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A new elliptic equation rational expansion method and its application to the shallow long wave approximate equations

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Abstract

A new elliptic equation rational expansion method is presented by a new general ansätz, which is a direct and unified algebraic method for constructing multiple and more general travelling wave solution for nonlinear partial differential equation and implemented in a computer algebraic system. The proposed method is applied to consider the shallow long wave approximate equation and obtains rich new families of the exact solutions, including rational form solitary wave, rational form triangular periodic, rational form Jacobi and Weierstrass doubly periodic solutions.

Keywords: Travelling wave solution; Elliptic equation rational expansion method; Rational form solitary wave solutions; Shallow long wave approximate equation

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1. Introduction

In the past decades, both mathematicians and physicists have devoted considerable effort to the study of solitons and related issue of the construction of solutions to nonlinear partial differential equation (NPDEs) [1–13]. Recently, Fan developed a new algebraic method [14] with symbolic computation for obtaining some various travelling wave solutions in a unified way and easily provides us with new and more general travelling wave solutions in terms of special functions such as hyperbolic, rational, triangular, Weierstrass and Jacobi elliptic double periodic functions. More recently, we extended the Fan's method to a generalized method [15] for finding the more general travelling wave solutions. On the other hand, we extend the generalized method to uniformly construct a series of soliton-like solutions and double-like periodic solutions for NPDEs [16]. As a result, we can not only successfully recover the previously known travelling wave solutions found by Fan's method but also obtain some new formal and more general solutions. Most recently, we present the Jacobi elliptic function rational expansion method [17] and the Riccati equation rational expansion method [18], in which the ansätzs in rational expansion, are presented.

Based on a more general ansätz and a general elliptic equation [19,20], we present a new subequation method, named elliptic equation rational expansion (EERE) method to obtain more types and general formal solutions which contain not only the results obtained by using the method [14] but also other types of solutions. For illustration, we apply the generalized method to solve and successfully construct new and more general solutions including rational form solitary wave, rational, rational form triangular periodic, rational form Jacobi and Weierstrass doubly periodic solutions for the shallow long wave approximate (SLA) equation, i.e.,

$$u_t - uu_x - v_x + \frac{u_{xx}}{2} = 0, \tag{1.1.1}$$

$$v_t - u_x v - uv_x - \frac{v_{xx}}{2} = 0, \tag{1.1.2}$$

which was found by Whitham [21] and Broer [22]. The symmetries and conservation laws of system (1.1) were discussed by Kuperschmidt [23]. By using the homogenous balance method, Zhang [24] obtained multiple soliton solutions of the system. Yan and Zhang [25] used sine–cosine method to obtain three families of soliton solutions. Chen and Zheng [26] used the Generalized extended tanh-function method to construct news explicit exact solutions of SLA equation. Wang et al. [27] used generally projective Riccati equation method to obtain new exact travelling wave solutions for SLA equation.

This paper is organized as follows. In Section 2, we summarize the EERE method. In Section 3, we apply the EERE method to SLA equation and bring out many rational formal rational form solitary wave, rational form triangular periodic, rational form Jacobi and Weierstrass doubly periodic solutions. Conclusions will be presented finally.

2. Elliptic equation rational expansion method

In the following we would like to outline the main steps of our method:

Step 1. For a given NPDE system with some physical fields $u_i(x, y, t)$ in three variables x, y, t,

$$F_i(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \ldots) = 0, \tag{2.1}$$

by using the wave transformation

$$u_i(x, y, t) = U_i(\xi), \quad \xi = k(x + ly + \lambda t), \tag{2.2}$$

where k, l and λ are constants to be determined later. Then the nonlinear partial differential equation (2.1) is reduced to a nonlinear ordinary differential equation (ODE):

$$G_i(U_i, U_i', U_i'', \ldots) = 0. \tag{2.3}$$

Step 2. We introduce a new ansätz in terms of finite rational formal expansion in the following forms:

$$U_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}\phi^j(\xi) + b_{ij}\phi^{j-1}(\xi)\phi'(\xi)}{(\mu\phi(\xi) + 1)^j} \tag{2.4}$$

and the new variable $\phi = \phi(\xi)$ satisfying

$$\phi'^2 = \left(\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)^2 = h_0 + h_1\phi + h_2\phi^2 + h_3\phi^3 + h_4\phi^4, \tag{2.5}$$

where h_ρ, a_{i0}, a_{ij} and b_{ij} ($\rho = 0, 1, \ldots, 4$; $i = 1, 2, \ldots$; $j = 1, 2, \ldots, m_i$) are constants to be determined later.

Step 3. The underlying mechanism for a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions to occur is that differ effects that act to change wave forms in many nonlinear equations, i.e. dispersion, dissipation and nonlinearity, either separately or various combination are able to balance out. We define the degree of $U_i(\xi)$ as $D[U_i(\xi)] = n_i$, which gives rise to the degrees of other expressions as

$$D[U_i^{(\alpha)}] = n_i + \alpha, \quad D[U_i^{\beta}(U_j^{(\alpha)})^s] = n_i\beta + (\alpha + n_j)s. \tag{2.6}$$

Therefore we can get the value of m_i in Eq. (2.4). If n_i is a nonnegative integer, then we first make the transformation $U_i = V_i^{n_i}$.

Step 4. Substitute Eq. (2.4) into Eq. (2.3) along with Eq. (2.5) and then set all coefficients of $\phi^p(\xi)\left(\sqrt{\sum_{\rho=0}^{4}h_\rho\phi^\rho}\right)^q$ ($p = 1,2,\ldots$; $q = 0,1$) of the resulting system's numerator to be zero to get an over-determined system of nonlinear algebraic equations with respect to k, μ, a_{i0}, a_{ij} and b_{ij} ($i = 1,2,\ldots$; $j = 1,2,\ldots,m_i$).

Step 5. Solving the over-determined system of nonlinear algebraic equations by use of *Maple*, we would end up with the explicit expressions for k, μ, a_{i0}, a_{ij} and b_{ij} ($i = 1,2,\ldots$; $j = 1,2,\ldots,m_i$).

Step 6. It is well known that the general solutions of Eq. (2.5) are

Case A. If $h_3 = h_4 = 0$, Eq. (2.5) possesses the following solutions:

$$\phi = \sqrt{h_0}\xi, \quad h_1 = h_2 = 0, \ h_0 > 0, \tag{2.7}$$

$$\phi = -\frac{h_0}{h_1} + \frac{1}{4}h_1\xi^2, \quad h_2 = 0, \ h_1 \neq 0, \tag{2.8}$$

$$\phi = -\frac{h_1}{2h_2} + \exp(\sqrt{h_2}\xi), \quad h_0 = \frac{h_1^2}{4h_2}, \ h_2 > 0, \tag{2.9}$$

$$\phi = -\frac{h_1}{2h_2} + \frac{h_1}{2h_2}\sin(\sqrt{-h_2}\xi), \quad h_0 = 0, \ h_2 < 0, \tag{2.10}$$

$$\phi = -\frac{h_1}{2h_2} + \frac{h_1}{2h_2}\sinh(\sqrt{h_2}\xi), \quad h_0 = 0, \ h_2 > 0. \tag{2.11}$$

Case B. If $h_1 = h_3 = 0$, Eq. (2.5) possesses the following solutions:

$$\phi = \sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi), \quad h_0 = 0, \ h_2 > 0, \ h_4 < 0, \tag{2.12}$$

$$\phi = \sqrt{-\frac{h_2}{2h_4}}\tanh\left(\sqrt{-\frac{h_2}{2}}\xi\right), \quad h_0 = \frac{h_2^2}{4h_4}, \ h_2 < 0, \ h_4 > 0, \tag{2.13}$$

$$\phi = \sqrt{-\frac{h_2}{h_4}}\sec(\sqrt{-h_2}\xi), \quad h_0 = 0, \ h_2 < 0; \ h_4 > 0, \tag{2.14}$$

$$\phi = \sqrt{\frac{h_2}{2h_4}}\tan\left(\sqrt{\frac{h_2}{2}}\xi\right), \quad h_0 = \frac{h_2^2}{4h_4}, \ h_2 > 0, \ h_4 > 0, \tag{2.15}$$

$$\phi = -\frac{1}{\sqrt{h_4}\xi}, \quad h_0 = h_2 = 0, \ h_4 > 0, \tag{2.16}$$

$$\phi = \operatorname{sn}(\xi), \quad h_0 = 1, \ h_2 = -(m^2+1), \ h_4 = m^2, \tag{2.17}$$

$$\phi = \operatorname{cd}(\xi), \quad h_0 = 1, \ h_2 = -(m^2+1), \ h_4 = m^2, \tag{2.18}$$

$$\phi = \operatorname{cn}(\xi), \quad h_0 = 1 - m^2, \ h_2 = 2m^2 - 1, \ h_4 = -m^2, \tag{2.19}$$

$$\phi = \operatorname{dn}(\xi), \quad h_0 = m^2 - 1, \ h_2 = 2 - m^2, \ h_4 = -1, \tag{2.20}$$

$$\phi = \operatorname{ns}(\xi), \quad h_0 = m^2, \ h_2 = -(m^2+1), \ h_4 = 1, \tag{2.21}$$

$$\phi = \mathrm{dc}(\xi), \quad h_0 = m^2, \ h_2 = -\left(m^2 + 1\right), \ h_4 = 1, \tag{2.22}$$

$$\phi = \mathrm{nc}(\xi), \quad h_0 = -m^2, \ h_2 = 2m^2 - 1, \ h_4 = 1 - m^2, \tag{2.23}$$

$$\phi = \mathrm{nd}(\xi), \quad h_0 = -1, \ h_2 = 2 - m^2, \ h_4 = 1 - m^2, \tag{2.24}$$

$$\phi = \mathrm{cs}(\xi), \quad h_0 = 1 - m^2, \ h_2 = 2 - m^2, \ h_4 = 1, \tag{2.25}$$

$$\phi = \mathrm{sc}(\xi), \quad h_0 = 1, \ h_2 = 2 - m^2, \ h_4 = 1 - m^2, \tag{2.26}$$

$$\phi = \mathrm{sd}(\xi), \quad h_0 = 1, \ h_2 = 2m^2 - 1, \ h_4 = m^2\left(m^2 - 1\right), \tag{2.27}$$

$$\phi = \mathrm{ds}(\xi), \quad h_0 = m^2\left(m^2 - 1\right), \ h_2 = 2m^2 - 1, \ h_4 = 1, \tag{2.28}$$

$$\phi = \mathrm{ns}(\xi) \pm \mathrm{cs}(\xi), \quad h_0 = \frac{1}{4}, \ h_2 = \frac{1 - 2m^2}{2}, \ h_4 = \frac{1}{4}, \tag{2.29}$$

$$\phi = \mathrm{nc}(\xi) \pm \mathrm{sc}(\xi), \quad h_0 = \frac{1 - m^2}{4}, \ h_2 = \frac{1 + m^2}{2}, \ h_4 = \frac{1 - m^2}{4}, \tag{2.30}$$

$$\phi = \mathrm{ns}(\xi) \pm \mathrm{ds}(\xi), \quad h_0 = \frac{m^2}{4}, \ h_2 = \frac{m^2 - 2}{2}, \ h_4 = \frac{1}{4}, \tag{2.31}$$

$$\phi = \mathrm{sn}(\xi) \pm i\mathrm{cn}(\xi), \quad h_0 = \frac{m^2}{4}, \ h_2 = \frac{m^2 - 2}{2}, \ h_4 = \frac{m^2}{4}, \tag{2.32}$$

where m is a modulus. The Jacobi elliptic functions are doubly periodic and possess properties of triangular functions:

$$\mathrm{sn}^2(\xi) + \mathrm{cn}^2(\xi) = 1, \quad \mathrm{dn}^2(\xi) = 1 - m^2\mathrm{sn}^2\xi,$$

$$\mathrm{sn}'(\xi) = \mathrm{cn}(\xi)\mathrm{dn}(\xi), \quad \mathrm{cn}'(\xi) = -\mathrm{sn}(\xi)\mathrm{dn}(\xi), \quad \mathrm{dn}'(\xi) = -m^2\mathrm{sn}(\xi)\mathrm{cn}(\xi).$$

When $m \to 1$, the Jacobi functions degenerate to the hyperbolic functions, i.e.

$$\mathrm{sn}(\xi) \to \tanh(\xi), \quad \mathrm{cn}(\xi) \to \mathrm{sech}(\xi).$$

When $m \to 0$, the Jacobi functions degenerate to the triangular functions, i.e.

$$\mathrm{sn}(\xi) \to \sin(\xi), \quad \mathrm{cn}(\xi) \to \cos(\xi).$$

The more detailed notations for the Weierstrass and Jacobi elliptic functions can be found in Refs. [19,20].

Case C. If $h_4 = 0$, Eq. (2.5) possesses the following solutions:

$$\phi = -\frac{h_2}{h_3}\mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right), \quad h_0 = h_1 = 0, \ h_2 > 0, \tag{2.33}$$

$$\phi = -\frac{h_2}{h_3}\sec^2\left(\frac{\sqrt{-h_2}}{2}\xi\right), \quad h_0 = h_1 = 0, \ h_2 < 0, \tag{2.34}$$

$$\phi = \frac{4}{h_3\xi^2}, \quad h_0 = h_1 = h_2 = 0, \tag{2.35}$$

$$\phi = \wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right), \quad h_2 = 0, \ h_3 > 0, \tag{2.36}$$

where $g_2 = -4\frac{h_1}{h_3}$ and $g_3 = -4\frac{h_0}{h_3}$ are called invariants of Weierstrass elliptic function.

Case D. If $h_0 = h_1 = 0$, Eq. (2.5) possesses the following solutions:

$$\phi = -\frac{h_2 \sec^2\left(\frac{\sqrt{-h_2}}{2}\xi\right)}{2\sqrt{-h_2 h_4}\tan\left(\frac{\sqrt{-h_2}}{2}\xi\right) + h_3}, \quad h_2 < 0, \tag{2.37}$$

$$\phi = \frac{h_2 \operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)}{2\sqrt{h_2 h_4}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3}, \quad h_2 > 0. \tag{2.38}$$

Thus according to Eqs. (2.2), (2.4), (2.7)–(2.38) and the conclusions in *Step 5*, we can obtain some rational formal travelling-wave solutions of Eq. (2.1).

Remark. In fact, we naturally present a more general ansätz, which reads,

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \times \frac{a_{ij}\phi^j(\xi) + b_{ij}\phi^{j-1}(\xi)\sqrt{\sum_{l=0}^{r} h_l \phi^l(\xi)} + c_{ij}\frac{\sqrt{\sum_{l=0}^{r} h_l \phi^l(\xi)}}{\phi^j(\xi)} + d_{ij}\phi^{-j}(\xi)}{\left(\mu_{j1}\phi(\xi) + \mu_{j2}\sqrt{\sum_{l=0}^{r} h_l \phi^l(\xi)} + \mu_{j3}\frac{\sqrt{\sum_{l=0}^{r} h_l \phi^l(\xi)}}{\phi(\xi)} + \mu_{j4}\phi^{-1}(\xi) + 1\right)^j}, \tag{2.39}$$

and the new variable $\phi = \phi(\xi)$ satisfying

$$(\phi'(\xi))^2 = \left(\frac{d\phi(\xi)}{d\xi}\right)^2 = \sum_{l=0}^{r} h_l \phi^l(\xi), \tag{2.40}$$

where a_{i0}, a_{ij}, b_{ij}, c_{ij}, d_{ij}, μ_{j1}, μ_{j2}, μ_{j3}, μ_{j4}, h_l ($i = 1, 2, \ldots;\ j = 1, 2, \ldots, m_i$; $l = 1, 2, \ldots, r$) and ξ are the differentiable functions to be determined later. Therefore, for some nonlinear equations, more types of nontravelling solutions, such as rational form soliton-like solutions, rational form triangular periodic-like, rational form Jacobi and Weierstrass doubly-like periodic solutions would be expected.

3. Exact solutions of the shallow long wave approximate equation

Let us consider the shallow long wave approximate (SLA) equation (1.1).

By considering the wave transformations $u(x,t) = U(\xi)$, $v(x,t) = V(\xi)$ and $\xi = k(x + \lambda t)$, we change Eq. (1.1) to the form

$$\lambda U' - UU' - V' + \frac{k}{2}U'' = 0, \tag{3.1.1}$$

$$\lambda V' - (UV)' - \frac{k}{2}V'' = 0. \tag{3.1.2}$$

According to the proposed method, we expand the solution of Eq. (3.1) in the form

$$\begin{cases} U(\xi) = a_0 + \sum_{j=1}^{m_u} \frac{a_j\phi^j(\xi)+b_j\phi^{j-1}(\xi)\phi'}{(\mu\phi(\xi)+1)^j}, \\ V(\xi) = A_0 + \sum_{j=1}^{m_v} \frac{A_j\phi^j(\xi)+B_j\phi^{j-1}(\xi)\phi'}{(\mu\phi(\xi)+1)^j}, \end{cases}$$

where $\phi(\xi)$ satisfies Eq. (2.5). Balancing the term V'' with term $(UV)'$ and the term V' with term UU' in Eq. (3.1) gives $m_u = 1$ and $m_v = 2$. So we have

$$U(\xi) = a_0 + \frac{a_1\phi(\xi) + b_1\phi'}{\mu\phi(\xi) + 1}, \tag{3.2.1}$$

$$V(\xi) = A_0 + \frac{A_1\phi(\xi) + B_1\phi'}{\mu\phi(\xi) + 1} + \frac{A_2\phi^2(\xi) + B_2\phi(\xi)\phi'}{(\mu\phi(\xi) + 1)^2}. \tag{3.2.2}$$

With the aid of the *Maple*, substituting (3.2) along with (2.5) into (3.1), yields a set of algebraic equations for $\phi^p(\xi)\left(\sqrt{\sum_{\rho=0}^4 h_\rho\phi^\rho}\right)^q$ $(p = 0, 1, \ldots;\ q = 0, 1)$. Setting the coefficients of these terms $\phi^i(\xi)$ to zero yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, A_0, A_1, B_1, A_2, B_2, μ and k.

By use of the *Maple* soft package "Charsets" by Dongming Wang, which is based on the Wu-elimination method [28], solving the over-determined algebraic equations, we get the following results:

$$\begin{aligned}
&k = \pm\frac{a_1}{\sqrt{-\mu^3h_1 + h_4 + \mu^2h_2 - h_3\mu + \mu^4h_0}},\ a_1 = a_1,\ b_1 = 0,\ \mu = \mu,\\
&a_0 = -\frac{4h_0\lambda\mu^4 - 4\mu^3h_0a_1 - 4\mu^3h_1\lambda + 3\mu^2h_1a_1 + 4\mu^2h_2\lambda - 2\mu a_1h_2 - 4\mu h_3\lambda + a_1h_3 + 4h_4\lambda}{4(\mu^3h_1 - h_4 - \mu^2h_2 + h_3\mu - \mu^4h_0)},\\
&A_0 = -\frac{(8h_0^2\mu^6 - 12h_1h_0\mu^5 + 12\mu^4h_0h_2 + 3\mu^4h_1^2 - 16\mu^3h_3h_0)a_1^2}{16(\mu^3h_1 - h_4 - \mu^2h_2 + h_3\mu - \mu^4h_0)^2}\\
&\qquad - \frac{(-4\mu^3h_1h_2 + 24h_4\mu^2h_0 + 6\mu^2h_1h_3 - 12h_4\mu h_1 - h_3^2 + 4h_4h_2)a_1^2}{16(\mu^3h_1 - h_4 - \mu^2h_2 + h_3\mu - \mu^4h_0)^2},\\
&A_1 = \frac{(-4\mu^3h_0 + 3\mu^2h_1 - 2h_2\mu + h_3)a_1^2}{4(\mu^3h_1 - h_4 - \mu^2h_2 + h_3\mu - \mu^4h_0)},\ B_1 = \pm\frac{a_1^2}{2\sqrt{-\mu^3h_1 + h_4 + \mu^2h_2 - h_3\mu + \mu^4h_0}},\\
&A_2 = -\frac{1}{2}a_1^2,\ B_2 = \mp\frac{a_1^2\mu}{2\sqrt{-\mu^3h_1 + h_4 + \mu^2h_2 - h_3\mu + \mu^4h_0}}.
\end{aligned} \tag{3.3}$$

Then according to Eqs. (3.4), we obtain the following solutions of the SLA equation.

Note: Since tan- and cot-type solution appear in pairs with tanh- and coth-type solutions, respectively, we omit them in this paper. In addition, some rational solutions are also omitted.

Family 1. When $h_3 = h_4 = 0$ and $h_0 = -\frac{h_1^2}{4h_2}$, we obtain the following solutions for the SLA equation:

$$u_{11} = a_0 + \frac{a_1(-h_1 + 2h_2\exp(\sqrt{h_2}\xi))}{\mu(-h_1 + 2h_2\exp(\sqrt{h_2}\xi)) + 2h_2}, \tag{3.4.1}$$

$$\begin{aligned} v_{11} = A_0 &+ \frac{(-\mu^3h_1^2 + 3\mu^2h_1h_2 - 2h_2^2\mu)a_1^2(-h_1 + 2h_2\exp(\sqrt{h_2}\xi))}{(4\mu^3h_1h_2 - 4\mu^2h_2^2 - \mu^4h_1^2)(\mu(-h_1 + 2h_2\exp(\sqrt{h_2}\xi)) + 2h_2)} \\ &- \frac{a_1^2(-h_1 + 2h_2\exp(\sqrt{h_2}\xi))^2}{2(\mu(-h_1 + 2h_2\exp(\sqrt{h_2}\xi)) + 2h_2)^2} \\ &\pm \frac{2a_1^2h_2\sqrt{h_2}\exp(\sqrt{h_2}\xi)}{\sqrt{-4\mu^3h_1 + 4\mu^2h_2 + \frac{\mu^4h_1^2}{h_2}}(\mu(-h_1 + 2h_2\exp(\sqrt{h_2}\xi)) + 2h_2)} \\ &\mp \frac{a_1^2\mu(-h_1 + 2h_2\exp(\sqrt{h_2}\xi))\sqrt{h_2}\exp(\sqrt{h_2}\xi)}{\sqrt{-4\mu^3h_1 + 4\mu^2h_2 + \frac{\mu^4h_1^2}{h_2}}(\mu(-h_1 + 2h_2\exp(\sqrt{h_2}\xi)) + 2h_2)^2}, \end{aligned} \tag{3.4.2}$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.3), μ, $h_2 > 0$, h_1, a_1 and λ are the arbitrary constants.

Family 2. When $h_0 = h_3 = h_4 = 0$, we obtain the following solutions for the SLA equation:

$$u_{21} = \frac{4\mu^3h_1\lambda - 3\mu^2h_1a_1 - 4\mu^2h_2\lambda + 2\mu a_1h_2}{4(\mu^3h_1 - \mu^2h_2)} + \frac{a_1(-h_1 + h_1\sinh(\sqrt{h_2}\xi))}{(\mu(-h_1 + h_1\sinh(\sqrt{h_2}\xi)) + 2h_2)}, \tag{3.5.1}$$

$$\begin{aligned} v_{21} = &-\frac{(3\mu^4h_1^2 - 4\mu^3h_1h_2)a_1^2}{16(\mu^3h_1 - \mu^2h_2)^2} + \frac{(3\mu^2h_1 - 2h_2\mu)a_1^2(-h_1 + h_1\sinh(\sqrt{h_2}\xi))}{4(\mu^3h_1 - \mu^2h_2)(\mu(-h_1 + h_1\sinh(\sqrt{h_2}\xi)) + 2h_2)} \\ &- \frac{a_1^2(-h_1 + h_1\sinh(\sqrt{h_2}\xi))^2}{(\mu(-h_1 + h_1\sinh(\sqrt{h_2}\xi)) + 2h_2)^2} \pm \frac{a_1^2h_1\sqrt{h_2}\cosh(\sqrt{h_2}\xi)}{2\sqrt{-\mu^3h_1 + \mu^2h_2}(\mu(-h_1 + h_1\sinh(\sqrt{h_2}\xi)) + 2h_2)} \\ &\mp \frac{a_1^2\mu(-h_1 + h_1\sinh(\sqrt{h_2}\xi))h_1\sqrt{h_2}\cosh(\sqrt{h_2}\xi)}{2\sqrt{-\mu^3h_1 + \mu^2h_2}(\mu(-h_1 + h_1\sinh(\sqrt{h_2}\xi)) + 2h_2)^2}, \end{aligned} \tag{3.5.2}$$

where $\xi = k(x + \lambda t)$, k and A_0 are determined by (3.3), $h_2 > 0$, h_1, μ, a_1 and λ are the arbitrary constants.

Family 3. When $h_0 = h_1 = h_3 = 0$, we obtain the following solutions for the SLA equation:

$$u_{31}=\frac{4\mu^2h_2\lambda-2\mu a_1h_2+4h_4\lambda}{4(h_4+\mu^2h_2)}+\frac{a_1\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi)}{\left(\mu\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi)+1\right)}, \tag{3.6.1}$$

$$\begin{aligned}v_{31}={}&-\frac{h_4h_2a_1^2}{4(h_4+\mu^2h_2)^2}+\frac{h_2\mu a_1^2\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi)}{2(h_4+\mu^2h_2)\left(\mu\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi)+1\right)}\\&+\frac{a_1^2h_2\operatorname{sech}^2(\sqrt{h_2}\xi)}{2h_4\left(\mu\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi)+1\right)^2}\pm\frac{a_1^2\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi)\tanh(\sqrt{h_2}\xi)\sqrt{h_2}}{2\sqrt{h_4+\mu^2h_2}\left(\mu\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi)+1\right)}\\&\mp\frac{a_1^2\mu h_2^{3/2}\operatorname{sech}^2(\sqrt{h_2}\xi)\tanh(\sqrt{h_2}\xi)}{2h_4\sqrt{h_4+\mu^2h_2}\left(\mu\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\xi)+1\right)^2},\end{aligned} \tag{3.6.2}$$

where $\xi = k(x + \lambda t)$, k is determined by (3.3), $h_2 > 0$, $h_4 < 0$, μ, a_1 and λ are the arbitrary constants.

Family 4. When $h_1 = h_3 = 0$ and $h_0 = -\frac{h_2^2}{4h_4}$, we obtain the following solutions for the SLA equation:

$$u_{41}=a_0+\frac{a_1\sqrt{-2\frac{h_2}{h_4}}\tanh\left(\frac{\sqrt{-2h_2}}{2}\xi\right)}{\mu\sqrt{-2\frac{h_2}{h_4}}\tanh\left(\frac{\sqrt{-2h_2}}{2}\xi\right)+2}, \tag{3.7.1}$$

$$\begin{aligned}v_{41}={}&A_0+\frac{(-\mu^3h_2^2-2h_2h_4\mu)a_1^2\sqrt{-2\frac{h_2}{h_4}}\tanh\left(\frac{\sqrt{-2h_2}}{2}\xi\right)}{(-4h_4^2-4\mu^2h_2h_4-\mu^4h_2^2)\left(\mu\sqrt{-2\frac{h_2}{h_4}}\tanh\left(\frac{\sqrt{-2h_2}}{2}\xi\right)+2\right)}\\&+\frac{a_1^2h_2\tanh^2\left(\frac{\sqrt{-2h_2}}{2}\xi\right)}{h_4\left(\mu\sqrt{-2\frac{h_2}{h_4}}\tanh\left(\frac{\sqrt{-2h_2}}{2}\xi\right)+2\right)^2}\\&\pm\frac{a_1^2\sqrt{-\frac{h_2}{h_4}}\operatorname{sech}^2\left(\frac{\sqrt{-2h_2}}{2}\xi\right)\sqrt{-h_2}}{\sqrt{4h_4+4\mu^2h_2+\frac{\mu^4h_2^2}{h_4}}\left(\mu\sqrt{-2\frac{h_2}{h_4}}\tanh\left(\frac{\sqrt{-2h_2}}{2}\xi\right)+2\right)}\\&\mp\frac{a_1^2\mu h_2\tanh\left(\frac{\sqrt{-2h_2}}{2}\xi\right)\operatorname{sech}^2\left(\frac{\sqrt{-2h_2}}{2}\xi\right)\sqrt{-2h_2}}{\sqrt{4h_4+4\mu^2h_2+\frac{\mu^4h_2^2}{h_4}}h_4\left(\mu\sqrt{-2\frac{h_2}{h_4}}\tanh\left(\frac{\sqrt{-2h_2}}{2}\xi\right)+2\right)^2},\end{aligned} \tag{3.7.2}$$

where $\xi = k(x + \lambda t)$, k, a_0, A_0 are determined by (3.3), $h_2 < 0$, $h_4 > 0$, μ, a_1 and λ are the arbitrary constants.

Family 5. When $h_0 = h_1 = h_4 = 0$, we obtain the following solutions for the SLA equation:

$$u_{51} = \frac{-4\mu^2 h_2\lambda + 2\mu a_1 h_2 + 4\mu h_3\lambda - a_1 h_3}{4(-\mu^2 h_2 + h_3\mu)} - \frac{a_1 h_2 \mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)}{-h_2\mu\,\mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + h_3}, \tag{3.8.1}$$

$$\begin{aligned} v_{51} = {} & \frac{h_3^2 a_1^2}{16(\mu^2 h_2 - h_3\mu)^2} + \frac{(-2h_2\mu + h_3)a_1^2 h_2 \mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)}{4(\mu^2 h_2 - h_3\mu)\left(-h_2\mu\,\mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + h_3\right)} \\ & - \frac{a_1^2 h_2^2 \mathrm{sech}^4\left(\frac{\sqrt{h_2}}{2}\xi\right)}{2\left(-h_2\mu\,\mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + h_3\right)^2} \\ & \pm \frac{a_1^2 h_2^{3/2} \mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right)}{2\sqrt{\mu^2 h_2 - h_3\mu}\left(-h_2\mu\,\mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + h_3\right)} \\ & \mp \frac{a_1^2 \mu h_2^{5/2} \mathrm{sech}^4\left(\frac{\sqrt{h_2}}{2}\xi\right)\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right)}{2\sqrt{\mu^2 h_2 - h_3\mu}\left(-h_2\mu\,\mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + h_3\right)^2}, \end{aligned} \tag{3.8.2}$$

where $\xi = k(x + \lambda t)$, k is determined by (3.3), $h_2 > 0$, h_3, μ, a_1 and λ are the arbitrary constants.

Family 6. When $h_0 = h_1 = 0$, we obtain the following solutions for the SLA equation:

$$u_{61} = \frac{4\mu^2 h_2\lambda - 2\mu a_1 h_2 - 4\mu h_3\lambda + a_1 h_3 + 4h_4\lambda}{4(h_4 + \mu^2 h_2 - h_3\mu)} + \frac{a_1 h_2 \mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)}{h_2\mu\,\mathrm{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + 2\sqrt{h_4 h_2}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3}, \tag{3.9.1}$$

$$
\begin{aligned}
v_{61} = a_0 &- \frac{(-2h_2\mu + h_3)a_1^2 h_2 \operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)}{4(h_4+\mu^2 h_2 - h_3\mu)\left(h_2\mu \operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + 2\sqrt{h_4h_2}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)} \\
&- \frac{a_1^2 h_2^2 \operatorname{sech}^4\left(\frac{\sqrt{h_2}}{2}\xi\right)}{2\left(h_2\mu \operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + 2\sqrt{h_4h_2}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)^2} \\
&\pm \frac{a_1^2 h_2^{3/2}\left(\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right)\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right)\left(2\sqrt{h_4h_2}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right) + \sqrt{h_4h_2}\operatorname{sech}^4\left(\frac{\sqrt{h_2}}{2}\xi\right)\right)}{2\sqrt{h_4+\mu^2h_2-h_3\mu}\left(h_2\mu \operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + 2\sqrt{h_4h_2}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)\left(2\sqrt{h_4h_2}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)} \\
&\pm \frac{a_1^2 h_2^{5/2}\mu \operatorname{sech}^4\left(\frac{\sqrt{h_2}}{2}\xi\right)\left(-2\sqrt{h_4h_2} + \sqrt{h_4h_2}\operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + h_3\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right)\right)}{2\sqrt{h_4+\mu^2h_2-h_3\mu}\left(h_2\mu \operatorname{sech}^2\left(\frac{\sqrt{h_2}}{2}\xi\right) + 2\sqrt{h_4h_2}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)^2\left(2\sqrt{h_4h_2}\tanh\left(\frac{\sqrt{h_2}}{2}\xi\right) - h_3\right)},
\end{aligned}
\tag{3.9.2}
$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.3), $h_2 > 0$, h_3, h_4, μ, a_1 and λ are the arbitrary constants.

Family 7. When $h_2 = h_4 = 0$, we obtain the following solutions for the SLA equation:

$$
\begin{aligned}
u_{71} = &\frac{-4h_0\lambda\mu^4 + 4\mu^3 h_0 a_1 + 4\mu^3 h_1\lambda - 3\mu^2 h_1 a_1 + 4\mu h_3\lambda - a_1 h_3}{4(\mu^3 h_1 + h_3\mu - \mu^4 h_0)} \\
&+ \frac{a_1 \wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right)}{\mu\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right) + 1},
\end{aligned}
\tag{3.10.1}
$$

$$
\begin{aligned}
v_{71} = A_0 &+ \frac{(-4\mu^3 h_0 + 3\mu^2 h_1 + h_3)a_1^2 \wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right)}{4(\mu^3 h_1 + h_3\mu - \mu^4 h_0)\left(\mu\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right) + 1\right)} - \frac{a_1^2 \wp^2\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right)}{2\left(\mu\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right) + 1\right)^2} \\
&\pm \frac{a_1^2\sqrt{h_0 + h_1\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right) + h_3\wp^3\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right)}}{2\sqrt{-\mu^3 h_1 - h_3\mu + \mu^4 h_0}\left(\mu\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right) + 1\right)} \\
&\mp \frac{a_1^2\mu\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right)\sqrt{h_0 + h_1\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right) + h_3\wp^3\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right)}}{2\sqrt{-\mu^3 h_1 - h_3\mu + \mu^4 h_0}\left(\mu\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right) + 1\right)^2},
\end{aligned}
\tag{3.10.2}
$$

where $\xi = k(x + \lambda t)$, $g_2 = -4\frac{h_1}{h_3}$, $g_3 = -4\frac{h_0}{h_3}$, k and A_0 are determined by (3.3), $h_3 > 0$, h_0, h_1, μ, a_1 and λ are the arbitrary constants.

Family 8. When $h_1 = h_3 = 0$, $h_0 = 1$, $h_2 = -(m^2 + 1)$ and $h_4 = m^2$, we obtain the following solutions for the SLA equation:

$$u_{81} = \frac{4\lambda\mu^4 - 4\mu^3 a_1 - 4\mu^2(m^2+1)\lambda + 2\mu a_1(m^2+1) + 4m^2\lambda}{4(m^2 - \mu^2(m^2+1) + \mu^4)} + \frac{a_1 \mathrm{sn}(\xi)}{\mu \mathrm{sn}(\xi) + 1}, \tag{3.11.1}$$

$$\begin{aligned} v_{81} = A_0 &+ \frac{(2\mu^3 - (m^2+1)\mu)a_1^2 \mathrm{sn}(\xi)}{2(m^2 + \mu^2(m^2+1) + \mu^4)(\mu \mathrm{sn}(\xi) + 1)} - \frac{a_1^2 \mathrm{sn}^2(\xi)}{2(\mu \mathrm{sn}(\xi) + 1)^2} \\ &\pm \frac{a_1^2 \mathrm{cn}(\xi)\mathrm{dn}(\xi)}{2\sqrt{m^2 - \mu^2(m^2+1) + \mu^4}(\mu \mathrm{sn}(\xi) + 1)} \\ &\mp \frac{a_1^2 \mu \mathrm{sn}(\xi)\mathrm{cn}(\xi)\mathrm{dn}(\xi)}{2\sqrt{m^2 - \mu^2(m^2+1) + \mu^4}(\mu \mathrm{sn}(\xi) + 1)^2}, \end{aligned} \tag{3.11.2}$$

where $\xi = k(x + \lambda t)$, k and A_0 are determined by (3.3), μ, a_1 and λ are the arbitrary constants.

Family 9. When $h_1 = h_3 = 0$, $h_0 = 1$, $h_2 = -(m^2 + 1)$ and $h_4 = m^2$, we obtain the following solutions for the SLA equation:

$$u_{91} = \frac{4\lambda\mu^4 - 4\mu^3 a_1 - 4\mu^2(m^2+1)\lambda + 2\mu a_1(m^2+1) + 4m^2\lambda}{4(m^2 - \mu^2(m^2+1) + \mu^4)} + \frac{a_1 \mathrm{cd}(\xi)}{\mu \mathrm{cd}(\xi) + 1}, \tag{3.12.1}$$

$$\begin{aligned} v_{91} = A_0 &+ \frac{(2\mu^3 - (m^2+1)\mu)a_1^2 \mathrm{cd}(\xi)}{2(m^2 - \mu^2(m^2+1) + \mu^4)(\mu \mathrm{cd}(\xi) + 1)} - \frac{a_1^2 \mathrm{cd}^2(\xi)}{2(\mu \mathrm{cd}(\xi) + 1)^2} \\ &\pm \frac{a_1^2 \mathrm{sn}(\xi)(1 - m^2)}{2\sqrt{m^2 - \mu^2(m^2+1) + \mu^4}\,\mathrm{dn}^2(\xi)(\mu \mathrm{cd}(\xi) + 1)} \\ &\mp \frac{a_1^2 \mu \mathrm{cd}(\xi)\mathrm{sn}(\xi)(1 - m^2)}{2\sqrt{m^2 - \mu^2(m^2+1) + \mu^4}\,\mathrm{dn}^2(\xi)(\mu \mathrm{cd}(\xi) + 1)^2}, \end{aligned} \tag{3.12.2}$$

where $\xi = k(x + \lambda t)$, k and A_0 are determined by (3.3), μ, a_1 and λ are the arbitrary constants.

Family 10. When $h_1 = h_3 = 0$, $h_0 = 1 - m^2$, $h_2 = 2m^2 - 1$ and $h_4 = -m^2$, we obtain the following solutions for the SLA equation:

$$u_{101} = a_0 + \frac{a_1 \mathrm{cn}(\xi)}{\mu \mathrm{cn}(\xi) + 1}, \tag{3.13.1}$$

$$\begin{aligned} v_{101} = A_0 &+ \frac{(2\mu^3(1-m^2) + (2m^2-1)\mu)a_1^2 \mathrm{cn}(\xi)}{2(-m^2 + \mu^2(2m^2-1) + \mu^4(1-m^2))(\mu \mathrm{cn}(\xi)+1)} \\ &- \frac{a_1^2 \mathrm{cn}^2(\xi)}{2(\mu \mathrm{cn}(\xi)+1)^2} \\ &\pm \frac{a_1^2 \mathrm{dn}(\xi)\mathrm{sn}(\xi)}{2\sqrt{-m^2 + \mu^2(2m^2-1) + \mu^4(1-m^2)}(\mu \mathrm{cn}(\xi)+1)} \\ &\mp \frac{a_1^2 \mu \mathrm{cn}(\xi)\mathrm{dn}(\xi)\mathrm{sn}(\xi)}{2\sqrt{-m^2 + \mu^2(2m^2-1) + \mu^4(1-m^2)}(\mu \mathrm{cn}(\xi)+1)^2}, \end{aligned} \tag{3.13.2}$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.3), μ, a_1 and λ are the arbitrary constants.

Family 11. When $h_1 = h_3 = 0$, $h_0 = m^2 - 1$, $h_2 = 2 - m^2$ and $h_4 = -1$, we obtain the following solutions for the SLA equation:

$$u_{111} = a_0 + \frac{a_1 \mathrm{dn}(\xi)}{\mu \mathrm{dn}(\xi) + 1}, \tag{3.14.1}$$

$$\begin{aligned} v_{111} = A_0 &+ \frac{(-2\mu^3(m^2-1) - (2-m^2)\mu)a_1^2 \mathrm{dn}(\xi)}{2(1 - \mu^2(2-m^2) - \mu^4(m^2-1))(\mu \mathrm{dn}(\xi)+1)} \\ &- \frac{a_1^2 \mathrm{dn}^2(\xi)}{2(\mu \mathrm{dn}(\xi)+1)^2} \\ &\pm \frac{a_1^2 m^2 \mathrm{cn}(\xi)\mathrm{sn}(\xi)}{2\sqrt{-1 + \mu^2(2-m^2) + \mu^4(m^2-1)}(\mu \mathrm{dn}(\xi)+1)} \\ &\mp \frac{a_1^2 \mu \mathrm{dn}(\xi) m^2 \mathrm{cn}(\xi)\mathrm{sn}(\xi)}{2\sqrt{-1 + \mu^2(2-m^2) + \mu^4(m^2-1)}(\mu \mathrm{dn}(\xi)+1)^2}, \end{aligned} \tag{3.14.2}$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.3), μ, a_1 and λ are the arbitrary constants.

Family 12. When $h_1 = h_3 = 0$, $h_0 = m^2$, $h_2 = -(1 + m^2)$ and $h_4 = 1$, we obtain the following solutions for the SLA equation:

$$\begin{aligned} u_{121} = &\frac{4m^2\lambda\mu^4 - 4\mu^3 m^2 a_1 + 4\mu^2(-m^2-1)\lambda - 2\mu a_1(-m^2-1) + 4\lambda}{4(1 - \mu^2(m^2+1) + \mu^4 m^2)} \\ &+ \frac{a_1 \mathrm{ns}(\xi)}{\mu \mathrm{ns}(\xi) + 1}, \end{aligned} \tag{3.15.1}$$

$$\begin{aligned} v_{121} &= A_0 + \frac{(2\mu^3 m^2 - (-m^2-1)\mu)a_1^2 \mathrm{ns}(\xi)}{2(1-\mu^2(m^2+1)+\mu^4 m^2)(\mu\mathrm{ns}(\xi)+1)} - \frac{a_1^2\mathrm{ns}^2(\xi)}{2(\mu\mathrm{ns}(\xi)+1)^2} \\ &\quad \pm \frac{a_1^2\mathrm{cn}(\xi)\mathrm{dn}(\xi)}{2\sqrt{1-\mu^2(m^2+1)+\mu^4 m^2}\mathrm{sn}^2(\xi)(\mu\mathrm{ns}(\xi)+1)} \\ &\quad \mp \frac{a_1^2\mu\mathrm{ns}(\xi)\mathrm{cn}(\xi)\mathrm{dn}(\xi)}{2\sqrt{1-\mu^2(m^2+1)+\mu^4 m^2}\mathrm{sn}^2(\xi)(\mu\mathrm{ns}(\xi)+1)^2}, \end{aligned} \tag{3.15.2}$$

where $\xi = k\,(x + \lambda t)$, k and A_0 are determined by (3.3), μ, a_1 and λ are the arbitrary constants.

Family 13. When $h_1 = h_3 = 0$, $h_0 = m^2$, $h_2 = -(1 + m^2)$ and $h_4 = 1$, we obtain the following solutions for the SLA equation:

$$u_{131} = a_0 + \frac{a_1\mathrm{dc}(\xi)}{\mu\mathrm{dc}(\xi)+1}, \tag{3.16.1}$$

$$\begin{aligned} v_{131} &= A_0 + \frac{(2\mu^3 m^2 - (m^2+1)\mu)a_1^2 \mathrm{dc}(\xi)}{2(1-\mu^2(m^2+1)+\mu^4 m^2)(\mu\mathrm{dc}(\xi)+1)} - \frac{a_1^2\mathrm{dc}^2(\xi)}{2(\mu\mathrm{dc}(\xi)+1)^2} \\ &\quad \pm \frac{a_1^2\mathrm{sn}(\xi)(1-m^2)}{2\sqrt{1-\mu^2(m^2+1)+\mu^4 m^2}\mathrm{cn}^2(\xi)(\mu\mathrm{dc}(\xi)+1)} \\ &\quad \mp \frac{a_1^2\mu\mathrm{dc}(\xi)\mathrm{sn}(\xi)(1-m^2)}{2\sqrt{1-\mu^2(m^2+1)+\mu^4 m^2}\mathrm{cn}^2(\xi)(\mu\mathrm{dc}(\xi)+1)^2}, \end{aligned} \tag{3.16.2}$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.3), μ, a_1 and λ are the arbitrary constants.

Family 14. When $h_1 = h_3 = 0$, $h_0 = -m^2$, $h_2 = 2m^2 - 1$ and $h_4 = 1 - m^2$, we obtain the following solutions for the SLA equation:

$$u_{141} = a_0 + \frac{a_1\mathrm{nc}(\xi)}{\mu\mathrm{nc}(\xi)+1}, \tag{3.17.1}$$

$$\begin{aligned} v_{141} &= A_0 + \frac{(-2\mu^3 m^2 + (2m^2-1)\mu)a_1^2 \mathrm{nc}(\xi)}{4(1-m^2+\mu^2(2m^2-1)-\mu^4 m^2)(\mu\mathrm{nc}(\xi)+1)} - \frac{a_1^2\mathrm{nc}^2(\xi)}{2(\mu\mathrm{nc}(\xi)+1)^2} \\ &\quad \pm \frac{a_1^2\mathrm{dn}(\xi)\mathrm{sn}(\xi)}{2\sqrt{1-m^2+\mu^2(2m^2-1)-\mu^4 m^2}\mathrm{cn}^2(\xi)(\mu\mathrm{nc}(\xi)+1)} \\ &\quad \mp \frac{a_1^2\mu\mathrm{nc}(\xi)\mathrm{dn}(\xi)\mathrm{sn}(\xi)}{2\sqrt{1-m^2+\mu^2(2m^2-1)-\mu^4 m^2}\mathrm{cn}^2(\xi)(\mu\mathrm{nc}(\xi)+1)^2}, \end{aligned} \tag{3.17.2}$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.3), μ, a_1 and λ are the arbitrary constants.

Family 15. When $h_1 = h_3 = 0$, $h_0 = -1$, $h_2 = 2 - m^2$ and $h_4 = m^2 - 1$, we obtain the following solutions for the SLA equation:

$$u_{151} = \frac{4\lambda\mu^4 - 4\mu^3 a_1 - 4\mu^2(2-m^2)\lambda + 2\mu a_1(2-m^2) - 4(m^2-1)\lambda}{4(1-m^2-\mu^2(2-m^2)+\mu^4)} + \frac{a_1 \mathrm{nd}(\xi)}{\mu \mathrm{nd}(\xi)+1}, \tag{3.18.1}$$

$$v_{151} = A_0 + \frac{(2\mu^3-(2-m^2)\mu)a_1^2 \mathrm{nd}(\xi)}{2(1-m^2-\mu^2(2-m^2)+\mu^4)(\mu \mathrm{nd}(\xi)+1)} - \frac{a_1^2 \mathrm{nd}^2(\xi)}{2(\mu \mathrm{nd}(\xi)+1)^2} \pm \frac{a_1^2 m^2 \mathrm{cn}(\xi)\mathrm{sn}(\xi)}{2\sqrt{-1+m^2+\mu^2(2-m^2)-\mu^4}\,\mathrm{dn}^2(\xi)(\mu \mathrm{nd}(\xi)+1)} \mp \frac{a_1^2 \mu \mathrm{nd}(\xi) m^2 \mathrm{cn}(\xi)\mathrm{sn}(\xi)}{2\sqrt{-1+m^2+\mu^2(2-m^2)-\mu^4}\,\mathrm{dn}^2(\xi)(\mu \mathrm{nd}(\xi)+1)^2}, \tag{3.18.2}$$

where $\xi = k(x + \lambda t)$, k and A_0 are determined by (3.3), μ, a_1 and λ are the arbitrary constants.

Family 16. When $h_1 = h_3 = 0$, $h_0 = 1 - m^2$, $h_2 = 2 - m^2$ and $h_4 = 1$, we obtain the following solutions for the SLA equation:

$$u_{161} = a_0 + \frac{a_1 \mathrm{cs}(\xi)}{\mu \mathrm{cs}(\xi)+1}, \tag{3.19.1}$$

$$v_{161} = A_0 + \frac{(2\mu^3(1-m^2)+(2-m^2)\mu)a_1^2 \mathrm{cs}(\xi)}{2(1+\mu^2(2-m^2)+\mu^4(1-m^2))(\mu \mathrm{cs}(\xi)+1)} - \frac{a_1^2 \mathrm{cs}^2(\xi)}{2(\mu \mathrm{cs}(\xi)+1)^2} \pm \frac{a_1^2 \mathrm{dn}(\xi)}{2\sqrt{1+\mu^2(2-m^2)+\mu^4(1-m^2)}\,\mathrm{sn}^2(\xi)(\mu \mathrm{cs}(\xi)+1)} \mp \frac{a_1^2 \mu \mathrm{cs}(\xi)\mathrm{dn}(\xi)}{2\sqrt{1+\mu^2(2-m^2)+\mu^4(1-m^2)}\,\mathrm{sn}^2(\xi)(\mu \mathrm{cs}(\xi)+1)^2}, \tag{3.19.2}$$

where $\xi = k\ (x + \lambda t)$, k, a_0 and A_0 are determined by (3.3), μ, a_1 and λ are the arbitrary constants.

Family 17. When $h_1 = h_3 = 0$, $h_0 = 1$, $h_2 = 2 - m^2$ and $h_4 = 1 - m^2$, we obtain the following solutions for the SLA equation:

$$u_{171} = \frac{4\lambda\mu^4 - 4\mu^3 a_1 + 4\mu^2(2-m^2)\lambda - 2\mu a_1(2-m^2) + 4(1-m^2)\lambda}{4(1-m^2+\mu^2(2-m^2)+\mu^4)} + \frac{a_1 \mathrm{sc}(\xi)}{\mu \mathrm{sc}(\xi)+1}, \tag{3.20.1}$$

$$
\begin{aligned}
v_{171} = A_0 &+ \frac{(2\mu^3 + (2-m^2)\mu)a_1^2\mathrm{sc}(\xi)}{2(1-m^2+\mu^2(2-m^2)+\mu^4)(\mu\mathrm{sc}(\xi)+1)} - \frac{a_1^2\mathrm{sc}^2(\xi)}{2(\mu\mathrm{sc}(\xi)+1)^2} \\
&\pm \frac{a_1^2\mathrm{dn}(\xi)}{2\sqrt{1-m^2+\mu^2(2-m^2)+\mu^4}\mathrm{cn}^2(\xi)(\mu\mathrm{sc}(\xi)+1)} \\
&\mp \frac{a_1^2\mu\mathrm{sc}(\xi)\mathrm{dn}(\xi)}{2\sqrt{1-m^2+\mu^2(2-m^2)+\mu^4}\mathrm{cn}^2(\xi)(\mu\mathrm{sc}(\xi)+1)^2}, \qquad (3.20.2)
\end{aligned}
$$

where $\xi = k(x + \lambda t)$, k and A_0 are determined by (3.3), μ, a_1 and λ are the arbitrary constants.

Family 18. When $h_1 = h_3 = 0$, $h_0 = 1$, $h_2 = 2m^2 - 1$ and $h_4 = m^2(m^2 - 1)$, we obtain the following solutions for the SLA equation:

$$
u_{181} = a_0 + \frac{a_1\mathrm{sd}(\xi)}{\mu\mathrm{sd}(\xi)+1}, \qquad (3.21.1)
$$

$$
\begin{aligned}
v_{181} = A_0 &+ \frac{(2\mu^3 + (2m^2-1)\mu)a_1^2\mathrm{sd}(\xi)}{2(m^2(m^2-1)+\mu^2(2m^2-1)+\mu^4)(\mu\mathrm{sd}(\xi)+1)} - \frac{a_1^2\mathrm{sd}^2(\xi)}{2(\mu\mathrm{sd}(\xi)+1)^2} \\
&\pm \frac{a_1^2\mathrm{cn}(\xi)}{2\sqrt{m^2(m^2-1)+\mu^2(2m^2-1)+\mu^4}\mathrm{dn}^2(\xi)(\mu\mathrm{sd}(\xi)+1)} \\
&\mp \frac{a_1^2\mu\mathrm{sd}(\xi)\mathrm{cn}(\xi)}{2\sqrt{m^2(m^2-1)+\mu^2(2m^2-1)+\mu^4}\mathrm{dn}^2(\xi)(\mu\mathrm{sd}(\xi)+1)^2}, \qquad (3.21.2)
\end{aligned}
$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.3), μ, a_1 and λ are the arbitrary constants.

Family 19. When $h_1 = h_3 = 0$, $h_0 = m^2(m^2 - 1)$, $h_2 = 2m^2 - 1$ and $h_4 = 1$, we obtain the following solutions for the SLA equation:

$$
u_{191} = a_0 + \frac{a_1\mathrm{ds}(\xi)}{\mu\mathrm{ds}(\xi)+1}, \qquad (3.22.1)
$$

$$
\begin{aligned}
v_{191} = A_0 &+ \frac{(2\mu^3 m^2(m^2-1) + (2m^2-1)\mu)a_1^2\mathrm{ds}(\xi)}{2(1+\mu^2(2m^2-1)+\mu^4 m^2(m^2-1))(\mu\mathrm{ds}(\xi)+1)} \\
&- \frac{a_1^2\mathrm{ds}^2(\xi)}{2(\mu\mathrm{ds}(\xi)+1)^2} \\
&\pm \frac{a_1^2\mathrm{cn}(\xi)}{2\sqrt{1+\mu^2(2m^2-1)+\mu^4 m^2(m^2-1)}\mathrm{sn}^2(\xi)(\mu\mathrm{ds}(\xi)+1)} \\
&\mp \frac{a_1^2\mu\mathrm{ds}(\xi)\mathrm{cn}(\xi)}{\sqrt{1+\mu^2(2m^2-1)+\mu^4 m^2(m^2-1)}\mathrm{sn}^2(\xi)(\mu\mathrm{ds}(\xi)+1)^2},
\end{aligned}
$$

(3.22.2)

where $\xi = k\ (x + \lambda t)$, k, a_0 and A_0 are determined by (3.3), μ, a_1 and λ are the arbitrary constants.

Family 20. When $h_1 = h_3 = 0$, $h_0 = 1/4$, $h_2 = \frac{1-2m^2}{2}$ and $h_4 = 1/4$, we obtain the following solutions for the SLA equation:

$$u_{201} = a_0 + \frac{a_1(\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi))}{\mu(\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi)) + 1}, \tag{3.23.1}$$

$$\begin{aligned} v_{201} = A_0 &+ \frac{(\mu^3 + (1-m^2)\mu)a_1^2(\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi))}{(1+2\mu^2(1-m^2)+\mu^4)(\mu(\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi))+1)} \\ &- \frac{a_1^2(\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi))^2}{2(\mu(\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi))+1)^2} \\ &\pm \frac{a_1^2(\mathrm{cn}(\xi)\mathrm{dn}(\xi) \pm \mathrm{dn}(\xi))}{\sqrt{1+2\mu^2(1-m^2)+\mu^4}\,\mathrm{sn}^2(\xi)(\mu(\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi))+1)} \\ &\pm \frac{a_1^2\mu(\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi))(\mathrm{cn}(\xi)\mathrm{dn}(\xi) \pm \mathrm{dn}(\xi))}{\sqrt{1+2\mu^2(1-m^2)+\mu^4}\,\mathrm{sn}^2(\xi)(\mu(\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi))+1)^2}, \end{aligned} \tag{3.23.2}$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.3), μ, a_1 and λ are the arbitrary constants.

Family 21. When $h_1 = h_3 = 0$, $h_0 = \frac{1-m^2}{4}$, $h_2 = \frac{1+m^2}{2}$ and $h_4 = \frac{1-m^2}{4}$, we obtain the following solutions for the SLA equation:

$$u_{211} = a_0 + \frac{a_1(\mathrm{nc}(\xi) \pm \mathrm{sc}(\xi))}{\mu(\mathrm{nc}(\xi) \pm \mathrm{sc}(\xi)) + 1}, \tag{3.24.1}$$

$$\begin{aligned} v_{211} = A_0 &+ \frac{(\mu^3(1-m^2) + (1+m^2)\mu)a_1^2(\mathrm{nc}(\xi) \pm \mathrm{sc}(\xi))}{(1-m^2+2\mu^2(1+m^2)+\mu^4(1-m^2))(\mu(\mathrm{nc}(\xi) \pm \mathrm{sc}(\xi))+1)} \\ &- \frac{a_1^2(\mathrm{nc}(\xi) \pm \mathrm{sc}(\xi))^2}{2(\mu(\mathrm{nc}(\xi) \pm \mathrm{sc}(\xi))+1)^2} \\ &\pm \frac{a_1^2(\mathrm{dn}(\xi)\mathrm{sn}(\xi) \pm \mathrm{dn}(\xi))}{\sqrt{1-m^2+2\mu^2(1+m^2)+\mu^4(1-m^2)}\,\mathrm{cn}^2(\xi)(\mu(\mathrm{nc}(\xi) \pm \mathrm{sc}(\xi,m))+1)} \\ &\pm \frac{a_1^2\mu(\mathrm{nc}(\xi) \pm \mathrm{sc}(\xi))(\mathrm{dn}(\xi)\mathrm{sn}(\xi) \pm \mathrm{dn}(\xi))}{\sqrt{1-m^2+2\mu^2(1+m^2)+\mu^4(1-m^2)}\,\mathrm{cn}^2(\xi)(\mu(\mathrm{nc}(\xi) \pm \mathrm{sc}(\xi,m))+1)^2}, \end{aligned} \tag{3.24.2}$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.3), μ, a_1 and λ are the arbitrary constants.

Family 22. When $h_1 = h_3 = 0$, $h_0 = m^4/4$, $h_2 = \frac{m^2-2}{2}$ and $h_0 = 1/4$, we obtain the following solutions for the SLA equation:

$$u_{221} = a_0 + \frac{a_1(\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi))}{\mu(\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi)) + 1}, \tag{3.25.1}$$

$$
\begin{aligned}
v_{221} = A_0 &+ \frac{(\mu^3 m^4 + (m^2-2)\mu)a_1^2(\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi))}{(1+2\mu^2(m^2-2)+\mu^4 m^4)(\mu(\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi))+1)} \\
&- \frac{a_1^2(\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi))^2}{2(\mu(\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi))+1)^2} \\
&\pm \frac{a_1^2(\mathrm{cn}(\xi)\mathrm{dn}(\xi) \pm \mathrm{cn}(\xi))}{\sqrt{1+2\mu^2(m^2-2)+\mu^4 m^4}\,\mathrm{sn}^2(\xi)(\mu(\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi))+1)} \\
&\pm \frac{a_1^2\mu(\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi))(\mathrm{cn}(\xi)\mathrm{dn}(\xi) \pm \mathrm{cn}(\xi))}{\sqrt{1+2\mu^2(m^2-2)+\mu^4 m^4}\,\mathrm{sn}^2(\xi)(\mu(\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi))+1)^2},
\end{aligned}
\tag{3.25.2}
$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.3), μ, a_1 and λ are the arbitrary constants.

Family 23. When $h_1 = h_3 = 0$, $h_0 = m^2/4$, $h_2 = \frac{m^2-2}{2}$ and $h_4 = m^2/4$, we obtain the following solutions for the SLA equation:

$$
u_{231} = a_0 + \frac{a_1(\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi))}{\mu(\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi))+1}, \tag{3.26.1}
$$

$$
\begin{aligned}
v_{231} = A_0 &+ \frac{(\mu^3 m^2 + (m^2-2)\mu)a_1^2(\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi))}{(m^2+2\mu^2(m^2-2)+\mu^4 m^2)(\mu(\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi))+1)} \\
&- \frac{a_1^2(\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi))^2}{2(\mu(\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi))+1)^2} \\
&\pm \frac{a_1^2(\mathrm{cn}(\xi)\mathrm{dn}(\xi) \mp \mathrm{idn}(\xi)\mathrm{sn}(\xi))}{\sqrt{m^2+2\mu^2(m^2-2)+\mu^4 m^2}(\mu(\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi))+1)} \\
&\pm \frac{a_1^2\mu(\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi))(\mathrm{cn}(\xi)\mathrm{dn}(\xi) \mp \mathrm{idn}(\xi)\mathrm{sn}(\xi))}{\sqrt{m^2+2\mu^2(m^2-2)+\mu^4 m^2}(\mu(\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi))+1)^2},
\end{aligned}
\tag{3.26.2}
$$

where $\xi = k(x + \lambda t)$, k, a_0 and A_0 are determined by (3.3), μ, a_1 and λ are the arbitrary constants.

4. Summary and conclusions

In summary, we have proposed a unified algebraic method: elliptic equation rational expansion (EERE) method with symbolic computation, which greatly exceeds the applicability of the existing the tanh method, the extended tanh method, the general tanh method, the projective Riccati equation method and the general projective Riccati equation method in obtaining multiple travelling wave solutions of general nonlinear evolution equations. With the aid of the *Maple*, we apply the EERE method to consider the shallow long wave approximate equation and obtain rich new families of the exact solutions, including rational form solitary wave, rational form triangular periodic, rational form Jacobi and Weierstrass doubly periodic solutions.

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