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Bright-Dark Mixed *N*-Soliton Solutions of the Multi-Component Mel'nikov System

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By virtue of the Kadomtsev–Petviashvili (KP) hierarchy reduction technique, we construct the general bright-dark mixed *N*-soliton solution to the multi-component Mel'nikov system. This multi-component system comprised of multiple (say M) short-wave components and one long-wave component with all possible combinations of nonlinearities including all-positive, all-negative and mixed types. Firstly, the two-bright-one-dark (2-b-1-d) and one-bright-two-dark (1-b-2-d) mixed *N*-soliton solutions in short-wave components of the three-component Mel'nikov system are derived in detail. Then we extend our analysis to the M-component Mel'nikov system to obtain its general mixed *N*-soliton solution. The formula obtained unifies the all-bright, all-dark and bright-dark mixed *N*-soliton solutions. For the collision of two solitons, an asymptotic analysis shows that for an M-component Mel'nikov system with $M \geq 3$, inelastic collision takes place, resulting in energy exchange among the short-wave components supporting bright solitons only if the bright solitons appear in at least two short-wave components. In contrast, the dark solitons in the short-wave components and the bright solitons in the long-wave component always undergo elastic collision which is only accompanied by a position shift.

1. Introduction

The study of multi-component nonlinear systems is of great interest as the interaction of multiple waves may result in some new physical phenomena.[1–4] Of particular interest is the multi-component generalization of the nonlinear Schrödinger (NLS) equation,[5–10] which has been considered as the generic model to describe the evolution of slowly varying wave packets in nonlinear wave systems. Moreover, the studies on nonlinear systems describing the interaction of long waves with short wave packets in nonlinear dispersive media have also received much attention in recent years.[11–16] Such systems have found wide applications in the fields of hydrodynamics, nonlinear optics, plasma physics and so on.[17–20]

It is desirable to extend the studies to multi-component cases since a variety of complex systems such as nonlinear optical fibers[3] and Bose–Einstein condensates[21] usually involve more than one component. In real physical systems, the nonlinearities can be positive or negative, depending on the physical situation.[22] For instance, in Bose–Einstein condensates,[21] the nonlinear coefficients take positive or negative values when the interaction between the atoms is repulsive or attractive. For some multi-component systems,[11,13,16] it has been found that the solitons exhibit certain energy-exchanging inelastic collision behaviors, which have not been found in the single-component counterparts and may be used to realize multi-state logic and soliton-collision based computing. In the current paper, we consider the multi-component generalization of the Mel'nikov system[23–26]

$$i\Phi_y = \Phi_{xx} + u\Phi, \tag{1}$$

$$u_{xt} + u_{xxxx} + 3(u^2)_{xx} - 3u_{yy} + \sigma(\Phi\Phi^*)_{xx} = 0, \tag{2}$$

where $\sigma = \pm 1$, u is the real long-wave amplitude and Φ is the complex short-wave amplitude; the asterisk means complex conjugate hereafter and the subscripts t and x, y denote partial differentiation with respect to time and space, respectively. This system is introduced by Mel'nikov and can be used to describe the interaction of long waves with short wave packets propagating on the x–y plane at an angle to each other. The Eqs. (1) and (2) can be extended to a multi-component case

$$i\Phi_y^{(k)} = \Phi_{xx}^{(k)} + u\Phi^{(k)}, \quad k = 1, 2, \ldots, M, \tag{3}$$

$$u_{xt} + u_{xxxx} + 3(u^2)_{xx} - 3u_{yy} + \left(\sum_{k=1}^{M} \sigma_k \Phi^{(k)}\Phi^{(k)*}\right)_{xx} = 0,$$
$$\sigma_k = \pm 1, \tag{4}$$

which describes the interaction of a long wave u with multiple (say M) short wave packets $\Phi^{(k)}$. The Eqs. (3) and (4) is referred to the M-component Mel'nikov system hereafter. Note that the Eqs. (3) and (4) supports all possible combinations of nonlinearities including all-positive, all-negative, and mixed types.

The Mel'nikov system (1)–(2) allows boomeron type solutions,[25] which describe the reflection of waves. In other words, the wave begins propagating in the opposite direction, and finally, goes back to where it started. Its multi-soliton solution is obtained in Ref. 26 through the matrices theory. Its bright- and dark-type soliton solutions have been derived from the Wronskian solutions of the KP hierarchy equations.[27] The Painlevé analysis and exponentially localized dromion type solutions of the system (1)–(2) are reported in Ref. 28. In a recent work, its rogue wave solution is obtained by using the Hirota's bilinear method.[29]

In our previous study,[30] the general *N*-dark soliton solutions of the multi-component Mel'nikov system (3)–(4) with all possible combinations of nonlinearities including all-positive, all-negative and mixed types are obtained through the KP hierarchy reduction technique. In the present paper, we continue to investigate the general bright-dark mixed *N*-soliton solutions of the multi-component Mel'nikov system (3)–(4). Moreover, the dynamics of single and two solitons are also discussed in detail. For the collision of two solitons in an M-component Mel'nikov system with $M \geq 3$, a detailed asymptotic analysis shows that inelastic collision takes place only if the bright parts of the mixed solitons appear in at least two short-wave components. Also, the inelastic collision results in energy exchange among the bright parts of the mixed solitons in short-wave components. In contrast, the dark parts of the mixed solitons in short-wave components and the bright solitons in the long-wave component always undergo standard elastic collision.

It is worth noting that the KP hierarchy reduction technique to derive soliton solutions of integrable systems is an effective and elegant method, which is first developed by the Kyoto school[31] in the 1970s. This method has been applied to get soliton solutions of the NLS equation, the modified KdV equation and the Davey–Stewartson (DS) equation. Additionally, the pseudo-reduction of the two-dimensional Toda lattice hierarchy to constrained KP systems with dark soliton solutions is introduced in Ref. 32, and the reduction to constrained KP systems with bright soliton solutions from multi-component KP hierarchy is established in Ref. 33. Based on this method, Ohta et al.[34] construct the general N-dark-dark soliton solutions for a two-coupled NLS equations (Manakov system). Also using this method, the general bright-dark mixed N-soliton solution of the vector NLS equations is investigated by Feng.[8] Most recently, this method is used to obtain the N-dark soliton[15] and bright-dark mixed N-soliton[16] solutions of the multi-component Yajima–Oikawa (YO) system. In some other recent works, the KP hierarchy reduction technique has also been applied to derive rogue wave solutions of integrable systems;[35–37] see also the literatures.[38–40]

This paper is organized as below. In Sect. 2, the general two-bright-one-dark (2-b-1-d) and one-bright-two-dark (1-b-2-d) mixed N-soliton solutions of the three-component Mel'nikov system are derived in detail. The dynamics of single and two solitons are also discussed. Section 3 is devoted to extend a similar analysis to obtain the general m-bright-$(M-m)$-dark mixed N-soliton solution of the M-component Mel'nikov system. The last section is allotted for a conclusion.

2. Bright-Dark Mixed *N*-Soliton Solution of the Three-Component Mel'nikov System

We first consider the general bright-dark mixed N-soliton solution to the three-component Mel'nikov system

$$i\Phi_y^{(k)} = \Phi_{xx}^{(k)} + u\Phi^{(k)}, \quad k = 1,2,3, \tag{5}$$

$$u_{xt} + u_{xxxx} + 3(u^2)_{xx} - 3u_{yy} + \left(\sum_{k=1}^{3}\sigma_k\Phi^{(k)}\Phi^{(k)*}\right)_{xx} = 0, \tag{6}$$

where $\sigma_k = \pm 1$ for $k = 1,2,3$. For the three-component Mel'nikov system, the mixed-type vector solitons in the short-wave components consist of two types: 2-b-1-d and 1-b-2-d. These two types of soliton solutions will be derived in the subsequent two subsections.

2.1 2-b-1-d mixed soliton solution

Without loss of generality, assuming the $\Phi^{(1)}$ and $\Phi^{(2)}$ components are of the bright type while the $\Phi^{(3)}$ component is of the dark type. We introduce the dependent variable transformations

$$\Phi^{(1)} = \frac{g^{(1)}}{f},$$
$$\Phi^{(2)} = \frac{g^{(2)}}{f},$$
$$\Phi^{(3)} = \rho_1 e^{i\theta_1}\frac{h^{(1)}}{f},$$
$$u = 2(\log f)_{xx}, \tag{7}$$

where $g^{(1)}$, $g^{(2)}$, and $h^{(1)}$ are complex functions; f is a real function; $\theta_1 = \alpha_1 x + \alpha_1^2 y + \beta_1(t)$, α_1, and ρ_1 are real constants, $\beta_1(t)$ is a real function. Then the three-component Mel'nikov system (5)–(6) is converted into the bilinear form

$$(D_x^2 - iD_y)g^{(k)}\cdot f = 0, \quad k = 1,2, \tag{8}$$

$$(D_x^2 + 2i\alpha_1 D_x - iD_y)h^{(1)}\cdot f = 0, \tag{9}$$

$$(D_x^4 + D_xD_t - 3D_y^2)f\cdot f = -\sum_{k=1}^{2}\sigma_k g^{(k)}g^{(k)*} + \sigma_3\rho_1^2(f^2 - h^{(1)}h^{(1)*}), \tag{10}$$

where the Hirota's bilinear operator D is defined as

$$D_x^l D_y^m D_t^n f(x,y,t)\cdot g(x,y,t) = \left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^l\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n f(x,y,t) \times g(x',y',t')|_{x=x',y=y',t=t'}. \tag{11}$$

In what follows, we proceed to show how the mixed N-soliton solution is derived through the KP hierarchy reduction technique. To this end, we consider a three-component KP hierarchy with one copy of the shifted singular point (c_1)

$$(D_{x_1}^2 - D_{x_2})\tau_{1,0}(k_1)\cdot\tau_{0,0}(k_1) = 0, \tag{12}$$

$$(D_{x_1}^2 - D_{x_2})\tau_{0,1}(k_1)\cdot\tau_{0,0}(k_1) = 0, \tag{13}$$

$$(D_{x_1}^2 - D_{x_2} + 2c_1D_{x_1})\tau_{0,0}(k_1+1)\cdot\tau_{0,0}(k_1) = 0, \tag{14}$$

$$(D_{x_1}^4 - 4D_{x_1}D_{x_3} + 3D_{x_2}^2)\tau_{0,0}(k_1)\cdot\tau_{0,0}(k_1) = 0, \tag{15}$$

$$D_{x_1}D_{y_1^{(1)}}\tau_{0,0}(k_1)\cdot\tau_{0,0}(k_1) = -2\tau_{1,0}(k_1)\tau_{-1,0}(k_1), \tag{16}$$

$$D_{x_1}D_{y_1^{(2)}}\tau_{0,0}(k_1)\cdot\tau_{0,0}(k_1) = -2\tau_{0,1}(k_1)\tau_{0,-1}(k_1), \tag{17}$$

$$(D_{x_1}D_{x_{-1}^{(1)}} - 2)\tau_{0,0}(k_1)\cdot\tau_{0,0}(k_1) = -2\tau_{0,0}(k_1+1)\tau_{0,0}(k_1-1). \tag{18}$$

Based on the Sato theory for the KP hierarchy,[31] the bilinear equations (12)–(18) have the Gram determinant tau function solution

$$\tau_{0,0}(k_1) = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \tag{19}$$

$$\tau_{1,0}(k_1) = \begin{vmatrix} A & I & \Omega^{\mathrm{T}} \\ -I & B & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & -\bar{\Psi} & 0 \end{vmatrix},$$
$$\tau_{-1,0}(k_1) = \begin{vmatrix} A & I & \mathbf{0}^{\mathrm{T}} \\ -I & B & \Psi^{\mathrm{T}} \\ -\bar{\Omega} & \mathbf{0} & 0 \end{vmatrix}, \tag{20}$$
$$\tau_{0,1}(k_1) = \begin{vmatrix} A & I & \Omega^{\mathrm{T}} \\ -I & B & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & -\bar{\Upsilon} & 0 \end{vmatrix},$$

$$\tau_{0,-1}(k_1) = \begin{vmatrix} A & I & \mathbf{0}^{\mathrm{T}} \\ -I & B & \Upsilon^{\mathrm{T}} \\ -\bar{\Omega} & \mathbf{0} & 0 \end{vmatrix}, \tag{21}$$

where $\mathbf{0}$ is an N-component zero-row vector; I is an $N \times N$ identity matrix; A and B are $N \times N$ matrices whose elements are defined respectively as

$$a_{ij}(k_1) = \frac{1}{p_i + \bar{p}_j}\left(-\frac{p_i - c_1}{\bar{p}_j + c_1}\right)^{k_1} e^{\xi_i + \bar{\xi}_j},$$
$$b_{ij} = \frac{1}{q_i + \bar{q}_j} e^{\eta_i + \bar{\eta}_j} + \frac{1}{r_i + \bar{r}_j} e^{\chi_i + \bar{\chi}_j},$$

meanwhile, Ω, Ψ, Υ, $\bar{\Omega}$, $\bar{\Psi}$, and $\bar{\Upsilon}$ are N-component row vectors

$$\Omega = (e^{\xi_1}, e^{\xi_2}, \ldots, e^{\xi_N}),$$
$$\Psi = (e^{\eta_1}, e^{\eta_2}, \ldots, e^{\eta_N}),$$
$$\Upsilon = (e^{\chi_1}, e^{\chi_2}, \ldots, e^{\chi_N}),$$
$$\bar{\Omega} = (e^{\bar{\xi}_1}, e^{\bar{\xi}_2}, \ldots, e^{\bar{\xi}_N}),$$
$$\bar{\Psi} = (e^{\bar{\eta}_1}, e^{\bar{\eta}_2}, \ldots, e^{\bar{\eta}_N}),$$
$$\bar{\Upsilon} = (e^{\bar{\chi}_1}, e^{\bar{\chi}_2}, \ldots, e^{\bar{\chi}_N}),$$

with

$$\xi_i = \frac{1}{p_i - c_1} x_{-1}^{(1)} + p_i x_1 + p_i^2 x_2 + p_i^3 x_3 + \xi_{i0},$$
$$\bar{\xi}_j = \frac{1}{\bar{p}_j + c_1} x_{-1}^{(1)} + \bar{p}_j x_1 - \bar{p}_j^2 x_2 + \bar{p}_j^3 x_3 + \bar{\xi}_{j0},$$
$$\eta_i = q_i y_1^{(1)} + \eta_{i0},$$
$$\bar{\eta}_j = \bar{q}_j y_1^{(1)} + \bar{\eta}_{j0},$$
$$\chi_i = r_i y_1^{(2)} + \chi_{i0},$$
$$\bar{\chi}_j = \bar{r}_j y_1^{(2)} + \bar{\chi}_{j0},$$

in which p_i, $\bar{p}_j$, q_i, $\bar{q}_j$, r_i, $\bar{r}_j$, ξ_{i0}, $\bar{\xi}_{j0}$, η_{i0}, $\bar{\eta}_{j0}$, χ_{i0}, $\bar{\chi}_{j0}$, and c_1 are complex constants.

The proof of the bilinear equations (12)–(18) can be shown by the Grammian technique,[41,42] which is omitted here. We first consider complex conjugate reduction by assuming x_1, x_3, $x_{-1}^{(1)}$, $y_1^{(1)}$, and $y_1^{(2)}$ are real; x_2 and c_1 are pure imaginary and by letting $p_j^* = \bar{p}_j$, $q_j^* = \bar{q}_j$, $r_j^* = \bar{r}_j$, $\xi_{j0}^* = \bar{\xi}_{j0}$, $\eta_{j0}^* = \bar{\eta}_{j0}$, and $\chi_{j0}^* = \bar{\chi}_{j0}$, then it is easy to check that

$$a_{ij}(k_1) = a_{ji}^*(k_1), \quad b_{ij} = b_{ji}^*.$$

Furthermore, we define

$$f = \tau_{0,0}(0), \quad g^{(1)} = \tau_{1,0}(0), \quad g^{(2)} = \tau_{0,1}(0), \quad h^{(1)} = \tau_{0,0}(1),$$

hence, f is real and

$$g^{(1)*} = -\tau_{-1,0}(0), \quad g^{(2)*} = -\tau_{0,-1}(0), \quad h^{(1)*} = \tau_{0,0}(-1),$$

thus, the bilinear equations (12)–(18) become

$$(D_{x_1}^2 - D_{x_2})g^{(k)} \cdot f = 0, \quad k = 1, 2, \tag{22}$$
$$(D_{x_1}^2 - D_{x_2} + 2c_1 D_{x_1})h^{(1)} \cdot f = 0, \tag{23}$$
$$(D_{x_1}^4 - 4D_{x_1}D_{x_3} + 3D_{x_2}^2)f \cdot f = 0, \tag{24}$$
$$D_{x_1}D_{y_1^{(k)}} f \cdot f = 2g^{(k)}g^{(k)*}, \quad k = 1, 2, \tag{25}$$
$$(D_{x_1}D_{x_{-1}^{(1)}} - 2)f \cdot f = -2h^{(1)}h^{(1)*}. \tag{26}$$

Using the independent variable transformations

$$x_1 = x, \quad x_2 = -iy, \quad x_3 = -8t, \tag{27}$$

i.e.,

$$\partial_x = \partial_{x_1}, \quad \partial_y = -i\partial_{x_2}, \quad \partial_t = -8\partial_{x_3}, \tag{28}$$

Eqs. (22)–(23) become Eqs. (8) and (9) by letting $c_1 = i\alpha_1$. Next, we show how to get Eq. (10) from Eqs. (24)–(26).

By row operations, f can be rewritten as

$$f = \begin{vmatrix} A' & I \\ -I & B' \end{vmatrix}, \tag{29}$$

where A' and B' are $N \times N$ matrices whose entries are

$$a'_{ij} = \frac{1}{p_i + p_j^*},$$
$$b'_{ij} = \frac{1}{q_i + q_j^*} e^{\eta_i + \eta_j^* + \xi_i^* + \xi_j} + \frac{1}{r_i + r_j^*} e^{\chi_i + \chi_j^* + \xi_i^* + \xi_j},$$

with

$$\eta_i + \xi_i^* = q_i y_1^{(1)} + \frac{1}{p_i^* + c_1} x_{-1}^{(1)} + p_i^* x_1 - p_i^{*2} x_2 + p_i^{*3} x_3 + \xi_{i0}^* + \eta_{i0},$$
$$\eta_j^* + \xi_j = q_j^* y_1^{(1)} + \frac{1}{p_j - c_1} x_{-1}^{(1)} + p_j x_1 + p_j^2 x_2 + p_j^3 x_3 + \xi_{j0} + \eta_{j0}^*,$$
$$\chi_i + \xi_i^* = r_i y_1^{(2)} + \frac{1}{p_i^* + c_1} x_{-1}^{(1)} + p_i^* x_1 - p_i^{*2} x_2 + p_i^{*3} x_3 + \xi_{i0}^* + \chi_{i0},$$
$$\chi_j^* + \xi_j = r_j^* y_1^{(2)} + \frac{1}{p_j - c_1} x_{-1}^{(1)} + p_j x_1 + p_j^2 x_2 + p_j^3 x_3 + \xi_{j0} + \chi_{j0}^*.$$

Consider the following reduction conditions

$$8p_i^{*3} = \sigma_1 q_i - \frac{\sigma_3 \rho_1^2}{p_i^* + c_1}, \quad 8p_j^3 = \sigma_1 q_j^* - \frac{\sigma_3 \rho_1^2}{p_j - c_1}, \tag{30}$$
$$8p_i^{*3} = \sigma_2 r_i - \frac{\sigma_3 \rho_1^2}{p_i^* + c_1}, \quad 8p_j^3 = \sigma_2 r_j^* - \frac{\sigma_3 \rho_1^2}{p_j - c_1}, \tag{31}$$

i.e.,

$$\frac{1}{q_i + q_j^*} = \frac{\sigma_1}{8(p_i^{*3} + p_j^3) + \dfrac{\sigma_3 \rho_1^2 (p_i^* + p_j)}{(p_i^* + i\alpha_1)(p_j - i\alpha_1)}}, \tag{32}$$
$$\frac{1}{r_i + r_j^*} = \frac{\sigma_2}{8(p_i^{*3} + p_j^3) + \dfrac{\sigma_3 \rho_1^2 (p_i^* + p_j)}{(p_i^* + i\alpha_1)(p_j - i\alpha_1)}}, \tag{33}$$

we have the relation

$$8\partial_{x_3} b'_{ij} = (\sigma_1 \partial_{y_1^{(1)}} + \sigma_2 \partial_{y_1^{(2)}} - \sigma_3 \rho_1^2 \partial_{x_{-1}^{(1)}}) b'_{ij}. \tag{34}$$

Equation (34) immediately leads to

$$8f_{x_3} = \sigma_1 f_{y_1^{(1)}} + \sigma_2 f_{y_1^{(2)}} - \sigma_3 \rho_1^2 f_{x_{-1}^{(1)}}, \tag{35}$$

and the derivation of (35) with respect to x_1 reads

$$8f_{x_1 x_3} = \sigma_1 f_{x_1 y_1^{(1)}} + \sigma_2 f_{x_1 y_1^{(2)}} - \sigma_3 \rho_1^2 f_{x_1 x_{-1}^{(1)}}. \tag{36}$$

On the other hand, Eqs. (25) and (26) can be expanded as

$$f_{x_1 y_1^{(1)}} f - f_{x_1} f_{y_1^{(1)}} = g^{(1)} g^{(1)*}, \quad f_{x_1 y_1^{(2)}} f - f_{x_1} f_{y_1^{(2)}} = g^{(2)} g^{(2)*}, \tag{37}$$

and

$$f_{x_1 x_{-1}^{(1)}} f - f_{x_1} f_{x_{-1}^{(1)}} - f^2 = -h^{(1)} h^{(1)*}, \tag{38}$$

respectively. By using relations (35) and (36), from (37) and (38), we arrive at

$$\begin{aligned} &-4D_{x_1} D_{x_3} f \cdot f \\ &\quad = -\sigma_1 g^{(1)} g^{(1)*} - \sigma_2 g^{(2)} g^{(2)*} + \sigma_3 \rho_1^2 (f^2 - h^{(1)} h^{(1)*}). \end{aligned} \tag{39}$$

Also by the transformations in (27), from Eqs. (24) and (39), Eq. (10) is immediately obtained.

Under the variable transformations (27), the variables $y_1^{(1)}$, $y_1^{(2)}$, and $x_{-1}^{(1)}$ become dummy variables, thus they can be treated as constants. Consequently, we can take $e^{\eta_i} = c_i^{(1)*}$, $e^{\eta_i^*} = c_i^{(1)}$, $e^{\chi_i} = c_i^{(2)*}$, and $e^{\chi_i^*} = c_i^{(2)}$; $(i = 1, 2, \ldots, N)$ and define $C_1 = -(c_1^{(1)}, c_2^{(1)}, \ldots, c_N^{(1)})$ and $C_2 = -(c_1^{(2)}, c_2^{(2)}, \ldots, c_N^{(2)})$, thus, we have obtained the 2-b-1-d mixed N-soliton solution of the three-component Mel'nikov system (5)–(6) as

$$f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \quad g^{(k)} = \begin{vmatrix} A & I & \Omega^{\mathrm{T}} \\ -I & B & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & C_k & 0 \end{vmatrix}, \quad h^{(1)} = \begin{vmatrix} A^{(1)} & I \\ -I & B \end{vmatrix}, \tag{40}$$

where the entries in A, $A^{(1)}$, and B are defined as

$$\begin{aligned} a_{ij} &= \frac{1}{p_i + p_j^*} e^{\xi_i + \xi_j^*}, \\ a_{ij}^{(1)} &= \frac{1}{p_i + p_j^*} \left(-\frac{p_i - i\alpha_1}{p_j^* + i\alpha_1} \right) e^{\xi_i + \xi_j^*}, \end{aligned} \tag{41}$$

$$\begin{aligned} b_{ij} = \left(\sum_{k=1}^{2} \sigma_k c_i^{(k)*} c_j^{(k)} \right) \Bigg[& 8(p_i^{*3} + p_j^3) \\ & + \frac{\sigma_3 \rho_1^2 (p_i^* + p_j)}{(p_i^* + i\alpha_1)(p_j - i\alpha_1)} \Bigg]^{-1}, \end{aligned} \tag{42}$$

respectively; Ω and C_k are N-component row vectors

$$\Omega = (e^{\xi_1}, e^{\xi_2}, \ldots, e^{\xi_N}), \quad C_k = -(c_1^{(k)}, c_2^{(k)}, \ldots, c_N^{(k)}), \tag{43}$$

with $\xi_i = p_i x - i p_i^2 y - 8 p_i^3 t + \xi_{i0}$, and p_i, ξ_{i0}, $c_i^{(k)}$ $(k = 1, 2;\ i = 1, 2, \ldots, N)$ are complex constants.

2.1.1 *One-soliton solution*

By taking $N = 1$ in formula (40), we can get the one-soliton solution. For this case, the tau functions can be rewritten as

$$f = 1 + E_{11^*} e^{\xi_1 + \xi_1^*}, \tag{44}$$

$$g^{(k)} = c_1^{(k)} e^{\xi_1}, \quad k = 1, 2, \tag{45}$$

$$h^{(1)} = 1 + F_{11^*} e^{\xi_1 + \xi_1^*}, \tag{46}$$

where

$$\begin{aligned} E_{11^*} = \left(\sum_{k=1}^{2} \sigma_k c_1^{(k)} c_1^{(k)*} \right) \Bigg[& 8(p_1 + p_1^*)(p_1^3 + p_1^{*3}) \\ & + \frac{\sigma_3 \rho_1^2 (p_1 + p_1^*)^2}{(p_1^* + i\alpha_1)(p_1 - i\alpha_1)} \Bigg]^{-1}, \\ F_{11^*} = -\frac{p_1 - i\alpha_1}{p_1^* + i\alpha_1} & E_{11^*}, \end{aligned}$$

with $\xi_1 = p_1 x - i p_1^2 y - 8 p_1^3 t + \xi_{10}$. Moreover, the one-soliton solution is nonsingular only when $E_{11^*} > 0$.

The 2-b-1-d mixed one-soliton solution can be expressed in the form

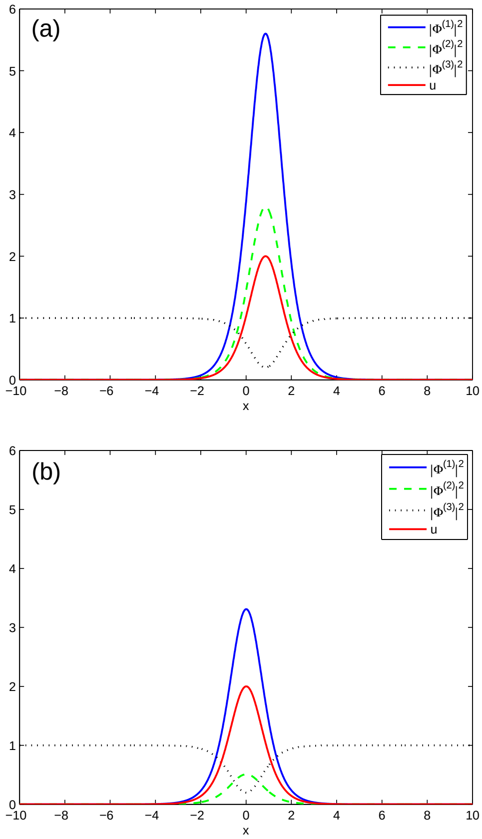

Fig. 1. (Color online) 2-b-1-d mixed one-soliton solution of the three-component Mel'nikov system (5)–(6).

$$\Phi^{(k)} = \frac{c_1^{(k)}}{2} e^{i\xi_{1I} - \eta_1} \operatorname{sech}(\xi_{1R} + \eta_1), \quad k = 1, 2, \tag{47}$$

$$\Phi^{(3)} = \frac{\rho_1}{2} e^{i\theta_1} [1 + e^{2i\phi_1} + (e^{2i\phi_1} - 1) \tanh(\xi_{1R} + \eta_1)], \tag{48}$$

$$u = 2 p_{1R}^2 \operatorname{sech}^2(\xi_{1R} + \eta_1), \tag{49}$$

where $e^{2\eta_1} = E_{11^*}$, $e^{2i\phi_1} = -(p_1 - i\alpha_1)/(p_1^* + i\alpha_1)$, $\xi_1 = \xi_{1R} + i\xi_{1I}$, the suffixes R and I denote the real and imaginary parts, respectively. Obviously, the amplitude of the bright soliton in the $\Phi^{(k)}$ component is $\frac{|c_1^{(k)}|}{2} e^{-\eta_1}$ while the amplitude of the bright soliton in the u component is $2p_{1R}^2$. For the dark soliton in the $\Phi^{(3)}$ component, $|\Phi^{(3)}|$ approaches $|\rho_1|$ as $x, y \to \pm\infty$. In addition, the intensity of the dark soliton is $|\rho_1| \cos \phi_1$. As the parameters $c_1^{(k)}$ appear explicitly in the amplitude of the bright parts of the mixed one-soliton, we can tune the intensity of the bright parts without altering the depth of the dark part. The mixed one-soliton at time $t = 0$ is displayed in Fig. 1 with the nonlinearities $(\sigma_1, \sigma_2, \sigma_3) = (1, -1, 1)$. The parameters used are $p_1 = 1 + \frac{1}{2} i$, $\alpha_1 = \rho_1 = 1$, $\xi_{10} = y = 0$, $c_1^{(2)} = 1 + i$, and (a) $c_1^{(1)} = 2$; (b) $c_1^{(1)} = 2 + 3i$. It can be seen that when the parameters $c_1^{(k)}$ take different values, the intensities of the bright solitons in the $\Phi^{(1)}$ and $\Phi^{(2)}$ components change, while the intensity of the bright soliton in the u component and the depth of the dark soliton remain unaltered.

2.1.2 *Two-soliton solution*

To obtain the two-soliton solution, we take $N = 2$ in formula (40). In this case, the tau functions can be rewritten as

$$f = 1 + E_{11^*}e^{\xi_1+\xi_1^*} + E_{12^*}e^{\xi_1+\xi_2^*} + E_{21^*}e^{\xi_2+\xi_1^*} + E_{22^*}e^{\xi_2+\xi_2^*} + E_{121^*2^*}e^{\xi_1+\xi_2+\xi_1^*+\xi_2^*}, \tag{50}$$

$$g^{(k)} = c_1^{(k)}e^{\xi_1} + c_2^{(k)}e^{\xi_2} + G_{121^*}^{(k)}e^{\xi_1+\xi_2+\xi_1^*} + G_{122^*}^{(k)}e^{\xi_1+\xi_2+\xi_2^*}, \quad k = 1, 2, \tag{51}$$

$$h^{(1)} = 1 + F_{11^*}^{(1)}e^{\xi_1+\xi_1^*} + F_{12^*}^{(1)}e^{\xi_1+\xi_2^*} + F_{21^*}^{(1)}e^{\xi_2+\xi_1^*} + F_{22^*}^{(1)}e^{\xi_2+\xi_2^*} + F_{121^*2^*}^{(1)}e^{\xi_1+\xi_2+\xi_1^*+\xi_2^*}, \tag{52}$$

where

$$E_{ij^*} = \left(\sum_{k=1}^{2}\sigma_k c_i^{(k)}c_j^{(k)*}\right)\left[8(p_i+p_j^*)(p_i^3+p_j^{*3}) + \frac{\sigma_3\rho_1^2(p_i+p_j^*)^2}{(p_i-i\alpha_1)(p_j^*+i\alpha_1)}\right]^{-1},$$

$$E_{121^*2^*} = |p_1-p_2|^2\left[\frac{E_{11^*}E_{22^*}}{(p_1+p_2^*)(p_2+p_1^*)} - \frac{E_{12^*}E_{21^*}}{(p_1+p_1^*)(p_2+p_2^*)}\right],$$

$$F_{ij^*}^{(1)} = -\frac{p_i-i\alpha_1}{p_j^*+i\alpha_1}E_{ij^*},$$

$$F_{121^*2^*}^{(1)} = \frac{(p_1-i\alpha_1)(p_2-i\alpha_1)}{(p_1^*+i\alpha_1)(p_2^*+i\alpha_1)}E_{121^*2^*},$$

$$G_{12i^*}^{(k)} = (p_1-p_2)\left(\frac{c_1^{(k)}E_{2i^*}}{p_1+p_i^*} - \frac{c_2^{(k)}E_{1i^*}}{p_2+p_i^*}\right),$$

with $\xi_i = p_i x - ip_i^2 y - 8p_i^3 t + \xi_{i0}$ for $i = 1, 2$.

To get nonsingular solution, the denominator f must be nonzero. For this purpose, we rewrite f as

$$f = 2e^{\xi_{1R}+\xi_{2R}}[e^{\zeta_R}\cos(\xi_{1I}-\xi_{2I}+\zeta_I) + e^{\eta_1+\eta_2}\cosh(\xi_{1R}-\xi_{2R}+\eta_1-\eta_2) + e^{\eta_3}\cosh(\xi_{1R}-\xi_{2R}+\eta_3)], \tag{53}$$

where

$$e^{2\eta_1} = E_{11^*},$$
$$e^{2\eta_2} = E_{22^*},$$
$$e^{2\eta_3} = E_{121^*2^*},$$
$$e^{\zeta_R+i\zeta_I} = E_{12^*}.$$

By combining the condition for the existence of a one-soliton solution, it is easy to show that $E_{ii^*} > 0$, $i = 1, 2$ is a necessary condition and $e^{\eta_1+\eta_2} + e^{\eta_3} > e^{\zeta_R}$ is a sufficient condition to guarantee a nonsingular two-soliton solution. Based on the above tau functions, the asymptotic analysis of the two-soliton solution can be performed as in Refs. 11 and 13; this will help us to understand the collision of two solitons. In particular, we study the interaction of two solitons in the x–y plane. A similar approach can also be applied to study the collision in other planes.

The asymptotic forms of two solitons s_1 and s_2 before and after collision can be deduced from the transformations in (7) and the tau functions in (50)–(52).

(a) Before collision ($x, y \to -\infty$)

Soliton s_1

$$\Phi_{1-}^{(k)} \simeq A_{1-}^{(k)}e^{i\xi_{1I}}\operatorname{sech}(\xi_{1R}+\eta_1), \quad k = 1, 2,$$

$$\Phi_{1-}^{(3)} \simeq \frac{\rho_1}{2}e^{i\theta_1}[1 + e^{2i\phi_1} + (e^{2i\phi_1}-1)\tanh(\xi_{1R}+\eta_1)],$$

$$u_{1-} \simeq 2p_{1R}^2\operatorname{sech}^2(\xi_{1R}+\eta_1).$$

Soliton s_2

$$\Phi_{2-}^{(k)} \simeq A_{2-}^{(k)}e^{i\xi_{2I}}\operatorname{sech}(\xi_{2R}+\eta_3-\eta_1), \quad k = 1, 2,$$

$$\Phi_{2-}^{(3)} \simeq \frac{\rho_1}{2}e^{i(\theta_1+2\phi_1)}[1 + e^{2i\phi_2} + (e^{2i\phi_2}-1)\tanh(\xi_{2R}+\eta_3-\eta_1)],$$

$$u_{2-} \simeq 2p_{2R}^2\operatorname{sech}^2(\xi_{2R}+\eta_3-\eta_1).$$

(b) After collision ($x, y \to +\infty$)

Soliton s_1

$$\Phi_{1+}^{(k)} \simeq A_{1+}^{(k)}e^{i\xi_{1I}}\operatorname{sech}(\xi_{1R}+\eta_3-\eta_2), \quad k = 1, 2,$$

$$\Phi_{1+}^{(3)} \simeq \frac{\rho_1}{2}e^{i(\theta_1+2\phi_2)}[1 + e^{2i\phi_1} + (e^{2i\phi_1}-1)\tanh(\xi_{1R}+\eta_3-\eta_2)],$$

$$u_{1+} \simeq 2p_{1R}^2\operatorname{sech}^2(\xi_{1R}+\eta_3-\eta_2).$$

Soliton s_2

$$\Phi_{2+}^{(k)} \simeq A_{2+}^{(k)}e^{i\xi_{2I}}\operatorname{sech}(\xi_{2R}+\eta_2), \quad k = 1, 2,$$

$$\Phi_{2+}^{(3)} \simeq \frac{\rho_1}{2}e^{i\theta_1}[1 + e^{2i\phi_2} + (e^{2i\phi_2}-1)\tanh(\xi_{2R}+\eta_2)],$$

$$u_{2+} \simeq 2p_{2R}^2\operatorname{sech}^2(\xi_{2R}+\eta_2).$$

In the above expressions, $e^{2i\phi_j} = -(p_j-i\alpha_1)/(p_j^*+i\alpha_1)$ for $j = 1, 2$; $(A_{1-}^{(1)}, A_{2-}^{(1)})$ are the amplitudes of the bright parts of the mixed two solitons s_1 and s_2 before interaction; $(A_{1+}^{(1)}, A_{2+}^{(1)})$ are the corresponding amplitudes after interaction. Here the superscript (subscript) of A represents the component (soliton) number and $-$ $(+)$ denotes the soliton before (after) collision, and the various amplitudes are given by

$$A_{1-}^{(k)} = \frac{c_1^{(k)}}{2\sqrt{E_{11^*}}},$$

$$A_{2-}^{(k)} = \frac{G_{121^*}^{(k)}}{2\sqrt{E_{11^*}E_{121^*2^*}}},$$

$$A_{1+}^{(k)} = \frac{G_{122^*}^{(k)}}{2\sqrt{E_{22^*}E_{121^*2^*}}},$$

$$A_{2+}^{(k)} = \frac{c_2^{(k)}}{2\sqrt{E_{22^*}}}.$$

What is more, there is a relation between the amplitudes before and after collision[13)]

$$A_{i+}^{(k)} = T_i^k A_{i-}^{(k)}, \quad i, k = 1, 2, \tag{54}$$

where the transition amplitudes T_i^k are given by

$$T_1^k = \left(\frac{p_1-p_2}{p_1^*-p_2^*}\right)\left(\frac{p_1^*+p_2}{p_1+p_2^*}\right)^{1/2}\left[\frac{(c_2^{(k)}/c_1^{(k)})r_1 - 1}{\sqrt{1-r_1r_2}}\right],$$

$$T_2^k = \left(\frac{p_1-p_2}{p_1^*-p_2^*}\right)\left(\frac{p_1^*+p_2}{p_1+p_2^*}\right)^{1/2}\left[\frac{\sqrt{1-r_1r_2}}{(c_1^{(k)}/c_2^{(k)})r_2 - 1}\right],$$

$$k = 1, 2,$$

with

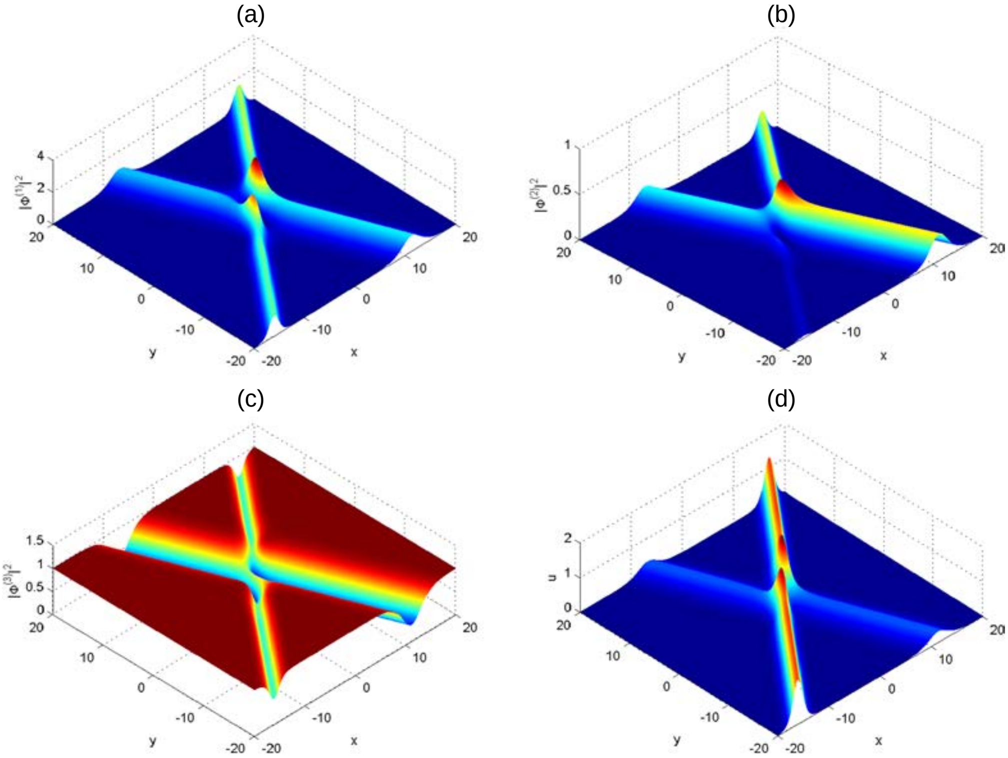

Fig. 2. (Color online) 2-b-1-d mixed two-soliton solution of the three-component Mel'nikov system (5)–(6) including inelastic collisions in the $\Phi^{(1)}$ and $\Phi^{(2)}$ components.

$$r_1 = \frac{p_1 + p_2^*}{p_2 + p_2^*}\frac{E_{12^*}}{E_{22^*}}, \quad r_2 = \frac{p_1^* + p_2}{p_1 + p_1^*}\frac{E_{21^*}}{E_{11^*}}.$$

The relation (54) means that the intensities of the bright parts of the mixed two solitons before and after collision alter in general. Only under the condition $\frac{|c_1^{(1)}|}{|c_2^{(1)}|} = \frac{|c_1^{(2)}|}{|c_2^{(2)}|}$, the transition amplitudes T_i^k become unimodular. This indicates that in general, the bright parts of the mixed two solitons exhibit energy-exchanging inelastic collision characterized with an intensity redistribution (energy sharing) among the bright parts of the mixed solitons in the $\Phi^{(1)}$ and $\Phi^{(2)}$ components. As the amplitudes of the dark parts of the mixed two solitons in the $\Phi^{(3)}$ component are the same and equal to ρ_1, the intensities of the dark parts remain unaltered after collision. Thus, the dark parts of the mixed two solitons undergo only elastic collision. What is more, both the bright and dark parts of the mixed two solitons undergo a position shift with the same magnitude but with opposite signs. The position shift of soliton s_1 (and s_2) is $\Lambda_1 = \eta_3 - \eta_1 - \eta_2$ (and $\Lambda_2 = -\Lambda_1$). In addition, similar analysis of the u component shows that standard elastic collision takes place between the two bright solitons in the u component. The collisions of two solitons of the three-component Mel'nikov system (5)–(6) with the nonlinearities $(\sigma_1, \sigma_2, \sigma_3) = (1, -1, 1)$ are depicted in Figs. 2 and 3 at time $t = 0$. The parameters used in Fig. 2 are $p_1 = \frac{2}{5} + \frac{1}{5}i$, $p_2 = \frac{4}{5} - \frac{2}{5}i$, $c_1^{(1)} = 1 + \frac{1}{2}i$, $c_2^{(1)} = \frac{5}{2}$, $c_1^{(2)} = \frac{1}{2}i$, $c_2^{(2)} = \frac{1}{4} - \frac{1}{4}i$, $\rho_1 = 1$, $\alpha_1 = \frac{1}{2}$, and $\xi_{10} = \xi_{20} = 0$, which correspond to inelastic collisions of the bright solitons in the $\Phi^{(1)}$ and $\Phi^{(2)}$ components. An example of elastic collision of the bright solitons in the $\Phi^{(1)}$ and $\Phi^{(2)}$ components is displayed in Fig. 3 with the parametric choice $c_1^{(1)} = 1$, $c_2^{(1)} = 2$, $c_1^{(2)} = \frac{1}{3}$, $c_2^{(2)} = \frac{2}{3}$ and the other parameters the same as in Fig. 2. In Figs. 2 and 3, (a) and (b) represent the collisions of two bright solitons in the $\Phi^{(1)}$ and $\Phi^{(2)}$ components, respectively; the collision of two dark solitons in the $\Phi^{(3)}$ component is displayed in (c); and (d) represents the collision of two bright solitons in the u component. The difference between Figs. 2 and 3 is in the inelastic and elastic collisions, which only appear in (a) and (b), respectively. It is obvious that in Figs. 2(a) and 2(b), the intensity of one soliton is suppressed while the intensity of the other soliton is enhanced after collision. The physical mechanism behind this interesting collision is attributed to an intensity redistribution among the components accompanied by a finite amplitude-dependent position shift.

2.2 *1-b-2-d mixed soliton solution*

In this case, assuming the $\Phi^{(1)}$ component is of the bright type while the $\Phi^{(2)}$ and $\Phi^{(3)}$ components are of the dark type, we introduce the dependent variable transformations

$$\begin{aligned}
\Phi^{(1)} &= \frac{g^{(1)}}{f}, \\
\Phi^{(2)} &= \rho_1 e^{i\theta_1}\frac{h^{(1)}}{f}, \\
\Phi^{(3)} &= \rho_2 e^{i\theta_2}\frac{h^{(2)}}{f}, \\
u &= 2(\log f)_{xx},
\end{aligned} \tag{55}$$

which convert the three-component Mel'nikov system (5)–(6) into the bilinear forms

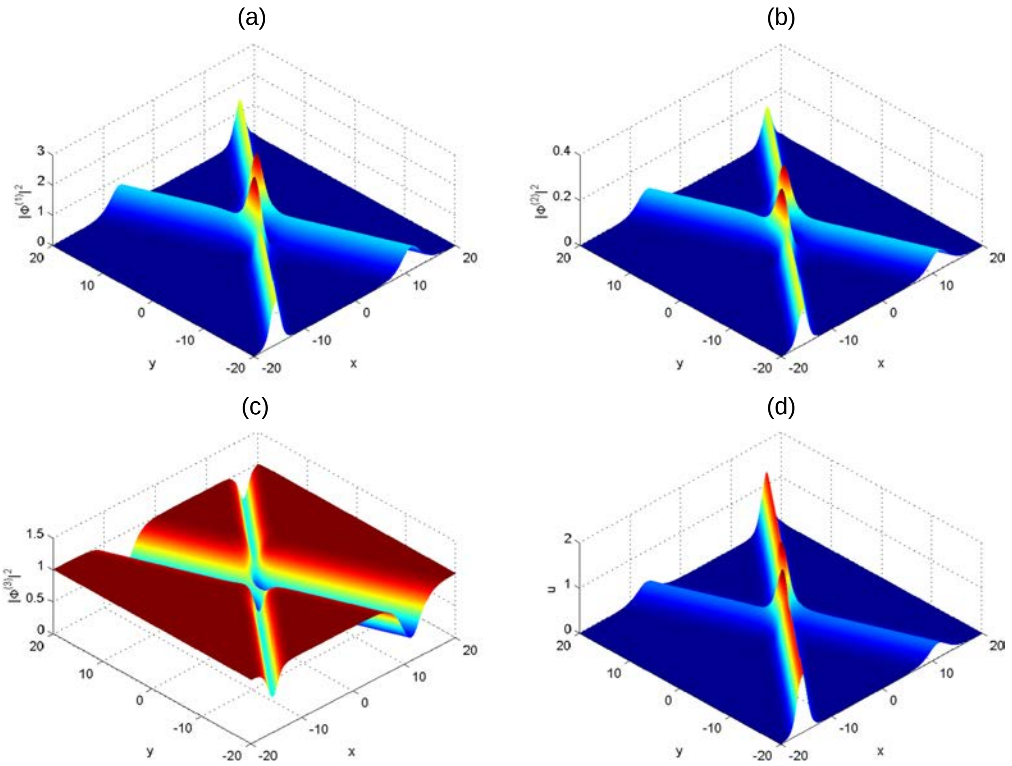

Fig. 3. (Color online) 2-b-1-d mixed two-soliton solution of the three-component Mel'nikov system (5)–(6) including elastic collisions in $\Phi^{(1)}$ and $\Phi^{(2)}$ components.

$$(D_x^2 - iD_y)g^{(1)} \cdot f = 0, \tag{56}$$

$$(D_x^2 + 2i\alpha_k D_x - iD_y)h^{(k)} \cdot f = 0, \quad k = 1, 2, \tag{57}$$

$$\begin{aligned}&(D_x^4 + D_x D_t - 3D_y^2)f \cdot f \\ &\quad = -\sigma_1 g^{(1)} g^{(1)*} + \sum_{k=1}^{2} \sigma_{k+1}\rho_k^2 (f^2 - h^{(k)}h^{(k)*}),\end{aligned} \tag{58}$$

where f is a real function; $g^{(1)}$, $h^{(1)}$, and $h^{(2)}$ are complex functions; $\theta_k = \alpha_k x + \alpha_k^2 y + \beta_k(t)$, α_k, and ρ_k $(k = 1, 2)$ are real constants, and $\beta_k(t)$ $(k = 1, 2)$ are real functions.

To construct the 1-b-2-d mixed N-soliton solution, we start with a two-component KP hierarchy with two copies of shifted singular points (c_1 and c_2)

$$(D_{x_1}^2 - D_{x_2})\tau_1(k_1, k_2) \cdot \tau_0(k_1, k_2) = 0, \tag{59}$$

$$(D_{x_1}^2 - D_{x_2} + 2c_1 D_{x_1})\tau_0(k_1 + 1, k_2) \cdot \tau_0(k_1, k_2) = 0, \tag{60}$$

$$(D_{x_1}^2 - D_{x_2} + 2c_2 D_{x_1})\tau_0(k_1, k_2 + 1) \cdot \tau_0(k_1, k_2) = 0, \tag{61}$$

$$(D_{x_1}^4 - 4D_{x_1}D_{x_3} + 3D_{x_2}^2)\tau_0(k_1, k_2) \cdot \tau_0(k_1, k_2) = 0, \tag{62}$$

$$D_{x_1}D_{y_1^{(1)}}\tau_0(k_1, k_2) \cdot \tau_0(k_1, k_2) = -2\tau_1(k_1, k_2)\tau_{-1}(k_1, k_2), \tag{63}$$

$$\begin{aligned}&(D_{x_1}D_{x_{-1}^{(1)}} - 2)\tau_0(k_1, k_2) \cdot \tau_0(k_1, k_2) \\ &\quad = -2\tau_0(k_1 + 1, k_2)\tau_0(k_1 - 1, k_2),\end{aligned} \tag{64}$$

$$\begin{aligned}&(D_{x_1}D_{x_{-1}^{(2)}} - 2)\tau_0(k_1, k_2) \cdot \tau_0(k_1, k_2) \\ &\quad = -2\tau_0(k_1, k_2 + 1)\tau_0(k_1, k_2 - 1).\end{aligned} \tag{65}$$

Based on the Sato theory for the KP hierarchy,[31] the bilinear equations (59)–(65) have the Gram determinant tau function solution

$$\tau_0(k_1, k_2) = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \tag{66}$$

$$\begin{aligned}\tau_1(k_1, k_2) &= \begin{vmatrix} A & I & \Omega^{\mathrm{T}} \\ -I & B & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & -\bar{\Psi} & 0 \end{vmatrix}, \\ \tau_{-1}(k_1, k_2) &= \begin{vmatrix} A & I & \mathbf{0}^{\mathrm{T}} \\ -I & B & \Psi^{\mathrm{T}} \\ -\bar{\Omega} & \mathbf{0} & 0 \end{vmatrix},\end{aligned} \tag{67}$$

where Ω, Ψ, $\bar{\Omega}$, and $\bar{\Psi}$ are the N-component row vectors defined previously; A and B are $N \times N$ matrices whose entries are

$$a_{ij}(k_1, k_2) = \frac{1}{p_i + \bar{p}_j}\left(-\frac{p_i - c_1}{\bar{p}_j + c_1}\right)^{k_1}\left(-\frac{p_i - c_2}{\bar{p}_j + c_2}\right)^{k_2} e^{\xi_i + \bar{\xi}_j},$$

$$b_{ij} = \frac{1}{q_i + \bar{q}_j} e^{\eta_i + \bar{\eta}_j},$$

with

$$\xi_i = \frac{1}{p_i - c_1}x_{-1}^{(1)} + \frac{1}{p_i - c_2}x_{-1}^{(2)} + p_i x_1 + p_i^2 x_2 + p_i^3 x_3 + \xi_{i0},$$

$$\bar{\xi}_j = \frac{1}{\bar{p}_j + c_1}x_{-1}^{(1)} + \frac{1}{\bar{p}_j + c_2}x_{-1}^{(2)} + \bar{p}_j x_1 - \bar{p}_j^2 x_2 + \bar{p}_j^3 x_3 + \bar{\xi}_{j0},$$

$$\eta_i = q_i y_1^{(1)} + \eta_{i0}, \quad \bar{\eta}_j = \bar{q}_j y_1^{(1)} + \bar{\eta}_{j0},$$

in which p_i, $\bar{p}_j$, q_i, $\bar{q}_j$, ξ_{i0}, $\bar{\xi}_{j0}$, η_{i0}, $\bar{\eta}_{j0}$, c_1, and c_2 are complex constants.

We also first consider complex conjugate reduction by assuming x_1, x_3, $x_{-1}^{(1)}$, $x_{-1}^{(2)}$, and $y_1^{(1)}$ are real; x_2, c_1, and c_2 are pure imaginary, defining $p_j^* = \bar{p}_j$, $q_j^* = \bar{q}_j$, $\xi_{j0}^* = \bar{\xi}_{j0}$, and $\eta_{j0}^* = \bar{\eta}_{j0}$. It is easy to check that

$$a_{ij}(k_1, k_2) = a_{ji}^*(-k_1, -k_2), \quad b_{ij} = b_{ji}^*.$$

Furthermore, by letting

$$f = \tau_0(0,0),$$
$$g^{(1)} = \tau_1(0,0),$$
$$h^{(1)} = \tau_0(1,0),$$
$$h^{(2)} = \tau_0(0,1),$$

thus, f is real and

$$g^{(1)*} = -\tau_{-1}(0,0), \quad h^{(1)*} = \tau_0(-1,0), \quad h^{(2)*} = \tau_0(0,-1).$$

Then the bilinear equations (59)–(65) become

$$(D_{x_1}^2 - D_{x_2})g^{(1)} \cdot f = 0, \tag{68}$$

$$(D_{x_1}^2 - D_{x_2} + 2c_k D_{x_1})h^{(k)} \cdot f = 0, \quad k = 1,2, \tag{69}$$

$$(D_{x_1}^4 - 4D_{x_1}D_{x_3} + 3D_{x_2}^2)f \cdot f = 0, \tag{70}$$

$$D_{x_1}D_{y_1^{(1)}} f \cdot f = 2g^{(1)}g^{(1)*}, \tag{71}$$

$$(D_{x_1}D_{x_{-1}^{(k)}} - 2)f \cdot f = -2h^{(k)}h^{(k)*}, \quad k = 1,2. \tag{72}$$

Also by the transformations in (27), Eqs. (68) and (69) are recast into Eqs. (56) and (57) along with $c_1 = i\alpha_1$ and $c_2 = i\alpha_2$. Thus, the remaining task is to derive Eq. (58) from Eqs. (70)–(72).

In a similar way, under the reduction conditions

$$8p_i^{*3} = \sigma_1 q_i - \frac{\sigma_2\rho_1^2}{p_i^* + c_1} - \frac{\sigma_3\rho_2^2}{p_i^* + c_2},$$
$$8p_j^3 = \sigma_1 q_j^* - \frac{\sigma_2\rho_1^2}{p_j - c_1} - \frac{\sigma_3\rho_2^2}{p_j - c_2}, \tag{73}$$

i.e.,

$$\frac{1}{q_i + q_j^*} = \frac{\sigma_1}{8(p_i^{*3} + p_j^3) + \dfrac{\sigma_2\rho_1^2(p_i^* + p_j)}{(p_i^* + c_1)(p_j - c_1)} + \dfrac{\sigma_3\rho_2^2(p_i^* + p_j)}{(p_i^* + c_2)(p_j - c_2)}}, \tag{74}$$

the following relation holds

$$8f_{x_3} = \sigma_1 f_{y_1^{(1)}} - \sigma_2\rho_1^2 f_{x_{-1}^{(1)}} - \sigma_3\rho_2^2 f_{x_{-1}^{(2)}}, \tag{75}$$

which also gives

$$8f_{x_1x_3} = \sigma_1 f_{x_1y_1^{(1)}} - \sigma_2\rho_1^2 f_{x_1x_{-1}^{(1)}} - \sigma_3\rho_2^2 f_{x_1x_{-1}^{(2)}}. \tag{76}$$

Similarly, Eqs. (71) and (72) can be expanded as

$$f_{x_1y_1^{(1)}} f - f_{x_1} f_{y_1^{(1)}} = g^{(1)}g^{(1)*}, \tag{77}$$

and

$$f_{x_1x_{-1}^{(1)}} f - f_{x_1} f_{x_{-1}^{(1)}} - f^2 = -h^{(1)}h^{(1)*},$$
$$f_{x_1x_{-1}^{(2)}} f - f_{x_1} f_{x_{-1}^{(2)}} - f^2 = -h^{(2)}h^{(2)*}, \tag{78}$$

respectively. By referring to the relations (75) and (76), from (77) and (78), we arrive at

$$-4D_{x_1}D_{x_3} f \cdot f = -\sigma_1 g^{(1)}g^{(1)*} + \sigma_2\rho_1^2(f^2 - h^{(1)}h^{(1)*}) + \sigma_3\rho_2^2(f^2 - h^{(2)}h^{(2)*}). \tag{79}$$

By using the transformations in (27), from Eqs. (70) and (79), Eq. (58) is immediately obtained.

To conclude, we have obtained the 1-b-2-d mixed N-soliton solution to the three-component Mel'nikov system (5)–(6) as

$$f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \quad g^{(1)} = \begin{vmatrix} A & I & \Omega^{\mathrm{T}} \\ -I & B & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & C_1 & 0 \end{vmatrix}, \quad h^{(k)} = \begin{vmatrix} A^{(k)} & I \\ -I & B \end{vmatrix}, \tag{80}$$

where the elements in A, $A^{(k)}$, and B are given by

$$a_{ij} = \frac{1}{p_i + p_j^*} e^{\xi_i + \xi_j^*},$$

$$a_{ij}^{(k)} = \frac{1}{p_i + p_j^*}\left(-\frac{p_i - i\alpha_k}{p_j^* + i\alpha_k}\right) e^{\xi_i + \xi_j^*}, \tag{81}$$

$$b_{ij} = \sigma_1 c_i^{(1)*} c_j^{(1)} \left[8(p_i^{*3} + p_j^3) + \sum_{l=1}^{2} \frac{\sigma_{l+1}\rho_l^2(p_i^* + p_j)}{(p_i^* + i\alpha_l)(p_j - i\alpha_l)}\right]^{-1}, \tag{82}$$

meanwhile, Ω and C_1 are

$$\Omega = (e^{\xi_1}, e^{\xi_2}, \ldots, e^{\xi_N}), \quad C_1 = -(c_1^{(1)}, c_2^{(1)}, \ldots, c_N^{(1)}), \tag{83}$$

with $\xi_i = p_i x - ip_i^2 y - 8p_i^3 t + \xi_{i0}$; p_i, ξ_{i0}, and $c_i^{(1)}$ ($k = 1,2$; $i = 1,2,\ldots,N$) are complex constants.

2.2.1 *One-soliton solution*

To get the one-soliton solution, we take $N = 1$ in formula (80). In this case, the tau functions can be rewritten as

$$f = 1 + E_{11^*} e^{\xi_1 + \xi_1^*}, \tag{84}$$

$$g^{(1)} = c_1^{(1)} e^{\xi_1}, \tag{85}$$

$$h^{(k)} = 1 + F_{11^*} e^{\xi_1 + \xi_1^*}, \quad k = 1,2, \tag{86}$$

where

$$E_{11^*} = \sigma_1 c_1^{(1)} c_1^{(1)*} \left[8(p_1 + p_1^*)(p_1^3 + p_1^{*3}) + \sum_{l=1}^{2} \frac{\sigma_{l+1}\rho_l^2(p_1 + p_1^*)^2}{(p_1 - i\alpha_l)(p_1^* + i\alpha_l)}\right]^{-1},$$

$$F_{11^*} = -\frac{p_1 - i\alpha_k}{p_1^* + i\alpha_k} E_{11^*},$$

with $\xi_1 = p_1 x - ip_1^2 y - 8p_1^3 t + \xi_{10}$. Note that this solution is nonsingular only when $E_{11^*} > 0$.

The 1-b-2-d mixed one-soliton solution can be expressed in the following form

$$\Phi^{(1)} = \frac{c_1^{(1)}}{2} e^{i\xi_{1I} - \eta_1} \operatorname{sech}[\xi_{1R} + \eta_1], \tag{87}$$

$$\Phi^{(k+1)} = \frac{\rho_k}{2} e^{i\theta_k}\{1 + e^{2i\phi_k} + (e^{2i\phi_k} - 1)\tanh[\xi_{1R} + \eta_1]\},$$
$$k = 1,2, \tag{88}$$

$$u = 2p_{1R}^2 \operatorname{sech}^2[\xi_{1R} + \eta_1], \tag{89}$$

where $e^{2\eta_1} = E_{11^*}$, $e^{2i\phi_k} = -(p_1 - i\alpha_k)/(p_1^* + i\alpha_k)$, and $\xi_1 = \xi_{1R} + i\xi_{1I}$. The amplitude of the bright part of the mixed single soliton in the $\Phi^{(1)}$ component is $\frac{|c_1^{(1)}|}{2} e^{-\eta_1}$ while the amplitude of the bright soliton in the u component is $2p_{1R}^2$. For the dark parts of the mixed single soliton in the $\Phi^{(2)}$ and

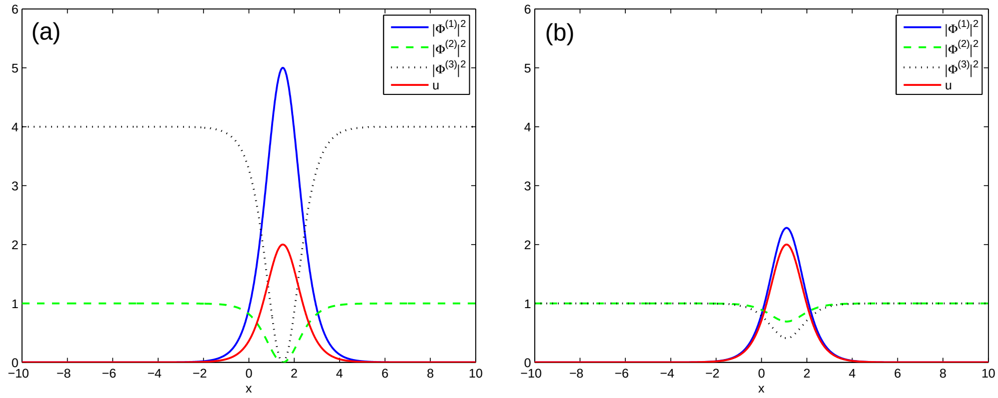

Fig. 4. (Color online) 1-b-2-d mixed one-soliton solution of the three-component Mel'nikov system (5)–(6); (a) degenerate case; (b) non-degenerate case.

$\Phi^{(3)}$ components, $|\Phi^{(k+1)}|$, $(k = 1, 2)$ approaches $|\rho_k|$ as $x, y \to \pm\infty$. What is more, the intensity of the dark soliton in the $\Phi^{(k+1)}$ component is $|\rho_k| \cos\phi_k$ for $k = 1, 2$. In addition, according to the values of α_1 and α_2, there exist two different cases: (i) $\alpha_1 = \alpha_2$ and (ii) $\alpha_1 \neq \alpha_2$. In the former case, we have $\phi_1 = \phi_2$, which implies that the dark solitons in the $\Phi^{(2)}$ and $\Phi^{(3)}$ components are proportional to each other. Therefore, this case is considered as degenerate. In the latter case, the condition $\phi_1 \neq \phi_2$ means that the dark solitons in the $\Phi^{(2)}$ and $\Phi^{(3)}$ components have different degrees of darkness at the center. In this case, $\Phi^{(2)}$ and $\Phi^{(3)}$ are not proportional to each other; thus, this is viewed as a non-degenerate case. Both the degenerate and non-degenerate cases are illustrated in Fig. 4 with the parametric choice $p_1 = 1 + \frac{1}{2}i$, $c_1^{(1)} = \rho_1 = 1$, $\xi_{10} = y = 0$, and (a) $\alpha_1 = \alpha_2 = \frac{1}{2}$, $\rho_2 = 2$; (b) $\alpha_1 = 2$, $\alpha_2 = -\frac{1}{3}$, $\rho_2 = 1$ at time $t = 0$ with the nonlinearities $(\sigma_1, \sigma_2, \sigma_3) = (1, -1, 1)$.

2.2.2 *Two-soliton solution*

To obtain the two-soliton solution, we take $N = 2$ in formula (80). In this case, the corresponding tau functions can be rewritten as

$$f = 1 + E_{11^*} e^{\xi_1+\xi_1^*} + E_{12^*} e^{\xi_1+\xi_2^*} + E_{21^*} e^{\xi_2+\xi_1^*} + E_{22^*} e^{\xi_2+\xi_2^*} + E_{121^*2^*} e^{\xi_1+\xi_2+\xi_1^*+\xi_2^*}, \quad (90)$$

$$g^{(1)} = c_1^{(1)} e^{\xi_1} + c_2^{(1)} e^{\xi_2} + G^{(1)}_{121^*} e^{\xi_1+\xi_2+\xi_1^*} + G^{(1)}_{122^*} e^{\xi_1+\xi_2+\xi_2^*}, \quad (91)$$

$$h^{(k)} = 1 + F^{(k)}_{11^*} e^{\xi_1+\xi_1^*} + F^{(k)}_{12^*} e^{\xi_1+\xi_2^*} + F^{(k)}_{21^*} e^{\xi_2+\xi_1^*} + F^{(k)}_{22^*} e^{\xi_2+\xi_2^*} + F^{(k)}_{121^*2^*} e^{\xi_1+\xi_2+\xi_1^*+\xi_2^*}, \quad k = 1, 2, \quad (92)$$

where

$$E_{ij^*} = \sigma_1 c_i^{(1)} c_j^{(1)*} \left[8(p_i + p_j^*)(p_i^3 + p_j^{*3}) + \sum_{l=1}^{2} \frac{\sigma_{l+1} \rho_l^2 (p_i + p_j^*)^2}{(p_i - i\alpha_l)(p_j^* + i\alpha_l)} \right]^{-1},$$

$$F^{(k)}_{ij^*} = -\frac{p_i - i\alpha_k}{p_j^* + i\alpha_k} E_{ij^*},$$

$$E_{121^*2^*} = |p_1 - p_2|^2 \left[\frac{E_{11^*} E_{22^*}}{(p_1 + p_2^*)(p_2 + p_1^*)} - \frac{E_{12^*} E_{21^*}}{(p_1 + p_1^*)(p_2 + p_2^*)} \right],$$

$$F^{(k)}_{121^*2^*} = \frac{(p_1 - i\alpha_k)(p_2 - i\alpha_k)}{(p_1^* + i\alpha_k)(p_2^* + i\alpha_k)} E_{121^*2^*},$$

$$G^{(1)}_{12i^*} = (p_1 - p_2) \left(\frac{c_1^{(1)} E_{2i^*}}{p_1 + p_i^*} - \frac{c_2^{(1)} E_{1i^*}}{p_2 + p_i^*} \right),$$

with $\xi_i = p_i x - i p_i^2 y - 8 p_i^3 t + \xi_{i0}$ for $i = 1, 2$.

Similarly, the nonsingular solution requires the denominator f to be nonzero. Also, we rewrite f as

$$f = 2e^{\xi_{1R}+\xi_{2R}} [e^{\zeta_R} \cos(\xi_{1I} - \xi_{2I} + \zeta_I) + e^{\eta_1+\eta_2} \cosh(\xi_{1R} - \xi_{2R} + \eta_1 - \eta_2) + e^{\eta_3} \cosh(\xi_{1R} - \xi_{2R} + \eta_3)], \quad (93)$$

where $e^{2\eta_1} = E_{11^*}$, $e^{2\eta_2} = E_{22^*}$, $e^{2\eta_3} = E_{121^*2^*}$, and $e^{\zeta_R + i\zeta_I} = E_{12^*}$. As in the above subsection, it is easy to see that $E_{ii^*} > 0$ $(i = 1, 2)$ is a necessary condition and $e^{\eta_1+\eta_2} + e^{\eta_3} > e^{\zeta_R}$ is a sufficient condition to guarantee a nonsingular two-soliton solution.

Following the method in the previous subsection, a similar asymptotic analysis of the collision between two solitons can also be performed, whose details are omitted here. The amplitudes of the bright parts of the mixed two solitons before and after collision are given by

$$(A^{(1)}_{1-}, A^{(1)}_{2-}, A^{(1)}_{1+}, A^{(1)}_{2+}) = \left(\frac{c_1^{(1)}}{2\sqrt{E_{11^*}}}, \frac{G^{(1)}_{121^*}}{2\sqrt{E_{11^*} E_{121^*2^*}}}, \frac{G^{(1)}_{122^*}}{2\sqrt{E_{22^*} E_{121^*2^*}}}, \frac{c_2^{(1)}}{2\sqrt{E_{22^*}}} \right). \quad (94)$$

After substituting the expressions for various quantities, a tedious calculation shows that $|A^{(1)}_{j-}| = |A^{(1)}_{j+}|$ for $j = 1, 2$, which indicates that the intensities of the bright parts of the colliding mixed two solitons are the same before and after collision. Similarly, the amplitudes of the dark parts of the mixed two solitons in the $\Phi^{(k+1)}$ component before and after interaction are the same and equal to ρ_k. In addition, it can also be seen that the bright solitons in the u component also undergo elastic collision. Hence, different from the 2-b-1-d case, the two solitons in all the components (short- and long-wave components) for the 1-b-2-d case always undergo standard elastic collision without energy exchange. Note that this phenomenon is consistent with the mixed soliton solution of the three-component YO system.[13,16] The mixed-type two

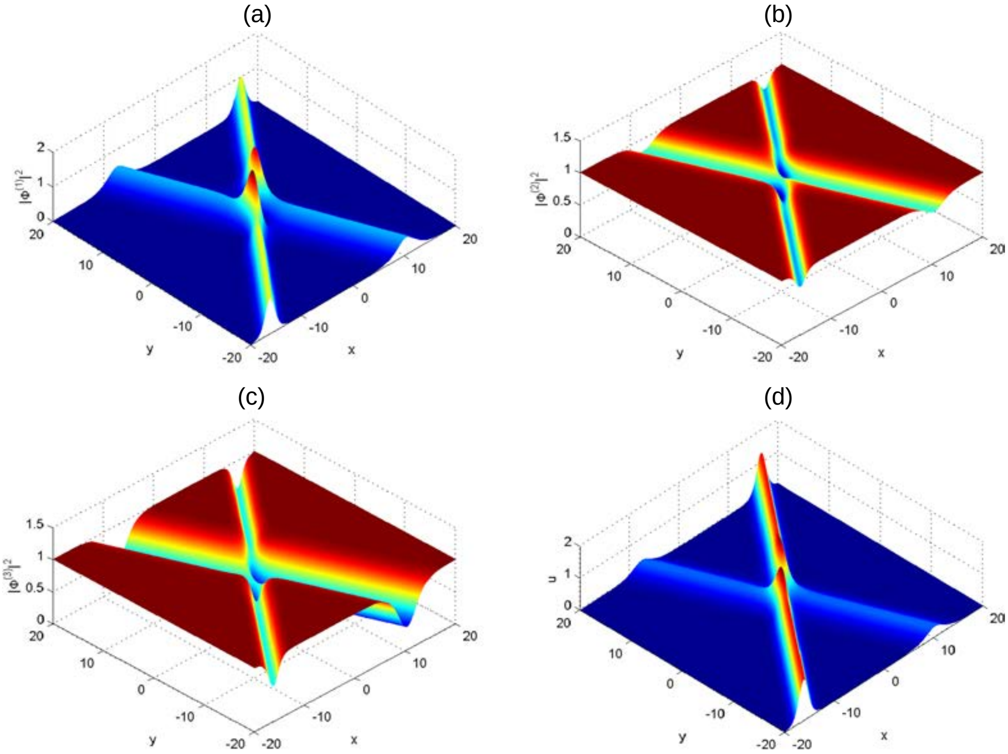

Fig. 5. (Color online) 1-b-2-d mixed two-soliton solution of the three-component Mel'nikov system (5)–(6).

solitons are displayed in Fig. 5 with the parametric choice $p_1 = \frac{2}{5} + \frac{1}{5}i$, $p_2 = \frac{4}{5} - \frac{2}{5}i$, $c_1^{(1)} = 1 + \frac{1}{2}i$, $c_2^{(1)} = \frac{5}{2}$, $\rho_1 = \rho_2 = \alpha_1 = 1$, $\alpha_2 = \frac{1}{2}$, and $\xi_{10} = \xi_{20} = 0$ at time $t = 0$ under the same nonlinear coefficients as previously. Figures 5(a) and 5(d) represent the collisions of two bright solitons in the $\Phi^{(1)}$ and u components, respectively; and the collisions of two dark solitons with different amplitudes in the $\Phi^{(2)}$ and $\Phi^{(3)}$ components are displayed in (b) and (c), respectively. From these figures, it is obvious that the solitons in all the components undergo elastic collision without a shape change which is only accompanied by a position shift.

3. Bright-Dark Mixed *N*-Soliton Solution of the Multi-Component Mel'nikov System

In this section, we consider the general bright-dark mixed N-soliton solution consisting of m bright solitons and $M - m$ dark solitons in the short-wave components to the M-component Mel'nikov system (3)–(4). To this end, the following dependent variable transformations are introduced

$$\Phi^{(k)} = \frac{g^{(k)}}{f}, \quad \Phi^{(l)} = \rho_l e^{i\theta_l}\frac{h^{(l)}}{f}, \quad u = 2(\log f)_{xx}, \tag{95}$$

where $\theta_l = \alpha_l x + \alpha_l^2 y + \beta_l(t)$; $k = 1, 2, \dots, m$; $l = 1, 2, \dots, M - m$; which convert Eqs. (3) and (4) into

$$(D_x^2 - iD_y)g^{(k)} \cdot f = 0, \quad k = 1, 2, \dots, m, \tag{96}$$

$$(D_x^2 + 2i\alpha_l D_x - iD_y)h^{(l)} \cdot f = 0, \quad l = 1, 2, \dots, M - m, \tag{97}$$

$$(D_x^4 + D_xD_t - 3D_y^2)f \cdot f = -\sum_{k=1}^{m}\sigma_k g^{(k)}g^{(k)*} + \sum_{l=1}^{M-m}\sigma_{l+m}\rho_l^2(f^2 - h^{(l)}h^{(l)*}). \tag{98}$$

In the same manner as for the three-component Mel'nikov system (5)–(6), one can show that the following tau functions satisfy bilinear equations (96)–(98) and thus provide the general mixed N-soliton solution to the M-component Mel'nikov system (3)–(4)

$$f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \quad g^{(k)} = \begin{vmatrix} A & I & \Omega^{\mathrm{T}} \\ -I & B & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & C_k & 0 \end{vmatrix}, \quad h^{(l)} = \begin{vmatrix} A^{(l)} & I \\ -I & B \end{vmatrix}, \tag{99}$$

where A, $A^{(l)}$, and B are $N \times N$ matrices whose entries are given respectively as

$$a_{ij} = \frac{1}{p_i + p_j^*}e^{\xi_i + \xi_j^*}, \quad a_{ij}^{(l)} = \frac{1}{p_i + p_j^*}\left(-\frac{p_i - i\alpha_l}{p_j^* + i\alpha_l}\right)e^{\xi_i + \xi_j^*},$$

$$b_{ij} = \left(\sum_{k=1}^{m}\sigma_k c_i^{(k)*}c_j^{(k)}\right)\left[8(p_i^{*3} + p_j^3) + \sum_{l=1}^{M-m}\frac{\sigma_{l+m}\rho_l^2(p_i^* + p_j)}{(p_i^* + i\alpha_l)(p_j - i\alpha_l)}\right]^{-1},$$

meanwhile, Ω and C_k are the N-component row vectors

$$\Omega = (e^{\xi_1}, e^{\xi_2}, \dots, e^{\xi_N}), \quad C_k = -(c_1^{(k)}, c_2^{(k)}, \dots, c_N^{(k)}),$$

with $\xi_i = p_i x - ip_i^2 y - 8p_i^3 t + \xi_{i0}$; p_i, ξ_{i0}, and $c_i^{(k)}$ ($i = 1, 2, \dots, N$) are complex constants.

Similarly, a necessary condition of the M-component Mel'nikov system (3)–(4) for the existence of a nonsingular mixed N-soliton solution is given by

$$\left(\sum_{k=1}^{m}\sigma_k|c_i^{(k)}|^2\right)\left(8(p_{iR}^2 - 3p_{iI}^2) + \sum_{l=1}^{M-m}\frac{\sigma_{l+m}\rho_l^2}{|p_i - i\alpha_l|^2}\right) > 0, \quad i = 1, 2, \dots, N. \tag{100}$$

The formula obtained provides a bright-dark mixed N-soliton solution to the M-component Mel'nikov system (3)–(4) with all possible combinations of nonlinearities, including all-positive, all-negative and mixed types. Moreover, as pointed out in Ref. 13, the arbitrariness of the nonlinearities σ_k increases the freedom, which results in rich soliton dynamics. In parallel with the vector NLS equation[8] and the multi-component YO system,[16] the expression of the general bright-dark mixed N-soliton solution also includes the all-bright and all-dark N-soliton solutions as special cases. For instance, the N-bright soliton solution can be directly obtained from the formula (99) by taking $m = M$; hence, it supports the same determinant form as the mixed N-soliton solution. Whereas, the expression of the N-dark soliton solution is different from that of the mixed N-soliton solution. As discussed in Refs. 8 and 16, it is known that the general N-dark soliton solution can alternatively take the same form as (99) except with matrix B redefined as an identity matrix ($b_{ij} = \delta_{ij}$) and the following constraints imposed on the parameters

$$8(p_i^{*2} - |p_i|^2 + p_i^2) + \sum_{l=1}^{M} \frac{\sigma_l \rho_l^2}{|p_i - i\alpha_l|^2} = 0, \quad i = 1, 2, \ldots, N. \tag{101}$$

For the collision of two solitons, a similar asymptotic analysis can be performed as in the above section, whose details are omitted here. It can be concluded that for an M-component Mel'nikov system (3)–(4) with $M \geq 3$, energy-exchanging inelastic collision is possible only if the bright parts of the mixed solitons appear in at least two short-wave components. What is more, the bright solitons in the short-wave components undergo elastic collision when $\frac{|c_1^{(1)}|}{|c_2^{(1)}|} = \frac{|c_1^{(2)}|}{|c_2^{(2)}|} = \cdots = \frac{|c_1^{(k)}|}{|c_2^{(k)}|}$, $k = 1, 2, \ldots, m$. Otherwise, they undergo energy-exchanging inelastic collision characterized by an intensity redistribution. Whereas, the dark solitons in the short-wave components and the bright solitons in the long-wave component always undergo elastic collision without a shape change which is only accompanied by a position shift.

4. Conclusion

The general bright-dark mixed N-soliton solutions of the multi-component Mel'nikov system with all possible combinations of nonlinearities are constructed by using the KP hierarchy reduction technique. Taking the three-component Mel'nikov system as a concrete example, its two kinds of mixed N-soliton solutions (2-b-1-d soliton and 1-b-2-d soliton in the three short-wave components) are derived from the tau functions of the KP hierarchy in detail. It is worth note that the derivation of the 2-b-1-d mixed N-soliton solution starts from a (2+1)-component KP hierarchy with one copy of a shifted singular point (c_1). In contrast, for the construction of the 1-b-2-d mixed N-soliton solution, we start from a (1+1)-component KP hierarchy with two copies of shifted singular points (c_1 and c_2). Hence, it is not difficult to conclude that the number of components in the KP hierarchy matches the number of short-wave components supporting bright solitons while the number of the copies of shifted singular points equals the number of short-wave components supporting dark solitons. This fact also can be referred to the constructions in Ref. 31. Then we extend our analysis to the M-component Mel'nikov system to obtain its m-bright-$(M - m)$-dark mixed N-soliton solution. It is obvious that the mixed soliton solution can be derived from the reduction of an $(m + 1)$-component KP hierarchy with $M - m$ copies of shifted singular points. The formula obtained also includes the general all-bright and all-dark N-soliton solutions as special cases.

Furthermore, the dynamics of single and two solitons are also discussed in detail. In particular, for the collision of two solitons, it has been shown that for an M-component Mel'nikov system with $M \geq 3$, if the bright solitons appear in at least two short-wave components, then interesting collision behaviors take place, resulting in energy exchange among the bright solitons in the short-wave components. Generally, after the inelastic collision of two solitons, the intensity of one soliton is enhanced while the intensity of the other soliton is suppressed, which can be observed in Figs. 2(a) and 2(b). Whereas, the dark solitons appearing in the short-wave components and the bright solitons appearing in the long-wave component always undergo elastic collision without a shape change while only accompanied by a position shift. This interesting feature is in parallel with the one- and two-dimensional multi-component YO system.[11,16]

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