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A multiple Riccati equations rational expansion method and novel solutions of the Broer–Kaup–Kupershmidt system

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Abstract

To construct exact solutions of nonlinear partial differential equation, a multiple Riccati equations rational expansion method (MRERE) is presented and a series of novel solutions of the Broer–Kaup–Kupershmidt system are found. The novel solutions obtained by MRERE method include solutions of hyperbolic (solitary) function and triangular periodic functions appearing at the same time.

1. Introduction

In the past decades, both mathematicians and physicists have devoted considerable effort to the study of solitons and related issue of the construction of solutions to nonlinear partial differential equations (PDEs) [1–10]. Recently, we present various rational expansion methods [10,11] to construct rational formal exact solutions of nonlinear PDEs. Advantages of these rational expansion methods are that the more general rational ansätz form or more subequation is used to reduce the target equation, the more general rational styles of exact solutions of nonlinear PDEs can be found. However in traditional subequation methods [6–11] the variables used in an ansätz always satisfy the same subequation or subequations. The present work is motivated by the desire to extend our work [10,11] to set up a new arithmetic, named multiple Riccati equations rational expansion method (MRERE), to construct new styles of solutions of nonlinear PDEs. We use two or more variables which satisfy different Riccati equations, in which different parameter is chosen independently. To our pleasantly surprised, in this way, we can construct many families of novel solution of some nonlinear PDEs, in which hyperbolic (solitary) function and triangular periodic functions can appear in a solution at same time.

We use this new arithmetic to the Broer–Kaup–Kupershmidt system to test the validity of our new method. As a result, we find some novel solutions. To my knowledge, these new styles of solutions do not be found before.

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2. Multiple Riccati equation rational expansion method

In the following we would like to outline the main steps of our method:

Step 1. Given a system of polynomial PDE with constant coefficients, with some physical fields $u_i(x,y,t)$ in three variables x, y, t,

$$\Delta(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \ldots) = 0, \tag{2.1}$$

use the wave transformation

$$u_i(x,y,t) = U_i(\xi), \quad \xi = k(x + ly + \lambda t), \tag{2.2}$$

where k, l and λ are constants to be determined later. Then the nonlinear partial differential system (2.1) is reduced to a nonlinear ordinary differential system:

$$\Theta(U_i, U_i', U_i'', \ldots) = 0. \tag{2.3}$$

Step 2. We introduce a new ansätz in terms of finite rational formal expansion in the following forms:

$$U_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} \phi^{r_{j1}}(\xi)\psi^{r_{j2}}(\xi)}{(\mu_1\phi(\xi) + \mu_2\psi(\xi) + 1)^j}, \tag{2.4}$$

where $a^j_{r_{j1}r_{j2}}$, μ_1 and μ_2 ($r_{jn} = 0,1,\ldots,j$; $n = 1,2$) are constants to be determined later and the new variables $\phi = \phi(\xi)$ and $\psi = \psi(\xi)$ satisfy the Riccati equation, i.e.,

$$\frac{d\phi}{d\xi} = h_1 + h_2\phi^2, \quad \frac{d\psi}{d\xi} = h_3 + h_4\psi^2, \tag{2.5}$$

where h_1, h_2, h_3 and h_4 are constants.

Step 3. Determine the m_i of the rational formal polynomial solutions (2.4) by, respectively, balancing the highest nonlinear terms and the highest-order partial derivative terms in the given system equations (see Refs. [6–11] for details), and then give the formal solutions.

Step 4. Substitute (2.4) into (2.3) along with (2.5) and then set all coefficients of $\phi^p(\xi)\psi^q(\xi)$, ($p = 0,1,2,\ldots$; $q = 0,1,2,\ldots$) of the resulting system's numerator to be zero to get an over-determined system of nonlinear algebraic equations with respect to k, μ_1, μ_2 and $a^j_{r_{j1}r_{j2}}$ ($r_{jn} = 0,1,\ldots,j$; $n = 1,2$).

Step 5. By solving the over-determined system of nonlinear algebraic equations by use of symbolic computation system *Maple*, we end up with the explicit expressions for k, μ_1, μ_2 and $a^j_{r_{j1}r_{j2}}$ ($r_{jn} = 0,1,\ldots,j$; $n = 1,2$).

Step 6. According to system (2.2), (2.4), the conclusions in *Step 5* and the general solutions of system (2.5) which can be seen in Appendix A, we can obtain rational formal exact solutions of system (2.1).

Remark 1. The MRERE method is more general than various existing methods [6–8] for finding exact solutions of nonlinear PDEs. The appeal and success of the method lies in the fact that writing the exact solutions of a nonlinear partial differential system as polynomials of ϕ and ψ whose derivations are in closed form, the equation can be changed into a nonlinear system of algebraic equations. The system can be solved with the help of symbolic computation. Note that: The projective Riccati equation expansion method which can also change an equation to a nonlinear system of algebraic equations is very like our method, but it dos not find the real reason why projective Riccati equation are in closed-form is that hyperbolic functions and triangular functions in solutions are in closed-form. So it is just particular case of our method.

Remark 2. We can easily see that when $h_1 \neq h_3$ or $h_2 \neq h_4$, ϕ and ψ satisfy the different Riccati equation, so hyperbolic functions and triangular functions can appear in a solution at the same time. For example, according to Appendix A, when $h_1 = -\frac{1}{2}$, $h_2 = \frac{1}{2}$ and $h_3 = h_4 = \pm\frac{1}{2}$, we can get following particular solution:

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} (\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))^{r_{j1}} (\sec(\xi) \pm \tan(\xi))^{r_{j2}}}{(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^j}, \tag{2.6.1}$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} (\coth(\xi) \pm \mathrm{csch}(\xi))^{r_{j1}} (\sec(\xi) \pm \tan(\xi))^{r_{j2}}}{(\mu_1(\coth(\xi) \pm \mathrm{csch}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^j}, \tag{2.6.2}$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} (\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))^{r_{j1}} (\csc(\xi) \pm \cot(\xi))^{r_{j2}}}{(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)^j}, \tag{2.6.3}$$

$$u_i = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} (\coth(\xi) \pm \mathrm{csch}(\xi))^{r_{j1}} (\csc(\xi) \pm \cot(\xi))^{r_{j2}}}{(\mu_1(\coth(\xi) \pm \mathrm{csch}(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)^j}. \tag{2.6.4}$$

These solutions have not been obtained by any other Riccati equation expansion methods or projective Riccati equation expansion methods.

Remark 3. Recently, there are a lot of papers of subequation method focus their attentions on generalize auxiliary equations or get more solutions of existing auxiliary equations to get more generalized solutions of target equations. But, to our knowledge, these methods can all be summarized as following rational expansion form:

$$u(\xi) = a_0 + \sum_{j=1}^{m} \frac{\sum_{r_1+\cdots+r_n=j} a^j_{r_1\cdots r_n} F_1^{r_1} \cdots F_n^{r_n}}{\sum_{r_1+\cdots+r_n=j} b^j_{r_1\cdots r_n} F_1^{l_1} \cdots F_n^{l_n} + b_0}, \tag{2.7}$$

where $a_0, a^j_{r_1\cdots r_n}, b^j_{r_1\cdots r_n}$ and ξ are differentiable function to be determined and $\frac{\mathrm{d}F_i}{\mathrm{d}\xi} = K_i(F_1,\ldots,F_n)$, where K_i are polynomial of F_i. It is clearly to see that (2.7) is also satisfying solving the recurrent relation or derivative relation for the terms of polynomial for computation closed. And according to this formula, we think we can get more generalized solutions of target equations by using simple auxiliary equation or simple solutions of auxiliary equation. This work will be continued in future.

3. Exact solutions of the Broer–Kaup–Kupershmidt system

Let us consider the Broer–Kaup–Kupershmidt (BKK) system,

$$\begin{aligned} &H_{ty} - H_{xxy} + 2(HH_x)_y + 2G_{xx} = 0, \\ &G_t + G_{xx} + 2(HG)_x = 0. \end{aligned} \tag{3.1}$$

The BKK system may be derived from the parameter dependent symmetry constraint of the Kadomtsev–Petviashvili (KP) equation [12]. For more details about the results about this system, the reader is advised to see the achievements in Refs. [12–16].

In order to get some families of rational formal wave solutions to the BKK system, by considering the wave transformations $H(x,y,t) = \mathscr{H}(\xi)$, $G(x,y,t) = \mathscr{G}(\xi)$ and $\xi = k(x + ly + \lambda t)$, we change (3.1) to the form

$$\begin{aligned} &\lambda l\mathscr{H}'' - lk\mathscr{H}''' + 2l(\mathscr{H}\mathscr{H}')' + 2\mathscr{G}'' = 0, \\ &\lambda\mathscr{G}' + k\mathscr{G}'' + 2(\mathscr{G}\mathscr{H})' = 0. \end{aligned} \tag{3.2}$$

For the BKK system, by balancing the highest nonlinear terms and the highest-order partial derivative terms in (3.2), we suppose (3.2) have the following formal travelling wave solution:

$$\begin{aligned} &\mathscr{H}(\xi) = a_0 + \frac{a_1\phi + b_1\psi}{\mu_1\phi + \mu_2\psi + 1}, \\ &\mathscr{G}(\xi) = A_0 + \frac{A_1\phi + B_1\psi}{\mu_1\phi + \mu_2\psi + 1} + \frac{A_2\phi^2 + B_2\phi\psi + C_1\psi^2}{(\mu_1\phi + \mu_2\psi + 1)^2}, \end{aligned} \tag{3.3}$$

where μ_1, μ_2, a_0, a_i, b_i, A_0, A_i, B_i and $C_1 (i = 1, 2)$ are constants to be determined later and the new variables ϕ and ψ satisfy (2.5).

With the aid of *Maple*, substituting (3.3) along with (2.5) into (3.2) and setting the coefficients of these terms $\phi^i\psi^j$ to be zero yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, A_0, A_1, B_1 A_2, B_2, C_1, μ_1, μ_2 and k.

By use of the *Maple* soft package "Charsets" by Dongming Wang, which is based on the Wu-elimination method, solving the over-determined algebraic equations, we get the following results:

$$A_1 = -\frac{\mu_1(2kh_1\mu_1 + 2a_0 + \lambda + 2k\mu_2h_3)b_1l}{2\mu_2}, \quad B_1 = -\frac{1}{2}(2kh_1\mu_1 + 2a_0 + \lambda + 2k\mu_2h_3)b_1l,$$
$$A_2 = \frac{lb_1\mu_1(h_2k\mu_2 + \mu_1^2\mu_2kh_1 + \mu_1\mu_2^2kh_3 - b_1\mu_1)}{2\mu_2^2}, \quad B_2 = \frac{(\mu_2kh_1\mu_1 - b_1 + \mu_2^2kh_3)lb_1\mu_1}{\mu_2}, \quad a_1 = \frac{b_1\mu_1}{\mu_2}$$
$$C_1 = \frac{1}{2}lb_1(h_4k - b_1 + \mu_2kh_1\mu_1 + \mu_2^2kh_3), \tag{3.4}$$

where a_0, A_0, b_1, μ_1, μ_2, k, l and λ are arbitrary constants.

According to (3.3), (3.4) and the general solutions of (2.5) listed in Appendix A, we will obtain the following wave solutions for BKK system.

Note that: Since the solutions obtained here are so many, we just list some new solutions for the BKK system to illustrate the efficiency of our method.

Family 1. When $h_1 = h_3 = \frac{1}{2}$ and $h_2 = h_4 = -\frac{1}{2}$, then we obtain following solutions:

$$H_1 = a_0 + \frac{b_1\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + b_1\mu_2(\coth(\xi) \pm \mathrm{csch}(\xi))}{\mu_2(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\coth(\xi) \pm \mathrm{csch}(\xi)) + 1)}, \tag{3.5.1}$$
$$\begin{aligned} G_1 = A_0 &- \frac{(k\mu_1 + 2a_0 + \lambda + k\mu_2)b_1l(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\coth(\xi) \pm \mathrm{csch}(\xi)))}{2\mu_2(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\coth(\xi) \pm \mathrm{csch}(\xi)) + 1)} \\ &+ \frac{lb_1\mu_1(-k\mu_2 + \mu_1^2\mu_2k + \mu_1\mu_2^2k - 2b_1\mu_1)(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))^2}{4\mu_2^2(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\coth(\xi) \pm \mathrm{csch}(\xi)) + 1)^2} \\ &+ \frac{(\mu_2k\mu_1 - 2b_1 + \mu_2^2k)lb_1\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))(\coth(\xi) \pm \mathrm{csch}(\xi))}{2\mu_2(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\coth(\xi) \pm \mathrm{csch}(\xi)) + 1)^2} \\ &+ \frac{lb_1(k - 2b_1 + \mu_2k\mu_1 + \mu_2^2k)(\coth(\xi) \pm \mathrm{csch}(\xi))^2}{2(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\coth(\xi) \pm \mathrm{csch}(\xi)) + 1)^2}, \end{aligned} \tag{3.5.2}$$

where $\xi = k(x + ly + \lambda t)$, a_0, A_0, b_1, μ_1, μ_2, k, l and λ are arbitrary constants.

Family 2. When $h_1 = h_2 = \pm\frac{1}{2}$ and $h_3 = h_4 = \pm\frac{1}{2}$, then we obtain following solutions:

$$H_2 = a_0 + \frac{b_1\mu_1(\sec(\xi) + \tan(\xi)) + b_1\mu_2(\csc(\xi) \pm \cot(\xi))}{\mu_2(\mu_1(\sec(\xi) \pm \tan(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)}, \tag{3.6.1}$$
$$\begin{aligned} G_2 = A_0 &- \frac{(\pm k\mu_1 + 2a_0 + \lambda \pm k\mu_2)b_1l(\mu_1(\sec(\xi) \pm \tan(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)))}{2\mu_2(\mu_1(\sec(\xi) \pm \tan(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)} \\ &+ \frac{lb_1\mu_1(\pm k\mu_2 \pm \mu_1^2\mu_2k \pm \mu_1\mu_2^2k - 2b_1\mu_1)(\sec(\xi) + \tan(\xi))^2}{4\mu_2^2(\mu_1(\sec(\xi) \pm \tan(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)^2} \\ &+ \frac{(\pm\mu_2k\mu_1 - 2b_1 \pm \mu_2^2k)lb_1\mu_1(\sec(\xi) \pm \tan(\xi))(\csc(\xi) \pm \cot(\xi))}{2\mu_2(\mu_1(\sec(\xi) \pm \tan(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)^2} \\ &+ \frac{lb_1(\pm k - 2b_1 \pm \mu_2k\mu_1 \pm \mu_2^2k)(\csc(\xi) \pm \cot(\xi))^2}{4(\mu_1(\sec(\xi) \pm \tan(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)^2}, \end{aligned} \tag{3.6.2}$$

where $\xi = k(x + ly + \lambda t)$, a_0, A_0, b_1, μ_1, μ_2, k, l and λ are arbitrary constants.

Family 3. When $h_1 = -\frac{1}{2}$, $h_2 = \frac{1}{2}$ and $h_3 = h_4 = \pm\frac{1}{2}$, then we obtain following solutions:

$$H_3 = a_0 + \frac{b_1\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + b_1\mu_2(\sec(\xi) \pm \tan(\xi))}{\mu_2(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)}, \tag{3.7.1}$$
$$\begin{aligned} G_3 = A_0 &- \frac{(-k\mu_1 + 2a_0 + \lambda \pm k\mu_2)b_1l(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)))}{2\mu_2(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)} \\ &+ \frac{lb_1\mu_1(k\mu_2 - \mu_1^2\mu_2k \pm \mu_1\mu_2^2k - 2b_1\mu_1)(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))^2}{4\mu_2^2(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^2} \\ &+ \frac{(-\mu_2k\mu_1 - 2b_1 \pm \mu_2^2k)lb_1\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))(\sec(\xi) \pm \tan(\xi))}{2\mu_2(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^2} \\ &+ \frac{lb_1(\pm k - 2b_1 - \mu_2k\mu_1 \pm \mu_2^2k)(\sec(\xi) \pm \tan(\xi))^2}{4(\mu_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^2}, \end{aligned} \tag{3.7.2}$$

where $\xi = k(x + ly + \lambda t)$, a_0, A_0, b_1, μ_1, μ_2, k, l and λ are arbitrary constants.

Family 4. When $h_1 = -\frac{1}{2}$, $h_2 = \frac{1}{2}$ and $h_3 = h_4 = \pm\frac{1}{2}$, then we obtain following solutions:

$$H_4 = a_0 + \frac{b_1\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + b_1\mu_2(\sec(\xi) \pm \tan(\xi))}{\mu_2(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)}, \tag{3.8.1}$$

$$\begin{aligned} G_4 = A_0 &- \frac{(-k\mu_1 + 2a_0 + \lambda \pm k\mu_2)b_1 l(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)))}{2\mu_2(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)} \\ &+ \frac{lb_1\mu_1(k\mu_2 - \mu_1^2\mu_2 k \pm \mu_1\mu_2^2 k - 2b_1\mu_1)(\coth(\xi) \pm \operatorname{csch}(\xi))^2}{4\mu_2^2(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^2} \\ &+ \frac{(-\mu_2 k\mu_1 - 2b_1 \pm \mu_2^2 k)lb_1\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi))(\sec(\xi) \pm \tan(\xi))}{2\mu_2(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^2} \\ &+ \frac{lb_1(\pm k - 2b_1 - \mu_2 k\mu_1 \pm \mu_2^2 k)(\sec(\xi) \pm \tan(\xi))^2}{(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)^2}, \end{aligned} \tag{3.8.2}$$

where $\xi = k(x + ly + \lambda t)$, a_0, A_0, b_1, μ_1, μ_2, k, l and λ are arbitrary constants.

Family 5. When $h_1 = -\frac{1}{2}$, $h_2 = \frac{1}{2}$ and $h_3 = h_4 = \pm\frac{1}{2}$, then we obtain following solutions:

$$H_5 = a_0 + \frac{b_1\mu_1(\tanh(\xi) \pm i\operatorname{sech}(\xi)) + b_1\mu_2(\csc(\xi) \pm \cot(\xi))}{\mu_2(\mu_1(\tanh(\xi) \pm i\operatorname{sech}(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)}, \tag{3.9.1}$$

$$\begin{aligned} G_5 = A_0 &- \frac{(-k\mu_1 + 2a_0 + \lambda \pm k\mu_2)b_1 l(\mu_1(\tanh(\xi) \pm i\operatorname{sech}(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)))}{2\mu_2(\mu_1(\tanh(\xi) \pm i\operatorname{sech}(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)} \\ &+ \frac{lb_1\mu_1(\mu_2 k - \mu_1^2\mu_2 k \pm \mu_1\mu_2^2 k - 2b_1\mu_1)(\tanh(\xi) \pm i\operatorname{sech}(\xi))^2}{4\mu_2^2(\mu_1(\tanh(\xi) \pm i\operatorname{sech}(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)^2} \\ &+ \frac{(-\mu_2 k\mu_1 - 2b_1 \pm \mu_2^2 k)lb_1\mu_1(\tanh(\xi) \pm i\operatorname{sech}(\xi))(\csc(\xi) \pm \cot(\xi))}{\mu_2(\mu_1(\tanh(\xi) \pm i\operatorname{sech}(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)^2} \\ &+ \frac{lb_1(\pm k - 2b_1 - \mu_2 k\mu_1 \pm \mu_2^2 k)(\csc(\xi) \pm \cot(\xi))^2}{(\mu_1(\tanh(\xi) \pm i\operatorname{sech}(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)^2}, \end{aligned} \tag{3.9.2}$$

where $\xi = k(x + ly + \lambda t)$, a_0, A_0, b_1, μ_1, μ_2, k, l and λ are arbitrary constants.

Family 6. When $h_1 = -\frac{1}{2}$, $h_2 = \frac{1}{2}$ and $h_3 = h_4 = \pm\frac{1}{2}$, then we obtain following solutions:

$$H_6 = a_0 + \frac{b_1\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + b_1\mu_2(\csc(\xi) \pm \cot(\xi))}{\mu_2(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)}, \tag{3.10.1}$$

$$\begin{aligned} G_6 = A_0 &- \frac{(-k\mu_1 + 2a_0 + \lambda \pm k\mu_2)b_1 l(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)))}{2\mu_2(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)} \\ &+ \frac{lb_1\mu_1(\mu_2 k - \mu_1^2\mu_2 k \pm \mu_1\mu_2^2 k - 2b_1\mu_1)(\coth(\xi) \pm \operatorname{csch}(\xi))^2}{4\mu_2^2(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)^2} \\ &+ \frac{(-\mu_2 k\mu_1 - 2b_1 \pm \mu_2^2 k)lb_1\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi))(\csc(\xi) \pm \cot(\xi))}{\mu_2(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)^2} \\ &+ \frac{lb_1(\pm k - 2b_1 - \mu_2 k\mu_1 \pm \mu_2^2 k)(\csc(\xi) \pm \cot(\xi))^2}{(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)^2}, \end{aligned} \tag{3.10.2}$$

where $\xi = k(x + ly + \lambda t)$, a_0, A_0, b_1, μ_1, μ_2, k, l and λ are arbitrary constants.

4. Conclusion

In this method, a new algebraic MRERE method is presented to find new exact solution of nonlinear PDEs. The Broer–Kaup–Kupershmidt system is chosen to illustrate the method such that some novel solutions are found, which include solutions of hyperbolic (solitary) function and triangular periodic functions appearing at the same time. Of course, the algorithm can also be applied to many nonlinear PDEs in mathematical physics. Further work is to extend the MRERE method to construct solutions of soliton-like solution and triangular periodic functions solution appearing in a solution at the same time.

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Appendix A

The general solutions of the Riccati Eq. (2.5)

$$\frac{dF}{d\xi} = R_1 + R_2F^2$$

are

(1) when $R_1 = \frac{1}{2}$ and $R_2 = -\frac{1}{2}$,
$F(\xi) = \tanh(\xi) \pm i\,\mathrm{sech}(\xi), \quad F(\xi) = \coth(\xi) \pm \mathrm{csch}(\xi)$,

(2) when $R_1 = R_2 = \pm\frac{1}{2}$,
$F(\xi) = \sec(\xi) \pm \tan(\xi), \quad F(\xi) = \csc(\xi) \pm \cot(\xi)$,

(3) when $R_1 = 1$ and $R_2 = -1$,
$F(\xi) = \tanh(\xi), \quad F(\xi) = \coth(\xi)$.

(4) when $R_1 = R_2 = 1$,
$F(\xi) = \tan(\xi)$,

(5) when $R_1 = R_2 = -1$,
$F(\xi) = \cot(\xi)$,

(6) when $R_1 = 0$ and $R_2 \neq 0$,
$$F(\xi) = -\frac{1}{R_2\xi + c_0},$$

where c_0 is an arbitrary constant.

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