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The Using of Conservation Laws in Symmetry-Preserving Difference Scheme*

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Abstract *In this paper, by means of the potential systems of the given nonlinear evolution equations, a procedure of symmetry preserving discretization of differential equations is presented. The specific process will be given detailed in section 2. This extended method is effective for discreting the high-order (high-dimensional) nonlinear evolution equations. As examples, the invariant difference models of the mKdV equation and the Boussinesq equation are constructed.*

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1 Introduction

Lie symmetries[1−8] are widely used in the analysis of nonlinear models. One of the main applications of the Lie group theory for differential equations is to get group-invariant solutions. Some effective methods have been introduced, such as the classical Lie group approach and the non-classical Lie group approach[9−13] etc. In many cases, the symmetries of differential equations are more important and better known than the equations themselves since these symmetries reflect fundamental physical laws. Thus, when discretizing, or other wise modifying dynamical equations, it is of interest to preserve the original symmetries.

Applications of Lie group theory to difference equations are much more recent.[14−18] We know that the structure of the admitted group essentially affects the construction of equations and grids. Group transformations can break the geometric structure of the mesh that influences the approximation and other properties of the lattice equation.[19−22] The problem of constructing discretization schemes that preserve symmetries of corresponding differential equations was first systematically addressed by Dorodnitsyn and his collaborators.[23−24] This approach is to start with a differential equation and to introduce a symmetry adapted mesh and difference equation in such a way that the symmetries of the original differential equation are preserved.[25−30]

In this paper, base on Dorodnitsyn's work, we give an extended method to construct the difference models of the nonlinear evolution equation which preserve Lie point symmetries of the original equation. The difference of this method to others is that we construct the difference models of the potential system instead of the original equation. The specific process is as follows: (i) For a given nonlinear evolution equation, the potential system of the original equation can be constructed firstly. One can see that the order of the potential system is lower than the original equation, this procedure can provide beneficial help for constructing the invariant difference models. (ii) Following Bluman's theory, each local symmetry of the potential system projects onto a local symmetry of the given system, we can construct the difference models of the potential system which preserve Lie point symmetries. (iii) We can construct the difference models of the original equation by a theorem if the difference models of the potential system are obtained. (iv) One can check that the difference models preserve Lie point symmetries of original equation.

This paper is arranged as follows: In Sec. 2, the overview of the discretization procedure is introduced detailedly. In Sec. 3, two kinds of difference models of the mKdV equation are obtained which the whole symmetries of the original differential equation are preserved. In Sec. 4, the same method is applied to the Boussinesq equation and the symmetry-preserving difference model is constructed. Finally, some conclusions and discussions are given in Sec. 5.

2 Overview of the Discretization Procedure

In this section, we introduce how a local Lie transformation group acts on nonlocal objects such as discrete variables, finite-difference derivatives, lattice spacings, etc.[24,27] At the end of this section, an extending method for constructing the difference models which preserve symmetries of the original differential equations will be given.

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2.1 *Definition of Finite-Difference Derivatives*

In contrast to differential operators, finite-difference operators are specified on a finite subset of the countable set of lattice points where the solution of the problem in question is to be sought. This nonlocality of operators (from the physical viewpoint) results in specific properties of finite-difference operators, properties which are absent in the local differential model.

Let Z be the space of sequences $(x, u, u_1, u_2, \ldots)$, where $x = \{x^i; i = 1, 2, \ldots, n\}$, $u = \{u^k; k = 1, 2, \ldots, m\}$. $u_1 = \{u_i^k\}$ is the set of $m \cdot n$ first partial derivatives, $u_2 = \{u_{i,j}^k\}$ is the set of second partial derivatives, etc.

We restrict to the case $n = 2$, i.e. $x = (x^1, x^2)$, and consider two types of differentiations,

$$D_1 = \frac{\partial}{\partial x^1} + u_1\frac{\partial}{\partial u} + u_{11}\frac{\partial}{\partial u_1} + u_{21}\frac{\partial}{\partial u_2} + \cdots\,, \tag{1}$$

$$D_2 = \frac{\partial}{\partial x^2} + u_2\frac{\partial}{\partial u} + u_{12}\frac{\partial}{\partial u_1} + u_{22}\frac{\partial}{\partial u_2} + \cdots\,, \tag{2}$$

where

$$u_1 = \frac{\partial u}{\partial x^1},\quad u_{11} = \frac{\partial^2 u}{\partial (x^1)^2},\quad u_{21} = \frac{\partial^2 u}{\partial x^2 \partial x^1},\ldots,$$

and summation over the omitted superscript k in Eqs. (1) and (2) is assumed.

Let us fix two arbitrary parameters $h_i > 0$, $i = 1, 2$, and use the tangent field (1) of the Taylor group to form a pair of operators,

$$\underset{\pm h}{S_i} = \mathrm{e}^{\pm h_i D_i} \equiv \sum_{s=0}^{\infty} \frac{(\pm h_i)^s}{s!} D_i^s\,,\quad i = 1, 2\,, \tag{3}$$

which will be called the right and left discrete shift operators.

Using $\underset{+h}{S_i}$ and $\underset{-h}{S_i}$, one can form a pair of right and left discrete (finite-difference) differentiation operators by setting

$$\underset{+h}{D_i} = \frac{1}{h_i}(\underset{+h}{S_i} - 1)\,,\quad \underset{-h}{S_i} = \frac{1}{h_i}(1 - \underset{-h}{S_i})\,.$$

One denotes sequences $(\underset{h}{u_1}, \underset{h}{u_{11}}, \underset{h}{u_{12}}, \ldots)$ of formal series by $\underset{h}{Z}$ and the product of the spaces $\underset{h}{Z}$ and $\tilde{Z}$ by

$$\underset{h}{\tilde{Z}} = (x, u, u_1, u_{11}, \ldots; \underset{h}{u_1}, \underset{h}{u_{11}}, \ldots)\,.$$

Before study the relations between one-parameter groups and difference meshes preserving their geometric structure under group transformations, we introduce the definition of potential system as following.

2.2 *Lie Point Symmetries of Potential System*

For an evolution equation,

$$f(t, x, u, u_x, \ldots) = 0\,, \tag{4}$$

we consider nonlocally related systems for a given system of differential equations using Bluman's method.[31] A useful conservation law yields potential variables and equivalent nonlocally related potential systems and subsystems for any given system.

Using the conservation law of the original equation, one can obtain the corresponding potential systems[31–33] with the following form,

$$\begin{aligned} w_x &= f_1(t, x, u, u_t, u_x, u_{xx}, \ldots)\,,\\ w_t &= f_2(t, x, u, u_t, u_x, u_{xx}, \ldots), \end{aligned} \tag{5}$$

where $w(x, t)$ is an auxiliary potential variable. That is to say, the integrable conditions of Eq. (5) $w_{xt} = w_{tx}$, is just the original equation.

Theorem 1[34] Each local symmetry of an underdetermined potential system (5) projects onto a local symmetry of the given system (4).

Let

$$X = \xi^t\frac{\partial}{\partial t} + \xi^x\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \varphi\frac{\partial}{\partial w}\,, \tag{6}$$

be the generator of a one-parameter transformation group of potential system.

The group generated by Eq. (6) transforms a point

$$(t, x, u, u_t, u_x, u_{xx}, \ldots, w_t, w_x)$$

to a new one $(t^*, x^*, u^*, u_t^*, u_x^*, u_{xx}^*, \ldots, w_t^*, w_x^*)$ together with Eq. (4). This situation changes when applying Lie points transformations to difference equations.

Using the theorem 1, one can obtain that

$$X' = \xi^t\frac{\partial}{\partial t} + \xi^x\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}\,, \tag{7}$$

is the generator of a one-parameter transformation group of system (4).

2.3 *Invariant Difference Meshes*

Let

$$F_i(z) = 0\,,\quad i = 1, 2\,, \tag{8}$$

where $F_i \in \underset{h}{A}$. In contrast to the point $(t, x, u, u_t, u_x, u_{xx}, \ldots, w)$, the difference stencil has its own geometrical structure. In the continous limit, all two equations reduce to the potential system. This equation is written on finitely many points of the difference mesh $\underset{h}{w}$, which may be uniform or nonuniform. We assume that the mesh is determined by the equation

$$\Omega_j(z) = 0\,,\quad (j = 1, 2)\,, \tag{9}$$

where $\Omega_j \in \underset{h}{A}$. The function Ω_j is uniquely determined by the "discretization" of the space of independent variables, which is obtained by the action of the operator $\underset{\pm h}{S^\alpha}$ on the "starting point" x_0. In the continous limit, the mesh equation $\Omega(z) = 0$ degenerates into an identity (for example, $0 = 0$).

The invariance of the difference equation (8) depends on the invariance of Eq. (9), since the latter must be included in the general condition of invariance:

$$XF_i(z)|_{(8),(9)} = 0\,,\quad X\Omega_i(z, h, \tau)|_{(8),(9)} = 0\,. \tag{10}$$

Proposition 1[16] For the mesh $\underset{h}{w}$ to remain uniform $(\tau_+^* = \tau_-^*, h_+^* = h_-^*)$ under the action of the transformation group, if it satisfies the following condition at each point $z \in \underset{h}{\tilde{Z}}$:

$$\underset{+h}{D}\,\underset{-h}{D}(\xi^t) = 0\,, \tag{11}$$

$$\underset{+h}{D}\,\underset{-h}{D}(\xi^x) = 0\,, \tag{12}$$

where $\underset{+h}{D}$ and $\underset{-h}{D}$ are right and left difference operators.

Proposition 2[16] For an orthogonal mesh $\underset{h}{w}$ to preserve its orthogonality in the plane (t,x) under any transformation of the group G with the operator (6), it is necessary and sufficient that the following condition should be satisfied at each mesh point:

$$\underset{\pm h}{D}(\xi^t) = -\underset{\pm h}{D}(\xi^x)\,. \tag{13}$$

2.4 *Construct the Difference Models*

If we have constructed the difference models of the potential system, the last step is to eliminate the potential variable in the difference models. Therefore, one can get the difference scheme of original equation. For this reason, we give a theorem.

Theorem 2 If the difference models of potential system (8), i.e., $F_i(z)=0$, $i=1,2$ admit the symmetries (6), then the difference model $\underset{+h}{D_1}F_1(z) + \underset{+h}{D_2}F_2(z) = 0$ admits the symmetries of original equation.

Proof It is easy to prove this theorem using the compatibility condition and the definition of discrete differentiation operators.

From the theorem 1, we know that the local symmetries (6) project onto local symmetries X'. So the difference modes which is obtained using the symmetries (6) and the theorem 2 must be admitted by the symmetries (7). Two examples will be given in the following two sections.

3 Invariant Model for the mKdV Equation

We consider the mKdV equation[35–36]

$$u_t + 6u^2u_x + u_{xxx} = 0\,, \tag{14}$$

which has been proposed as model equation for the weakly non-linear long waves which occur in many different physical systems.

The first step is to give the potential systems of Eq. (14). We seek the local conservation law multipliers with the form $\Lambda = \Lambda(x,t,U)$ of the Eq. (14). In terms of Euler operator

$$E_U = \frac{\partial}{\partial U} - D_x\frac{\partial}{\partial U_x} - D_t\frac{\partial}{\partial U_t} - D_{xxx}\frac{\partial}{\partial U_{xxx}}\,,$$

the determining Eq. (14) for the multipliers become

$$E_U(\Lambda(x,t,U)(U_t + 6U^2U_x + U_{xxx})) \equiv 0\,,$$

where $U(x,t)$ is an arbitrary function. Above equation split with respect to U_t, U_x, U_{xxx} to yield the overdetermined linear PDE system. The solutions of the determining system are the two local multipliers given by $\Lambda_1 = 1, \Lambda_2 = u$.

Each of the multipliers determines a nontrivial local conservation law with the following forms,

$$D_t(u) + D_x(2u^3 + u_{xx}) = 0\,,$$

$$D_t\Big(\frac{1}{2}u^2\Big) + D_x\Big(\frac{3}{2}u^4 + uu_{xx} - \frac{1}{2}u_x^2\Big) = 0\,.$$

One can introduce two potential variables $w, \tilde{w}$, and obtain two potential systems as follows,

$$w_t = -2u^3 - u_{xx}\,, \quad w_x = u\,, \tag{15}$$

$$\tilde{w}_x = \frac{1}{2}u^2\,, \quad \tilde{w}_t = -\frac{3}{2}u^4 - uu_{xx} + \frac{1}{2}u_x^2\,. \tag{16}$$

Next, the difference models of the potential systems (15)–(16) will be constructed.

Case 1 We first construct the difference model of the system (15). Using the Lie Group method, the Eqs. (15) admits a 4-parameter symmetry group G. This group can be represented by the following infinitesimal operators:

$$X_1 = 3t\partial t + x\partial x - u\partial u\,, \quad X_2 = \partial t\,,$$
$$X_3 = \partial w\,, \quad X_4 = \partial x\,, \tag{17}$$

in this case, there are virtually no constraints on the mesh and the difference equation. The set (17) satisfies all conditions (11)–(12). So one can use an orthogonal grid which is uniform in both t and x directions. It implies that the mesh equation has the form $h^+ = h^-$, and satisfies the second condition of Eq. (10). We omit here and the following chapters. Let us consider the set of operators (17) in the space $(t, x, u, w, h, \tau, u_-, u_+, w_-, w_+, \hat{w}, \hat{w}_-)$ that corresponds to the stencil shown in Fig. 1.

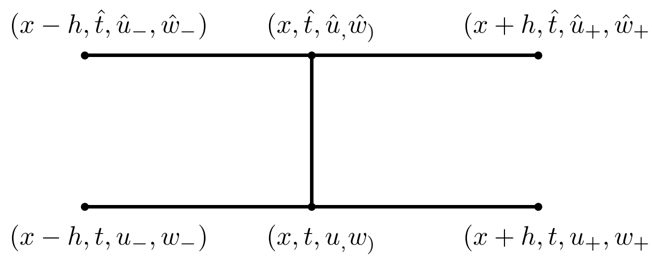

Fig. 1 The stencil of the orthogonal mesh.

The corresponding discrete subspace is twelve-dimensional: $M \sim (t, x, u, w, h, \tau, u_-, u_+, w_-, w_+, \hat{w}, \hat{w}_-)$, where $\tau = \hat{t} - t$. The prolonged operator (17) in this subspace has the form,

$$\begin{aligned} prX = {} & \xi^t\frac{\partial}{\partial t} + \xi^x\frac{\partial}{\partial x} + (\hat{\xi}^t - \xi^t)\frac{\partial}{\partial \tau} \\ & + (\hat{\xi}^x - \xi^x)\frac{\partial}{\partial h} + \eta^u\frac{\partial}{\partial u} + \eta^u_+\frac{\partial}{\partial u_+} \\ & + \eta^w\frac{\partial}{\partial w} + \eta^w_+\frac{\partial}{\partial w_+} + \eta^w_-\frac{\partial}{\partial w_-} \\ & + \eta^w_+\frac{\partial}{\partial w_+} + \hat{\eta}^w_+\frac{\partial}{\partial \hat{w}_+} + \hat{\eta}^w_-\frac{\partial}{\partial \hat{w}_-}\,, \end{aligned}$$

where

$$\hat{f} = f(t+\tau, x, u, w)\,, \quad f_\pm = f(t, x \pm h, u, w)\,.$$

The number of functionally independent invariants is given by $l = \dim M - \operatorname{rank} Z$, $l \geq 0$, with $\dim M = 12$ and the matrix Z composed by the coefficients of the prolonged on the space M operators

$$Z = \begin{pmatrix} \xi_1^t & \xi_1^x & (\hat{\xi}_1^t - \xi_1^t) & ((\xi_1^x)_+ - \xi_1^x) & \eta_1^u & \cdots & (\hat{\eta}_1^w)_- \\ \vdots & & & & & & \vdots \\ \xi_n^t & \xi_n^x & (\hat{\xi}_n^t - \xi_n^t) & ((\xi_n^x)_+ - \xi_n^x) & \eta_n^u & \cdots & (\hat{\eta}_n^w)_- \end{pmatrix}.$$

Having found the finite-difference invariants as solutions of linear equations

$$\mathrm{pr}X_i I(t, x, u, w, h, \tau, u_-, u_+, w_-, w_+, \hat{w}, \hat{w}_-) = 0\,, \quad i = 1, 2, \ldots, n\,. \tag{18}$$

there are eight difference invariants by solving above equations,

$$I_1 = uh\,, \quad I_2 = u_+ h\,, \quad I_3 = u_- h\,, \quad I_4 = \tau u^3\,, \quad I_5 = \frac{h^3}{\tau}\,, \quad I_6 = \tau \overset{3}{u_+}\,, \quad I_7 = w_+ - w\,, \quad I_8 = \hat{w} - w\,. \tag{19}$$

By means of the invariants (19), one could write the following explicit scheme for Eq. (17):

$$I_7 = I_1\,, \quad \frac{I_8}{I_3} = -\frac{(I_2 - I_1) - (I_1 - I_3)}{I_5} - 2I_4\,,$$

i.e.

$$\underset{h}{w} = u\,, \quad \underset{\tau}{w} = -\frac{1}{h}(\underset{h}{u_x} - \underset{h}{u_{\bar{x}}}) - 2u^3\,, \tag{20}$$

Using the difference operators $\underset{+\tau}{D}, \underset{+h}{D}$ onto the Eqs. (20) respectively. Following the theorem 2, one could obtain the invariant difference equation of explicit scheme for Eq. (14):

$$\underset{\tau}{u_t} = -\underset{h}{u_3} - 6u^2 \underset{h}{u_1} - 6hu \underset{h}{u_1^2} - 2h^2 \underset{h}{u_1^3}\,, \tag{21}$$

where

$$\underset{h}{u_1} = \frac{u_+ - u}{h}\,, \quad \underset{\tau}{u_t} = \frac{\hat{u} - u}{\tau}\,, \quad \underset{h}{u_3} = \underset{+h}{D}\,\underset{+h}{D}\,\underset{+h}{D}(u)\,,$$

In the continuous limit, one obtains $u_t = -u_{xxx} - 6u^2 u_x$.

Let us represent the operators (17) in $\underset{h}{Z}$ by extending X_1 to h and τ as follows:

$$\tilde{X}_1 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} - 2\,\underset{h}{u_x}\frac{\partial}{\partial \underset{h}{u_x}} - 3\,\underset{h}{u_{xx}}\frac{\partial}{\partial \underset{h}{u_{xx}}} - 4\,\underset{h}{u_{xxx}}\frac{\partial}{\partial \underset{h}{u_{xxx}}} + 3\tau\frac{\partial}{\partial \tau} + h\frac{\partial}{\partial h}\,,$$

$$\tilde{X}_2 = \frac{\partial}{\partial t}\,, \quad \tilde{X}_3 = \frac{\partial}{\partial w}\,, \quad \tilde{X}_4 = \frac{\partial}{\partial x}\,. \tag{22}$$

We check the invariance criterion (10) for the Eq. (21) and the operators (22)

$$\tilde{X}_1(\underset{\tau}{u_t} + \underset{h}{u_3} + 6u^2 \underset{h}{u_1} + 6hu \underset{h}{u_1^2} + 2h^2 \underset{h}{u_1^3})|_{(21)} = 0\,,$$
$$\tilde{X}_2(\underset{\tau}{u_t} + \underset{h}{u_3} + 6u^2 \underset{h}{u_1} + 6hu \underset{h}{u_1^2} + 2h^2 \underset{h}{u_1^3})|_{(21)} = 0\,,$$
$$\tilde{X}_3(\underset{\tau}{u_t} + \underset{h}{u_3} + 6u^2 \underset{h}{u_1} + 6hu \underset{h}{u_1^2} + 2h^2 \underset{h}{u_1^3})|_{(21)} = 0\,,$$
$$\tilde{X}_4(\underset{\tau}{u_t} + \underset{h}{u_3} + 6u^2 \underset{h}{u_1} + 6hu \underset{h}{u_1^2} + 2h^2 \underset{h}{u_1^3})|_{(21)} = 0\,.$$

From above identical equation, one can obtain that the difference equation (21) admits the group G with operator (17). i.e., the original continuous symmetries are preserved in discrete models.

Case 2 The difference model for the system (16): Using the Lie Group method, the Eqs. (16) admits a 4-parameter symmetry group G'. And the infinitesimal operators have the following forms:

$$X_1 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} - \tilde{w}\frac{\partial}{\partial \tilde{w}}\,,$$
$$X_2 = \frac{\partial}{\partial t}\,, \quad X_3 = \frac{\partial}{\partial \tilde{w}}\,, \quad X_4 = \frac{\partial}{\partial x}\,. \tag{23}$$

The set Eq. (23) satisfied all conditions (11)–(12). So, one can use an orthogonal grid which is uniform in both t and x directions. Let us consider the set of operators (23) in the space $(t, x, u, w, h, \tau, u_-, u_+, w_-, w_+, \hat{\tilde{w}}, \hat{\tilde{w}}_-)$ that corresponds to the stencil shown in Fig. 1.

There are eight difference invariants of the Lie algebra (23) in the prolonged space:

$$I_1 = uh\,, \quad I_2 = u_+ h\,, \quad I_3 = u_- h\,, \quad I_4 = \tau u^3\,, \quad I_5 = \frac{h^3}{\tau}\,,$$
$$I_6 = \frac{\tilde{w}_+ - \tilde{w}}{u}\,, \quad I_7 = \frac{\hat{\tilde{w}} - \tilde{w}}{u}\,, \quad I_8 = \tau \overset{3}{u_+}\,. \tag{24}$$

By means of the invariants (24), one could write the following explicit scheme for Eq. (14):

$$I_6 = \frac{1}{2}I_1\,,$$
$$I_7 = -\frac{3}{2}I_4 - \frac{(I_2 - I_1) - (I_1 - I_3)}{I_5} + \frac{1}{2}\frac{(I_2 - I_1)(I_1 - I_3)I_4}{I_1^4}\,, \tag{25}$$

i.e.

$$\underset{h}{\tilde{w}_x} = \frac{1}{2}u^2\,, \quad \underset{\tau}{\tilde{w}_t} = -\frac{3}{2}u^4 - u\,\underset{h}{u_2} + \frac{1}{2}\,\underset{h}{u_x}\,\underset{h}{u_{\bar{x}}}\,. \tag{26}$$

Using the difference operators $\underset{+\tau}{D}, \underset{+h}{D}$ onto Eqs. (26) respectively, one can obatin that $\underset{t\tau}{\tilde{w}} = \underset{\tau h}{\tilde{w}}$ with help of the theorem 2. So we could write the invariant difference equation of explicit scheme for Eq. (24):

$$\underset{\tau}{u_t}\,u + \frac{\tau}{2}\,\underset{\tau}{u_t^2} = -6u^3 \underset{h}{u_1} - 9u^2 \underset{h}{u_1^2}\,h - 6u \underset{h}{u_1^3}\,h^2 - \frac{3}{2}\underset{h}{u_1^4}\,h^3 - u\,\underset{h}{u_3} - h\,\underset{h}{u_1}\,\underset{h}{u_3}\,, \tag{27}$$

where

$$\underset{h}{u_1} = \frac{u_+ - u}{h}\,, \quad \underset{\tau}{u_t} = \frac{\hat{u} - u}{\tau}\,.$$

It is easy to check the invariance criterion (10) for Eq. (27) by extending operators (22) to h and τ i.e., the difference equation (27) admits the group G', here we omit. In the continuous limit, one obtains $uu_t = -uu_{xxx} - 6u^3u_x$, i.e. the mKdV equation (14).

Remark Here we obtain two difference equations of mKdV equation by using different potential systems. We point out that the invariant approximation is still not unique. For example, by extending the stencil (i.e., by increasing the number of mesh points involved in the approximation) we can find invariant approximations of any higher order.

4 Invariant Model for the Boussinesq Equation

The equation

$$u_{tt} + u_{xx} + (u^2)_{xx} + u_{xxxx} = 0\,, \tag{28}$$

was introduced to Boussinesq in 1871 to describe the propagation of long waves in shallow water. The Boussinesq equation[37−38] also arises in several other physical applications including one-dimensional nonlinear lattice waves, vibrations in a nonlinear string, and ion sound waves in a plasma.

The potential system of Eq. (28) has the following form,

$$w_x = u_t\,, \quad v_x = w\,, \quad v_t = -u - u^2 - u_{xx}\,, \tag{29}$$

where $w(x,t)$, $v(x,t)$ are potential variables.

Using the Lie Group method, the system(28) admits a 5-parameter symmetry group G''. And the infinitesimal operators are:

$$X_1 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - (1+2u)\frac{\partial}{\partial u} + (t-2v)\frac{\partial}{\partial v} - 3w\frac{\partial}{\partial w}\,,$$

$$X_2 = \frac{\partial}{\partial x}\,, \quad X_3 = \frac{\partial}{\partial t}\,, \quad X_4 = x\frac{\partial}{\partial v} + \frac{\partial}{\partial w}\,, \quad X_5 = \frac{\partial}{\partial v}\,. \tag{30}$$

The set (30) satisfies all conditions (11)–(12). So, one can use an orthogonal grid which is uniform in both t and x directions. Consider the set of operators (30) in the space $(t, x, u, v, w, h, \tau, u_-, u_+, \hat{u}, w_+, v_{+,}\hat{v})$ that corresponds to the stencil shown in Fig. 2.

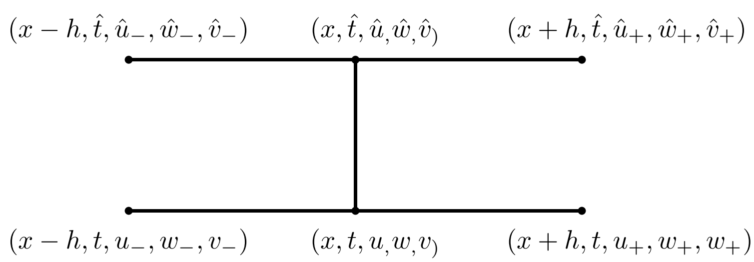

Fig. 2 The difference stencil for Eq. (33).

There are eight difference invariants of the Lie algebra (30) by soling Eqs. (18):

$$I_1 = \frac{1+2u_+}{2+4u}\,, \quad I_2 = \frac{1+2u_-}{2+4u}\,,$$

$$I_3 = \frac{1+2\hat{u}}{2+4u}\,, \quad I_4 = \frac{w_+ - w}{(1+2u)^{3/2}}\,,$$

$$I_5 = h\sqrt{1+2u}\,, \quad I_6 = \frac{-wh - v + v^+}{4+8u}\,,$$

$$I_7 = \tau + 2\tau u\,, \quad I_8 = \frac{-\tau - 4v + 4\hat{v}}{4+8u}\,. \tag{31}$$

By means of the invariants (31), one could write the following explicit scheme for Eq. (29):

$$\frac{I_4}{I_5} = \frac{I_3 - 1/2}{I_7}\,, \quad I_6 = 0\,, \quad \frac{I_8}{I_7} = -\frac{1}{4} - \frac{I_1 + I_2 - 1}{I_5^2}\,, \tag{32}$$

i.e.

$$\underset{h}{w_x} = \underset{\tau}{u_t}\,, \quad \underset{h}{v_x} = w\,, \quad \underset{\tau}{v_t} = -u - u^2 - \underset{h}{u_2}\,. \tag{33}$$

Using the compatibility

$$\underset{h}{v_{xt}} = \underset{h}{v_{tx}}\,, \quad \underset{h}{w_{xt}} = \underset{h}{w_{tx}}\,, \quad \underset{h}{u_{xt}} = \underset{h}{u_{tx}}$$

and the theorem 1, one can obtain the invariant difference equation of explicit scheme:

$$\begin{aligned} \underset{\tau}{u_2} + \tau \underset{\tau}{u_3} = &-(\underset{h}{u_2} + h\underset{h}{u_3}) - 2\underset{h}{u_1^2} - 2u(\underset{h}{u_2} + h\underset{h}{u_3}) \\ &- 2h\underset{h}{u_1}(\underset{h}{u_2} + h\underset{h}{u_3}) - h^2(\underset{h}{u_2} + h\underset{h}{u_3})^2 \\ &- 2h\underset{h}{u_1}(\underset{h}{u_2} + h\underset{h}{u_3}) - (\underset{h}{u_4} + h\underset{h}{u_5})\,. \end{aligned} \tag{34}$$

The invariance criterion (10) for Eq. (34) can be easily checked. The model admits the group G'' i.e., the original continuous symmetries are preserved in discrete models. In the continuous limit, one can get the equation $u_{tt} = -u_{xx} - (u^2)_{xx} - u_{xxxx}$. We point that the difference model of the Boussinesq equation is obtained that the whole symmetries of the original differential equation are preserved.

5 Conclusions

It is well known that one system of differential equations can be approximated by many difference schemes. A finite-difference modeling always involves the problem of selecting the schemes that are in some respect advantageous. The selection criteria are often given by fundamental physical principles present in the original model, such as conservation laws, variational principles, the existence of physically meaningful exact solutions, etc.

Our attention is focused on the problem of constructing difference equations and meshes such that the difference models preserve the symmetries of the original continuous system. In this paper, we construct the symmetry-preserving difference models of the potential system instead of the original differential equation firstly. Then, the difference models of original differential equation are obtained by using a theorem. This extended method has many benefits, for example, the order of the potential system is lower than the original differential equation, one can construct difference models of low-order potential system instead of the high-order original equation, etc. We give two examples in the paper, i.e. the mKdV equation and the Boussinesq equation. Through the verification, the difference models that we obtained inherited the symmetries of the original PDEs.

We have constructed the symmetry-preserving difference models of two equations. The next step is to study their properties, for example, the error analysis, stability analysis and numerical examples, etc. Above topics will be discussed in the future series research works.

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