) \) satisfy the Riccati equation, i.e.,

\[
\frac{d\phi}{d\xi} = h_1 + h_2 \phi^2, \quad \frac{d\psi}{d\xi} = h_3 + h_4 \phi^2,
\]

(7)

where \( h_1, h_2, h_3, \) and \( h_4 \) are constants.

Step 3 Determine the \( m_i \) in the rational formal polynomial solutions (6) by respectively balancing the highest nonlinear terms and the highest-order partial derivative terms in the given system equations (see Refs. [3] ∼ [5] for details), and then form the general solutions.

Step 4 Substitute Eq. (6) into Eq. (5) along with Eq. (7) and then set all coefficients of \( \phi^p(\xi)\psi^q(\xi), \quad (p = 0, 1, 2, \ldots; q = 0, 1, 2, \ldots) \) of the resulting system's numerator to be zero to get an over-determined system of nonlinear algebraic equations with respect to \( k, \mu_1, \mu_2 \) and \( a_{i1r_1, i2r_2 = j} \) \((r_n = 1, 2, \ldots; j = 1, 2, \ldots; m_i = 1, 2; i = 1, 2, \ldots)\).

Step 5 By solving the over-determined system of nonlinear algebraic equations by use of symbolic computation system Maple, we end up with the explicit expressions for \( k, \mu_1, \mu_2 \), and \( a_{i1r_1, i2r_2 = j} \) \((r_n = 1, 2, \ldots; j = 1, 2, \ldots; m_i = 1, 2; i = 1, 2, \ldots)\).

Step 6 The general solutions of the Riccati equation (7)

\[
\frac{dF}{d\xi} = R_1 + R_2 F^2
\]

are

1) when \( R_1 = 1/2 \) and \( R_2 = -1/2 \),

\[
F(\xi) = \tanh(\xi) \pm i \text{sech}(\xi),
\]

\[
F(\xi) = \coth(\xi) \pm \text{csch}(\xi).
\]

(8)

2) when \( R_1 = R_2 = \pm 1/2 \),

\[
F(\xi) = \sec(\xi) \pm \tan(\xi), \quad F(\xi) = \csc(\xi) \pm \cot(\xi).
\]

(9)

3) when \( R_1 = 1 \) and \( R_2 = -1 \),

\[
F(\xi) = \tanh(\xi), \quad F(\xi) = \coth(\xi).
\]

(10)

4) when \( R_1 = R_2 = 1 \),

\[
F(\xi) = \tan(\xi).
\]

(11)

5) when \( R_1 = R_2 = -1 \),

\[
F(\xi) = \cot(\xi).
\]

(12)

6) when \( R_1 = 0 \) and \( R_2 \neq 0 \),

\[
F(\xi) = -\frac{1}{R_2 \xi + c_0}.
\]

(13)

According to system (4) and (6), the conclusions in Step 5 and the general solutions (8) ∼ (13), we can obtain many types of complexiton solution of system (3) as following.

(i) When \( h_1 = 1, h_2 = -1, \) and \( h_3 = h_4 = \pm 1, \) we can get combining tanh (coth) and tanh function solutions and combining tanh (coth) and sec function solutions:

\[
u_i = a_{i0} + \sum_{j=1}^{m_i} a_{i1r_1, i2r_2 = j} (\mu_1 \tanh(\xi) + \mu_2 \tan(\xi) + 1)^{r_1, r_2} \tanh(\xi)^{r_1, r_2} (\xi)\tanh(\xi)^{r_1, r_2} (\xi),
\]

(14)

\[
u_i = a_{i0} + \sum_{j=1}^{m_i} a_{i1r_1, i2r_2 = j} (\mu_1 \tanh(\xi) + \mu_2 \sec(\xi) + 1)^{r_1, r_2} \tanh(\xi)^{r_1, r_2} (\xi)\sec(\xi)^{r_1, r_2} (\xi),
\]

(15)

\[
u_i = a_{i0} + \sum_{j=1}^{m_i} a_{i1r_1, i2r_2 = j} (\mu_1 \coth(\xi) + \mu_2 \tan(\xi) + 1)^{r_1, r_2} \coth(\xi)^{r_1, r_2} (\xi)\tan(\xi)^{r_1, r_2} (\xi),
\]

(16)

\[
u_i = a_{i0} + \sum_{j=1}^{m_i} a_{i1r_1, i2r_2 = j} (\mu_1 \coth(\xi) + \mu_2 \sec(\xi) + 1)^{r_1, r_2} \coth(\xi)^{r_1, r_2} (\xi)\sec(\xi)^{r_1, r_2} (\xi),
\]

(17)

(ii) When \( h_1 = 1, h_2 = -1, \) and \( h_3 = h_4 = \pm 1/2, \) we can get combining tanh (coth), sec, and tan function solutions and combining tanh (coth), csc, sec, cot function solutions:

\[
u_i = a_{i0} + \sum_{j=1}^{m_i} a_{i1r_1, i2r_2 = j} (\mu_1 \tanh(\xi) + \mu_2 \sec(\xi) \pm \tan(\xi) + 1)^{r_1, r_2} \tanh(\xi)^{r_1, r_2} (\xi)\sec(\xi)^{r_1, r_2} (\xi),
\]

(18)

\[
u_i = a_{i0} + \sum_{j=1}^{m_i} a_{i1r_1, i2r_2 = j} (\mu_1 \tanh(\xi) + \mu_2 \sec(\xi) \pm \tan(\xi) + 1)^{r_1, r_2} \tanh(\xi)^{r_1, r_2} (\xi)\sec(\xi)^{r_1, r_2} (\xi),
\]

(19)

\[
u_i = a_{i0} + \sum_{j=1}^{m_i} a_{i1r_1, i2r_2 = j} (\mu_1 \coth(\xi) + \mu_2 \sec(\xi) \pm \tan(\xi) + 1)^{r_1, r_2} \coth(\xi)^{r_1, r_2} (\xi)\sec(\xi)^{r_1, r_2} (\xi),
\]

(20)
(ii) When \( h_1 = 1, h_2 = -1, h_3 = 0 \), and \( h_4 \neq 0 \), we can get combining \( \tanh(\csc, \coth) \) and rational function solutions:

\[
u_i = a_0 + \sum_{j=1}^{m_i} \frac{a^2_{r_1 r_2}}{(\mu_1 \coth(\xi) + \mu_2 (\sec(\xi) \pm \tan(\xi)) + 1)^j}, \tag{21}
\]

\[
u_i = a_0 + \sum_{j=1}^{m_i} \frac{a^2_{r_1 r_2}}{(\mu_1 \coth(\xi) + \mu_2 (\sec(\xi) \pm \tan(\xi)) + 1)^j}, \tag{22}
\]

\[
u_i = a_0 + \sum_{j=1}^{m_i} \frac{a^2_{r_1 r_2}}{(\mu_1 \coth(\xi) + \mu_2 (\sec(\xi) \pm \tan(\xi)) + 1)^j}. \tag{23}
\]

(iv) When \( h_1 = -1/2, h_2 = 1/2, h_3 = h_4 = \pm 1 \), we can get combining \( \tanh, \sec \) and \( \tan, \cot \) function solutions:

\[
u_i = a_0 + \sum_{j=1}^{m_i} \frac{a^2_{r_1 r_2}}{(\mu_1 \coth(\xi) \pm \csc(\xi) \pm \sec(\xi)) + 1)^j}, \tag{24}
\]

\[
u_i = a_0 + \sum_{j=1}^{m_i} \frac{a^2_{r_1 r_2}}{(\mu_1 \coth(\xi) \pm \csc(\xi) \pm \sec(\xi)) + 1)^j}, \tag{25}
\]

\[
u_i = a_0 + \sum_{j=1}^{m_i} \frac{a^2_{r_1 r_2}}{(\mu_1 \coth(\xi) \pm \csc(\xi) \pm \sec(\xi)) + 1)^j}. \tag{26}
\]

(v) When \( h_1 = -1/2, h_2 = 1/2, h_3 = h_4 = \pm 1/2 \), we can get combining \( \tanh, \sec \) (coth, csc), \sec and \( \tan, \cot \) function solutions:

\[
u_i = a_0 + \sum_{j=1}^{m_i} \frac{a^2_{r_1 r_2}}{(\mu_1 \coth(\xi) \pm \csc(\xi) \pm \sec(\xi)) + 1)^j}, \tag{27}
\]

\[
u_i = a_0 + \sum_{j=1}^{m_i} \frac{a^2_{r_1 r_2}}{(\mu_1 \coth(\xi) \pm \csc(\xi) \pm \sec(\xi)) + 1)^j}, \tag{28}
\]

\[
u_i = a_0 + \sum_{j=1}^{m_i} \frac{a^2_{r_1 r_2}}{(\mu_1 \coth(\xi) \pm \csc(\xi) \pm \sec(\xi)) + 1)^j}. \tag{29}
\]

\[
u_i = a_0 + \sum_{j=1}^{m_i} \frac{a^2_{r_1 r_2}}{(\mu_1 \coth(\xi) \pm \csc(\xi) \pm \sec(\xi)) + 1)^j}, \tag{30}
\]

\[
u_i = a_0 + \sum_{j=1}^{m_i} \frac{a^2_{r_1 r_2}}{(\mu_1 \coth(\xi) \pm \csc(\xi) \pm \sec(\xi)) + 1)^j}. \tag{31}
\]

(vi) When \( h_1 = -1/2, h_2 = 1/2, h_3 = 0 \) and \( h_4 \neq 0 \), we can get combining \( \tanh, \sec \) (coth, csc), rational function solutions:

\[
u_i = a_0 + \sum_{j=1}^{m_i} \frac{a^2_{r_1 r_2}}{(\mu_1 \coth(\xi) \pm \csc(\xi) \pm \sec(\xi)) + 1)^j}, \tag{32}
\]

\[
u_i = a_0 + \sum_{j=1}^{m_i} \frac{a^2_{r_1 r_2}}{(\mu_1 \coth(\xi) \pm \csc(\xi) \pm \sec(\xi)) + 1)^j}. \tag{33}
\]

(vii) When \( h_1 = h_2 = \pm 1, h_3 = 0 \), and \( h_4 \neq 0 \), we can get combining \( \tan \) (cot), rational function solutions:

\[
u_i = a_0 + \sum_{j=1}^{m_i} \frac{a^2_{r_1 r_2}}{(\mu_1 \coth(\xi) \pm \csc(\xi) \pm \sec(\xi)) + 1)^j}. \tag{34}
\]
wave solutions for (1+1)-dimensional DLWE. where \(a\) can be traced back to the works of Broer, fundamental interest in fluid dynamics. There are an amount of papers devoted to this equation. water wave model in a harbor and coastal design. Therefore, finding more types of exact solutions of Eq. (38) is of good understanding of all the solutions of Eq. (38) is very helpful for coastal and civil engineers to apply the nonlinear solutions:

\[
u = \frac{\sum_{j=1}^{m} \frac{a_{r_{1}r_{2}} \cot r_{1} (\xi) \left[-1 / (h_{4} \xi + c_{0})\right]^{r_{2}}}{(\mu_{1} \cot (\xi) - \mu_{2} / (h_{4} \xi + c_{0}) + 1)^{j}}}{(\mu_{1} \cot (\xi) - \mu_{2} / (h_{4} \xi + c_{0}) + 1)^{j}},
\]

(viii) When \(h_{1} = h_{2} = \pm 1 / 2, h_{3} = 0,\) and \(h_{4} \neq 0,\) we can get combining tan (cot), sec (esc), rational function solutions:

\[
u = \frac{\sum_{j=1}^{m} \frac{a_{r_{1}r_{2}} \sec (\xi) \left(\pm \tan \left(\frac{\xi}{\mu_{1}}\right) - \mu_{2} / (h_{4} \xi + c_{0}) + 1\right)^{j}}{(\mu_{1} (\sec (\xi) + \tan (\xi) - \mu_{2} / (h_{4} \xi + c_{0}) + 1)^{j}}}{(\mu_{1} (\sec (\xi) + \tan (\xi) - \mu_{2} / (h_{4} \xi + c_{0}) + 1)^{j}}}
\]

\[
u = \frac{\sum_{j=1}^{m} \frac{a_{r_{1}r_{2}} \csc (\xi) \left(\pm \cot (\xi)\right)^{j}}{(\mu_{1} \csc (\xi) - \mu_{2} / (h_{4} \xi + c_{0}) + 1)^{j}}}{(\mu_{1} \csc (\xi) - \mu_{2} / (h_{4} \xi + c_{0}) + 1)^{j}}}
\]

Remark 1 We can easily see that when \(h_{1} \neq h_{3}\) or \(h_{2} \neq h_{4},\) \(\phi\) and \(\psi\) satisfy the different Riccati equations, so hyperbolic functions and triangular functions can appear in one solution at the same time. These solutions have not been obtained by any other Riccati equation expansion methods or projective Riccati equation expansion methods.

3 Complexiton Solutions of (1+1)-Dimensional Dispersive Long Wave Equation

Let us consider the (1+1)-dimensional dispersive long wave equation (DLWE),

\[
(38)
\end{equation}
\]

where \(v\) is the elevation of the water wave, \(v\) is the surface velocity of water along \(x\)-direction. The equation system (38) can be traced back to the works of Broer, Kaup, Jaulent–Miodek, Martinez, Kupershmidt, etc. A good understanding of all the solutions of Eq. (38) is very helpful for coastal and civil engineers to apply the nonlinear water wave model in a harbor and coastal design. Therefore, finding more types of exact solutions of Eq. (38) is of fundamental interest in fluid dynamics. There are an amount of papers devoted to this equation. In order to get some families of rational form wave solutions to the (1+1)-dimensional DLWE, by considering the wave transformations \(w(x, t) = W(\xi), u(x, t) = V(\xi),\) where \(\xi = k(x + \lambda t),\) we change Eq. (38) to the form

\[
(39)
\end{equation}
\]

For the (1+1)-dimensional DLWE, by balancing the highest nonlinear terms and the highest order partial derivative terms in Eq. (39), We suppose that equation (39) has the following formal travelling wave solution:

\[
V(\xi) = a_{0} + \frac{a_{1} \phi + b_{1} \psi}{\mu_{1} \phi + \mu_{2} \psi + 1}, \quad W(\xi) = A_{0} + \frac{A_{1} \phi + B_{1} \psi}{\mu_{1} \phi + \mu_{2} \psi + 1} + \frac{A_{2} \phi^{2} + B_{2} \phi \psi + C_{1} \psi^{2}}{\mu_{1} \phi + \mu_{2} \psi + 1}\]

(40)

where \(\mu_{1}, \mu_{2}, a_{0}, a_{1}, b_{1}, A_{0}, A_{1}, B_{1}, A_{2}, C_{1}, \mu_{1}, \mu_{2}, k,\) and \(\lambda\).

By use of the Maple soft package “Charsets” by Dongming Wang, which is based on the Wu-elimination method, solving the over-determined algebraic equations, we get the following results. Note that since the solutions obtained here are so many, we just list some new solutions for the (1+1)-dimensional DLWE to illustrate the efficiency of our method.

\[
B_{1} = \frac{b_{1} A_{1}}{a_{1}}, \quad A_{2} = - \frac{a_{1}^{2}}{2}, \quad B_{2} = -a_{1} b_{1}, \quad \lambda = - \frac{A_{1} + a_{0} a_{1}}{a_{1}}.
\]

(41)

where \(a_{0}, a_{1}, b_{1}, A_{1}, C_{1},\) and \(k\) are arbitrary constants.

According to Eqs. (40) and (41) and the general solutions of Eq. (7) listed in Sec. 2, we will obtain the following wave solutions for (1+1)-dimensional DLWE.

Family 1 When \(h_{1} = 1, h_{2} = -1,\) and \(h_{2} = h_{4} = \pm 1,\) then we can get combining tanh and tan function solution:

\[
\nu = a_{0} + \frac{a_{1} \tanh (\xi)}{\mu_{1} \tanh (\xi) + \mu_{2} \tanh (\xi) + 1} + \frac{b_{1} \tanh (\xi)}{\mu_{1} \tanh (\xi) + \mu_{2} \tanh (\xi) + 1},
\]

\[
w_{1} = A_{0} + \frac{A_{1} \tanh (\xi)}{\mu_{1} \tanh (\xi) + \mu_{2} \tanh (\xi) + 1} + \frac{b_{1} A_{1} \tanh (\xi)}{\mu_{1} \tanh (\xi) + \mu_{2} \tanh (\xi) + 1} - \frac{a_{1} b_{1} \tanh (\xi) \tanh (\xi) + \mu_{2} \tanh (\xi) + 1}}{2 (\mu_{1} \tanh (\xi) + \mu_{2} \tanh (\xi) + 1)^{2}}
\]

\[
\quad - \frac{a_{1} b_{1} \tanh (\xi) \tanh (\xi) + \mu_{2} \tanh (\xi) + 1}}{2 (\mu_{1} \tanh (\xi) + \mu_{2} \tanh (\xi) + 1)^{2}},
\]

where \(\xi = k(x + \lambda t), \lambda = -(A_{1} + a_{0} a_{1}) / a_{1}, a_{0}, A_{0}, a_{1}, b_{1}, A_{1}, C_{1},\) and \(k\) are arbitrary constants.
**Family 2** When \( h_1 = 1, h_2 = -1, \) and \( h_3 = h_4 = \pm 1/2, \) then we can get combining \( \tanh, \sec \) and \( \tan \) function solution:

\[
v_2 = a_0 + b_1 \frac{a_1 \tanh(\xi) + \mu_2 (\sec(\xi) \pm \tan(\xi)) + 1}{\mu_1 \tanh(\xi) + \mu_2 (\sec(\xi) \pm \tan(\xi)) + 1},
\]

\[
w_2 = A_0 + b_1 \frac{A_1 \tanh(\xi) + \mu_2 (\sec(\xi) \pm \tan(\xi)) + 1}{\mu_1 \tanh(\xi) + \mu_2 (\sec(\xi) \pm \tan(\xi)) + 1} - \frac{a_1^2 \tanh^2(\xi) + \mu_2 (\sec(\xi) \pm \tan(\xi)) + 1^2}{b_1^2 (\sec(\xi) \pm \tan(\xi))^2} - \frac{2(\mu_1 \tanh(\xi) + \mu_2 (\sec(\xi) \pm \tan(\xi)) + 1^2)}{(\mu_1 \tanh(\xi) + \mu_2 (\sec(\xi) \pm \tan(\xi)) + 1)^2},
\]

where \( \xi = k(x + \lambda t), \lambda = -(A_1 + a_0 a_1)/a_1, a_0, A_0, a_1, b_1, A_1, C_1, \) and \( k \) are arbitrary constants.

**Family 3** When \( h_1 = 1, h_2 = -1, h_3 = 0, \) and \( h_4 \neq 0, \) then we can get combining \( \tanh \) and rational function solution:

\[
v_3 = a_0 + \frac{a_1 \tanh(\xi)(h_4 \xi + c_0) - b_1}{\mu_1 \tanh(\xi)(h_4 \xi + c_0) - \mu_2 - h_4 \xi + c_0},
\]

\[
w_3 = A_0 + \frac{A_1 a_1 \tanh(\xi)(h_4 \xi + c_0) + b_1 A_1}{a_1 (\mu_1 \tanh(\xi)(h_4 \xi + c_0) - \mu_2 - h_4 \xi + c_0)} - \frac{a_1^2 \tanh^2(\xi)(h_4 \xi + c_0)^2}{2(\mu_1 \tanh(\xi)(h_4 \xi + c_0) - \mu_2 - h_4 \xi + c_0)^2} - \frac{b_1^2}{2(\mu_1 \tanh(\xi)(h_4 \xi + c_0) - \mu_2 - h_4 \xi + c_0)^2},
\]

where \( \xi = k(x + \lambda t), \lambda = -(A_1 + a_0 a_1)/a_1, h_4 \neq 0, a_0, A_0, a_1, b_1, A_1, C_1, \) and \( k \) are arbitrary constants.

**Family 4** When \( h_1 = -1/2, h_2 = 1/2 \) and \( h_3 = h_4 = \pm 1, \) then we can get combining \( \tanh, \sech \) and \( \tan \) function solution:

\[
v_4 = a_0 + \frac{a_1 (\tanh(\xi) \pm \i)(h_4 \xi + c_0) + b_1 \tan(\xi)}{\mu_1 (\tanh(\xi) \pm \i)(h_4 \xi + c_0) + \mu_2 \tan(\xi) + 1},
\]

\[
w_4 = A_0 + \frac{A_1 a_1 (\tanh(\xi) \pm \i)(h_4 \xi + c_0) + b_1 A_1 \tan(\xi)}{a_1(\mu_1 (\tanh(\xi) \pm \i)(h_4 \xi + c_0) - \mu_2 \tan(\xi) + 1)} - \frac{a_1^2 (\tanh(\xi) \pm \i)^2}{2(\mu_1 (\tanh(\xi) \pm \i) + \mu_2 \tan(\xi) + 1)^2} - \frac{b_1^2 (\tan(\xi) \pm \i)^2}{2(\mu_1 (\tanh(\xi) \pm \i) + \mu_2 \tan(\xi) + 1)^2},
\]

where \( \xi = k(x + \lambda t), \lambda = -(A_1 + a_0 a_1)/a_1, a_0, A_0, a_1, b_1, A_1, C_1, \) and \( k \) are arbitrary constants.

**Family 5** When \( h_1 = -1/2, h_2 = 1/2, \) and \( h_3 = h_4 = \pm 1/2, \) then we combine \( \tanh, \sec, \sech \) and \( \tan \) function solution:

\[
v_5 = a_0 + \frac{a_1 (\tanh(\xi) \pm \i) \sec(\xi)}{\mu_1 (\tanh(\xi) \pm \i) \sec(\xi) + \mu_2 (\sec(\xi) \pm \tan(\xi)) + 1},
\]

\[
w_5 = A_0 + \frac{A_1 (\tanh(\xi) \pm \i) \sec(\xi)}{a_1(\mu_1 (\tanh(\xi) \pm \i) \sec(\xi) + \mu_2 (\sec(\xi) \pm \tan(\xi)) + 1)} - \frac{a_1^2 (\tanh(\xi) \pm \i)^2}{2(\mu_1 (\tanh(\xi) \pm \i) \sec(\xi) + \mu_2 (\sec(\xi) \pm \tan(\xi)) + 1)^2} - \frac{b_1^2 (\sec(\xi) \pm \tan(\xi))^2}{2(\mu_1 (\tanh(\xi) \pm \i) \sec(\xi) + \mu_2 (\sec(\xi) \pm \tan(\xi)) + 1)^2},
\]

where \( \xi = k(x + \lambda t), \lambda = -(A_1 + a_0 a_1)/a_1, a_0, A_0, a_1, b_1, A_1, C_1, \) and \( k \) are arbitrary constants.

**Family 6** When \( h_1 = -1/2, h_2 = 1/2, h_3 = 0, \) and \( h_4 \neq 0, \) then we can get combining \( \tanh, \sec, \) rational function solution:

\[
v_6 = a_0 + \frac{a_1 (\tanh(\xi) \pm \i)(h_4 \xi + c_0) - b_1}{\mu_1 (\tanh(\xi) \pm \i)(h_4 \xi + c_0) - \mu_2 + h_4 \xi + c_0},
\]

\[
w_6 = A_0 + \frac{A_1 a_1 (\tanh(\xi) \pm \i)(h_4 \xi + c_0) - b_1 A_1}{a_1(\mu_1 (\tanh(\xi) \pm \i)(h_4 \xi + c_0) - \mu_2 + h_4 \xi + c_0)}.\]
where \( \xi = k(x + \lambda t) \), \( \lambda = -(A_1 + a_0a_1)/a_1 \), \( h_4 \neq 0 \), \( c_0 \), \( a_0 \), \( A_0 \), \( a_1 \), \( b_1 \), \( A_1 \), \( C_1 \), and \( k \) are arbitrary constants.

**Family 7** When \( h_1 = h_2 = \pm 1 \), \( h_3 = 0 \), and \( h_4 \neq 0 \), then we can get combining tan and rational function solution:

\[
v_{7} = a_0 + \frac{a_1 \tan(\xi) + \tan(\xi)}{a_1 \tan(\xi) + \tan(\xi),}
\]

\[
w_{7} = A_0 + \frac{(a_1 \tan(\xi)(h_4 \xi + c_0) - \mu \xi + h_4 \xi + c_0)}{a_1 \tan(\xi)(h_4 \xi + c_0) - \mu \xi + h_4 \xi + c_0} - \frac{a_1 \tan^2(\xi)(h_4 \xi + c_0)}{2 \mu(\tan(\xi)(h_4 \xi + c_0) - \mu \xi + h_4 \xi + c_0)^2}
\]

where \( \xi = k(x + \lambda t) \), \( \lambda = -(A_1 + a_0a_1)/a_1 \), \( h_4 \neq 0 \), \( c_0 \), \( a_0 \), \( A_0 \), \( a_1 \), \( b_1 \), \( A_1 \), \( C_1 \), and \( k \) are arbitrary constants.

**Family 8** When \( h_1 = h_2 = \pm 1/2 \), \( h_3 = 0 \), and \( h_4 \neq 0 \), then we can get combining tan, sec, rational function solution:

\[
v_{8} = a_0 + \frac{a_1 \sec(\xi) \pm \tan(\xi)}{a_1 \sec(\xi) \pm \tan(\xi),}
\]

\[
w_{8} = A_0 + \frac{(a_1 \sec(\xi) \pm \tan(\xi))(h_4 \xi + c_0) - \mu \xi + h_4 \xi + c_0)}{a_1 \sec(\xi) \pm \tan(\xi)(h_4 \xi + c_0) - \mu \xi + h_4 \xi + c_0} - \frac{a_1 \sec^2(\xi) \pm \tan(\xi)^2(h_4 \xi + c_0)^2}{2 \mu(\sec(\xi) \pm \tan(\xi))(h_4 \xi + c_0) - \mu \xi + h_4 \xi + c_0)^2}
\]

where \( \xi = k(x + \lambda t) \), \( \lambda = -(A_1 + a_0a_1)/a_1 \), \( h_4 \neq 0 \), \( c_0 \), \( a_0 \), \( A_0 \), \( a_1 \), \( b_1 \), \( A_1 \), \( C_1 \), and \( k \) are arbitrary constants.

At the same time, we can also get some new solutions which are not the complexiton solutions and cannot be obtained by other tanh method, such as

**Family 9** When \( h_1 = h_3 = 1/2 \) and \( h_2 = h_4 = -1/2 \), then we obtain the following solutions

\[
v_{9} = a_0 + \frac{a_1 \tan(\xi) \pm \i sech(\xi)) + b_1 (coth(\xi) \pm csch(\xi))}{a_1 \tan(\xi) \pm \i sech(\xi)) + b_1 (coth(\xi) \pm csch(\xi)) + 1}
\]

\[
w_{9} = A_0 + \frac{(a_1 \tan(\xi) \pm \i sech(\xi)) + b_1 (coth(\xi) \pm csch(\xi))}{a_1 \tan(\xi) \pm \i sech(\xi)) + b_1 (coth(\xi) \pm csch(\xi)) + 1} - \frac{a_1^2 \tan(\xi) \pm \i sech(\xi))}{2 \mu_1(\tan(\xi) \pm \i sech(\xi)) \pm \i sech(\xi)) + 1}\]

\[
- \frac{a_1 b_1 (\tan(\xi) \pm \i sech(\xi)) (coth(\xi) \pm csch(\xi))}{(\mu_1(\tan(\xi) \pm \i sech(\xi)) \pm \i sech(\xi)) + 1} + b_1^2 \frac{(coth(\xi) \pm csch(\xi))}{2}
\]

where \( \xi = k(x + \lambda t) \), \( \lambda = -(A_1 + a_0a_1)/a_1 \), \( h_4 \neq 0 \), \( c_0 \), \( a_0 \), \( A_0 \), \( a_1 \), \( b_1 \), \( A_1 \), \( C_1 \), and \( k \) are arbitrary constants.

**Family 10** When \( h_1 = h_2 = \pm 1/2 \) and \( h_3 = h_4 = \pm 1/2 \), then we obtain the following solutions

\[
v_{10} = a_0 + \frac{a_1 \sec(\xi) \pm \tan(\xi)) + b_1 (csc(\xi) \pm cot(\xi))}{a_1 \sec(\xi) \pm \tan(\xi)) + b_1 (csc(\xi) \pm cot(\xi)) + 1}
\]

\[
w_{10} = A_0 + \frac{(A_1 a_1 \sec(\xi) \pm \tan(\xi)) + b_1 A_1 (csc(\xi) \pm cot(\xi))}{a_1 (a_1 \sec(\xi) \pm \tan(\xi)) + b_1 (csc(\xi) \pm cot(\xi)) + 1}
\]
\[
\begin{align*}
&= a_1^2(\sec(\xi) \pm \tan(\xi))^2
- \frac{2(\mu_1(\sec(\xi) \pm \tan(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi) + 1))^2}{\mu_1(\sec(\xi) \pm \tan(\xi)))(\csc(\xi) \pm \cot(\xi))} \\
&\quad - \frac{a_1b_1(\sec(\xi) \pm \tan(\xi))}{\mu_1(\sec(\xi) \pm \tan(\xi)))(\csc(\xi) \pm \cot(\xi))} \\
&\quad - \frac{b_1^2(\csc(\xi) \pm \cot(\xi))^2}{\mu_2(\csc(\xi) \pm \cot(\xi)) + 1} \\
&= \frac{2(\mu_1(\sec(\xi) \pm \tan(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)^2}{\mu_1(\sec(\xi) \pm \tan(\xi)))(\csc(\xi) \pm \cot(\xi))}.
\end{align*}
\]

where \(\xi = k(x + \lambda t)\), \(\lambda = -(A_1 + a_0a_1)/a_1\), \(a_0\), \(A_0\), \(a_1\), \(b_1\), \(A_1\), \(C_1\), and \(k\) are arbitrary constants.

4 Discussions and Conclusion

Although our method cannot recover all complexiton solutions obtained by Ma’s method and Lou’s method, other new types of complexiton solutions cannot be found by Ma’s method and Lou’s method. In particular, our method is a unified straightforward and pure algebraic algorithm to integrable equations and nonintegrable equations, which is implemented in a computer algebraic system. The complexiton solutions presented above do not satisfy the definition by Ma,[7] but one of the basic character of the solutions by Ma[7] and the solutions by us are the same, i.e. possessing various combination of trigonometric periodic and hyperbolic function solutions, various combination of trigonometric periodic and rational function solutions, various combination of hyperbolic and rational function solutions. Of course our method can also be extended to other integrable systems and nonintegrable systems. Our further work is finding different travelling wave complexiton solutions by this direct and unified method for nonlinear evolution equations.

References