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Similarity Reductions of (2+1)-Dimensional Multi-component Broer–Kaup System*

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Abstract *Painlevé property of the (2+1)-dimensional multi-component Broer–Kaup (BK) system is considered by using the standard Weiss–Kruskal approaches. Applying the Clarkson and Kruskal (CK) direct method to the (2+1)-dimensional multi-component BK system, some types of similarity reductions are obtained. By solving the reductions, one can get the solutions of the (2+1)-dimensional multi-component BK system.*

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1 Introduction

Nonlinear partial differential equations widely describe complex phenomena in various fields of sciences, such as physics, chemistry, biology, etc. Then solving nonlinear problems gets important and significant in nonlinear science. To find some similarity solutions of a nonlinear physics problem, the Lie group approach is one of the most important methods. One may use the classical Lie group approach[1] and the nonclassical Lie group approach.[2] However, both the classical Lie group approach and the nonclassical Lie group approach are quite complicated in practical calculations. The direct method, which was presented first by Clarkson and Kruskal,[3] has been widely used to find the similarity solutions for many important mathematical and physical models. On the one hand, the direct method can simplify the calculation. On the other hand, one believes that the results obtained by the CK direct method contain those obtained by the classical Lie group approach and the results of the nonclassical Lie approach include those of the direct method.[4] It is known that possessing Painlevé property is also one of the most important integrable properties, which means that the solutions of the model are single-valued about an arbitrary singularity manifold and various other integrable properties are linked with the Painlevé property.[5−7] There are some kinds of approaches to study the Painlevé property of a nonlinear evolution equation, such as the ARS algorithm,[5] the standard Weiss–Kruskal approaches,[6] and the Conte's invariant method.[7]

In Ref. [8], Lou and Hu used the Darboux-transformation-related symmetry constraints of the Kadomtsev–Petviashvili (KP) equation to get some integrable (1+1)-dimensional and (2+1)-dimensional multi-component BK system. In this paper, first, using the standard Weiss–Kruskal approaches, we try to investigate the Painlevé property of the (2+1)-dimensional multi-component BK system. Second, applying the CK direct method, we construct similarity solutions of the (2+1)-dimensional multi-component BK system:

$$G_{jy} = -G_{jxx} - 2(G_jH_j)_x\,, \tag{1}$$

$$H_{jy} = H_{jxx} - 2H_jH_{jx} - 2\sum_{i=1}^{N}\partial_z^{-1}G_{ixx}\,,\quad j = 1, 2, \ldots, N. \tag{2}$$

Differentiating Eq. (2) with respect to z, one can get

$$H_{jyz} = H_{jxxz} - 2H_{jx}H_{jz} - 2H_jH_{jxz} - 2\sum_{i=1}^{N}G_{ixx}\,. \tag{3}$$

Now we consider Eqs. (1) and (3).

2 Painlevé Property of (2+1)-Dimensional Multi-component Broer–Kaup System

Using the standard Weiss–Kruskal approaches, we give the form of the expansion about the singular manifold,

$$H_j = \phi^{\alpha_j}\sum_{k=0}^{\infty}h_{jk}\phi^k\,,\quad G_j = \phi^{\beta_j}\sum_{k=0}^{\infty}g_{jk}\phi^k\,, \tag{4}$$

where $h_{jk} = h_{jk}(x, y, z)$, $g_{jk} = g_{jk}(x, y, z)$ and $\phi = \phi(x, y, z)$ are analytic functions of (x, y, z). Substituting Eq. (4) into Eqs. (1) and (3) leads to

$$\alpha_j = -1,\quad \beta_j = -2,\quad h_{j0} = \phi_x,\quad \sum_{j=1}^{N}g_{j0} = -\phi_x\phi_z,$$

and the recursion relations have the forms

$$2(i-3)\phi_xg_{j0}h_{ji} + i(i-3)\phi_x^2g_{ji} = f_1(h_{jl}, g_{jl})\,, \tag{5}$$

$$-(i-2)(i-3)^2\phi_x^2\phi_zh_{ji} + 2(i-2)(i-3)\phi_x^2 \times\sum_{j=1}^{N}g_{ji} = f_2(h_{jl}, g_{jl})\,, \tag{6}$$

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where f_1 and f_2 are complicated functions of h_{jl}, g_{jl} ($l \leq i-1$) and the derivatives of the singularity ϕ. From Eqs. (5) and (6), the resonance values of i are given by

$$\begin{vmatrix} a_1 & 0 & \cdots & 0 & b & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 & 0 & b & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_N & 0 & 0 & \cdots & b \\ c & 0 & \cdots & 0 & d & d & \cdots & d \\ 0 & c & \cdots & 0 & d & d & \cdots & d \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & c & d & d & \cdots & d \end{vmatrix}$$
$$= (i+1)i^{N-1}(i-2)^N(i-3)^{3N-1}(i-4)\phi_x^{4N}\phi_z^N = 0\,,$$

where $a_k = 2(i-3)\phi_x g_{k0}$, $b = i(i-3)\phi_x^2$, $c = -(i-2)(i-3)^2\phi_x^2\phi_z$, $d = 2(i-2)(i-3)\phi_x^2$, i.e. $i = -1, 0, \ldots, 0, 2, \ldots, 2, 3, \ldots, 3, 4, \ldots, 4$.

Resonances are those values of i at which it is possible to introduce arbitrary functions into the expansions (4). For each nontrivial resonance there occurs a compatibility condition that must be satisfied if the solution has a single-valued expansion. After detailed calculation, we find the compatibility conditions at $i = 0, \ldots, 0, 2, \ldots, 2, 3, \ldots, 3, 4, \ldots, 4$ are satisfied identically for Eqs.(5) and (6). The resonance at $i = -1$ corresponds to the arbitrary function ϕ defining the singularity manifold for the (2+1)-dimensional multi-component BK system. According to the Weiss–Kruskal approach, equations (1) and (3) possess Painlevé properties. It is said that the (2+1)-dimensional multi-component BK system is integrable under the meaning that it possesses Painlevé property.

3 Similarity Reduction and Solutions of (2+1)-Dimensional Multi-component Broer–Kaup System

To Eqs. (1) and (3), suppose that the solution has the following form

$$H_j = \alpha_j(x,y,z) + \beta_j(x,y,z)P_j(\xi,\eta)\,, \tag{7}$$
$$G_j = a_j(x,y,z) + b_j(x,y,z)Q_j(\xi,\eta)\,, \tag{8}$$

where $\alpha_j(x,y,z)$, $\beta_j(x,y,z)$, $\xi = \xi(x,y,z)$, $\eta = \eta(x,y,z)$, $a_j(x,y,z)$, and $b_j(x,y,z)$ are functions of (x,y,z) to be determined. Substituting Eqs. (7) and (8) into Eqs. (1) and (3) yields

$$\begin{aligned}&\delta_0 + \delta_1 P_j + \delta_2 Q_j + \delta_3 P_{j\xi} + \delta_4 P_{j\eta} + \delta_5 Q_{j\xi} + \delta_6 Q_{j\eta} \\ &\quad + \delta_7 Q_{j\xi\xi} + \delta_8 Q_{j\xi\eta} + \delta_9 Q_{j\eta\eta} + \delta_{10} P_j Q_j + \delta_{11} P_{j\xi} Q_j \\ &\quad + \delta_{12} P_{j\eta} Q_j + \delta_{13} P_j Q_{j\xi} + \delta_{14} P_j Q_{j\eta} = 0\,, \end{aligned}\tag{9}$$

$$\begin{aligned}&\sum_{i=0}^{N}(\gamma_{0i} + \gamma_{1i}P_j + \gamma_{2i}P_{j\xi} + \gamma_{3i}P_{j\eta} + \gamma_{4i}P_{j\xi\xi} \\ &\quad + \gamma_{5i}P_{j\xi\eta} + \gamma_{6i}P_{j\eta\eta} + \gamma_{7i}P_{j\xi\xi\xi} + \gamma_{8i}P_{j\xi\xi\eta} + \gamma_{9i}P_{j\xi\eta\eta} \\ &\quad + \gamma_{10i}P_{j\eta\eta\eta} + \gamma_{11i}P_j^2 + \gamma_{12i}P_jP_{j\xi} + \gamma_{13i}P_jP_{j\eta} \\ &\quad + \gamma_{14i}P_{j\xi}^2 + \gamma_{15i}P_{j\xi}P_{j\eta} + \gamma_{16i}P_{j\eta}^2 + \gamma_{17i}P_jP_{j\xi\xi} \\ &\quad + \gamma_{18i}P_jP_{j\xi\eta} + \gamma_{19i}P_jP_{j\eta\eta} + \gamma_{20i}Q_i) = 0\,, \end{aligned}\tag{10}$$

where $\delta_i (i = 0, 1, \ldots, 14)$ and $\gamma_{ji} (j = 0, 1, \ldots, 20;\ i = 0, 1, \ldots, N)$ are given by

$$\begin{aligned}&\delta_0 = a_{jy} + a_{jxx} + 2\alpha_j a_{jx} + 2\alpha_{jx} a_j\,, \\ &\delta_1 = 2a_{jx}\beta_j + 2a_j\beta_{jx}\,, \\ &\delta_2 = b_{jy} + b_{jxx} + 2\alpha_j b_{jx} + 2\alpha_{jx} a_j\,, \\ &\delta_3 = 2a_j\beta_j\xi_x\,, \quad \delta_4 = 2a_j\beta_j\eta_x\,, \\ &\delta_5 = b_j\xi_y + 2b_{jx}\xi_x + b_j\xi_{xx} + 2\alpha_j b_j\xi_x\,, \\ &\delta_6 = b_j\eta_y + 2b_{jx}\eta_x + b_j\eta_{xx} + 2\alpha_j b_j\eta_x\,, \\ &\delta_7 = b_j\xi_x^2\,, \quad \delta_8 = 2b_j\xi_x\eta_x\,, \\ &\delta_9 = b_j\eta_x^2\,, \quad \delta_{10} = 2\beta_j b_{jx} + 2\beta_{jx} b_j\,, \\ &\delta_{11} = 2\beta_j b_j\xi_x\,, \quad \delta_{12} = 2\beta_j b_j\eta_x\,, \\ &\delta_{13} = 2\beta_j b_j\xi_x\,, \quad \delta_{14} = 2\beta_j b_j\eta_x\,. \end{aligned}\tag{11}$$

For γ_{ji}, if j is fixed, we can take $\gamma_j = (\gamma_{j0}, \gamma_{j1}, \ldots, \gamma_{jN})$. So

$$\begin{aligned}
\gamma_0 &= (\gamma_{00}, \gamma_{01}, \ldots, \gamma_{0N}) = (\alpha_{jyz} - \alpha_{jxxz} + 2\alpha_{jx}\alpha_{jz} + 2\alpha_j\alpha_{jxz}, 2a_{1xx}, 2a_{2xx}, \ldots, 2a_{Nxx})\,, \\
\gamma_1 &= (\gamma_{10}, \gamma_{11}, \ldots, \gamma_{1N}) = (\beta_{jyz} - \beta_{jxxz} + 2\alpha_{jx}\beta_{jz} + 2\alpha_{jz}\beta_{jx} + 2\alpha_j\beta_{jxz} + 2\alpha_{jxz}\beta_j)(1, 0, 0, \ldots, 0)\,, \\
\gamma_2 &= (\gamma_{20}, \gamma_{21}, \ldots, \gamma_{2N}) = [\beta_{jy}\xi_z + \beta_{jz}\xi_y + \beta_j\xi_{yz} - \beta_{jxx}\xi_z - 2\beta_{jxz}\xi_x - 2\beta_{jx}\xi_{xz} - \beta_{jz}\xi_{xx} \\
&\quad - \beta_j\xi_{xxz} + 2\beta_j(\alpha_{jx}\xi_z + \alpha_{jz}\xi_x) + 2\alpha_j(\beta_{jx}\xi_z + \beta_{jz}\xi_x + \beta_j\xi_{xz})](1, 0, 0, \ldots, 0)\,, \\
\gamma_3 &= (\gamma_{30}, \gamma_{31}, \ldots, \gamma_{3N}) = [\beta_{jy}\eta_z + \beta_{jz}\eta_y + \beta_j\eta_{yz} - \beta_{jxx}\eta_z - 2\beta_{jxz}\eta_x - 2\beta_{jx}\eta_{xz} - \beta_{jz}\eta_{xx} \\
&\quad - \beta_j\eta_{xxz} + 2\beta_j(\alpha_{jx}\eta_z + \alpha_{jz}\eta_x) + 2\alpha_j(\beta_{jx}\eta_z + \beta_{jz}\eta_x + \beta_j\eta_{xz})](1, 0, 0, \ldots, 0)\,, \\
\gamma_4 &= (\gamma_{40}, \gamma_{41}, \ldots, \gamma_{4N}) = [(\beta_j\xi_y - 2\beta_{jx}\xi_x - \beta_j\xi_{xx})\xi_z - (\beta_{jz}\xi_x + 2\beta_j\xi_{xz} - 2\alpha_j\beta_j\xi_z)\xi_x](1, 0, 0, \ldots, 0)\,, \\
\gamma_5 &= (\gamma_{50}, \gamma_{51}, \ldots, \gamma_{5N}) = [\beta_j(\xi_y\eta_z + \xi_z\eta_y) - \beta_j(\xi_{xx}\eta_z + \xi_z\eta_{xx} + 2\xi_{xz}\eta_x + 2\xi_x\eta_{xz}) \\
&\quad -2\beta_{jx}(\xi_x\eta_z + \xi_z\eta_x) - 2\beta_{jz}\xi_x\eta_x + 2\alpha_j\beta_j(\xi_x\eta_z + \xi_z\eta_x)](1, 0, 0, \ldots, 0)\,, \\
\gamma_6 &= (\gamma_{60}, \gamma_{61}, \ldots, \gamma_{6N}) = [\beta_j(\eta_y\eta_z - \eta_{xx}\eta_z - 2\eta_x\eta_{xz}) - 2\beta_{jx}\eta_x\eta_z - \beta_{jz}\eta_x^2 + 2\alpha_j\beta_j\eta_x\eta_z](1, 0, 0, \ldots, 0)\,, \\
\gamma_7 &= (\gamma_{70}, \gamma_{71}, \ldots, \gamma_{7N}) = -\beta_j\xi_x^2\xi_z(1, 0, 0, \ldots, 0)\,, \\
\gamma_8 &= (\gamma_{80}, \gamma_{81}, \ldots, \gamma_{8N}) = -\beta_j\xi_x(2\xi_z\eta_x + \xi_x\eta_z)(1, 0, 0, \ldots, 0)\,, \\
\gamma_9 &= (\gamma_{90}, \gamma_{91}, \ldots, \gamma_{9N}) = -\beta_j\eta_x(2\xi_x\eta_z + \xi_z\eta_x)(1, 0, 0, \ldots, 0)\,, \\
\gamma_{10} &= (\gamma_{100}, \gamma_{101}, \ldots, \gamma_{10N}) = -\beta_j\eta_x^2\eta_z(1, 0, 0, \ldots, 0)\,, \\
\gamma_{11} &= (\gamma_{110}, \gamma_{111}, \ldots, \gamma_{11N}) = 2(\beta_{jx}\beta_{jz} + \beta_j\beta_{jxz})(1, 0, 0, \ldots, 0)\,,
\end{aligned}$$

$$
\begin{aligned}
\gamma_{12} &= (\gamma_{120}, \gamma_{121}, \ldots, \gamma_{12N}) = 2\beta_j(\beta_{jx}\xi_z + \beta_{jz}\xi_x + \beta_{jz}\xi_x + \beta_j\xi_{xz})(1, 0, 0, \ldots, 0)\,,\\
\gamma_{13} &= (\gamma_{130}, \gamma_{131}, \ldots, \gamma_{13N}) = 2\beta_j(\beta_{jx}\eta_z + \beta_{jz}\eta_x + \beta_{jz}\mu_x + \beta_j\eta_{xz})(1, 0, 0, \ldots, 0)\,,\\
\gamma_{14} &= (\gamma_{140}, \gamma_{141}, \ldots, \gamma_{14N}) = 2\beta_j^2\xi_x\xi_z(1, 0, 0, \ldots, 0)\,,\\
\gamma_{15} &= (\gamma_{150}, \gamma_{151}, \ldots, \gamma_{15N}) = 2\beta_j^2(\xi_z\eta_x + \xi_x\eta_z)(1, 0, 0, \ldots, 0)\,,\\
\gamma_{16} &= (\gamma_{160}, \gamma_{161}, \ldots, \gamma_{16N}) = 2\beta_j^2\eta_x\eta_z(1, 0, 0, \ldots, 0)\,,\\
\gamma_{17} &= (\gamma_{170}, \gamma_{171}, \ldots, \gamma_{17N}) = 2\beta_j^2\xi_x\xi_z(1, 0, 0, \ldots, 0)\,,\\
\gamma_{18} &= (\gamma_{180}, \gamma_{181}, \ldots, \gamma_{18N}) = 2\beta_j^2(\xi_z\eta_x + \xi_x\eta_z)(1, 0, 0, \ldots, 0)\,,\\
\gamma_{19} &= (\gamma_{190}, \gamma_{191}, \ldots, \gamma_{19N}) = 2\beta_j^2\eta_x\eta_z(1, 0, 0, \ldots, 0)\,,\\
\gamma_{20} &= (\gamma_{200}, \gamma_{201}, \ldots, \gamma_{20N}) = (0, 2b_{1xx}, 2b_{2xx}, \ldots, 2b_{Nxx})\,,\\
\gamma_{21} &= (\gamma_{210}, \gamma_{211}, \ldots, \gamma_{21N}) = (0, 2(2b_{1x}\xi_x + b_1\xi_{xx}), 2(2b_{2x}\xi_x + b_2\xi_{xx}), \ldots, 2(2b_{Nx}\xi_x + b_N\xi_{xx}))\,,\\
\gamma_{22} &= (\gamma_{220}, \gamma_{221}, \ldots, \gamma_{22N}) = (0, 2(2b_{1x}\eta_x + b_1\eta_{xx}), 2(2b_{2x}\eta_x + b_2\eta_{xx}), \ldots, 2(2b_{Nx}\eta_x + b_N\eta_{xx}))\,,\\
\gamma_{23} &= (\gamma_{230}, \gamma_{231}, \ldots, \gamma_{23N}) = (0, 2b_1\xi_x^2, 2b_2\xi_x^2, \ldots, 2b_N\xi_x^2)\,,\\
\gamma_{24} &= (\gamma_{240}, \gamma_{241}, \ldots, \gamma_{24N}) = (0, 4b_1\xi_x\eta_x, 4b_2\xi_x\eta_x, \ldots, 4b_N\xi_x\eta_x)\,,\\
\gamma_{25} &= (\gamma_{250}, \gamma_{251}, \ldots, \gamma_{25N}) = (0, 2b_1\eta_x^2, 2b_2\eta_x^2, \ldots, 2b_N\eta_x^2)\,.
\end{aligned} \tag{12}
$$

Equations (9) and (10) are PDEs of P and Q with respect to ξ and η only for the coefficients of different derivatives and with the powers of P and Q being functions of ξ and η. That is to say, the constrained conditions

$$\delta_i = \delta_j\Delta_i(\xi, \eta)\,, \quad (i = 0, 1, \ldots, 14)\,, \tag{13}$$

$$\gamma_{ji} = \gamma_{j'i'}\Gamma_{ji}(\xi, \eta);\ (i = 1, 2, \ldots, N;\ j = 1, 2, \ldots, 25). \tag{14}$$

For some fixed non-zero δ_j and $\gamma_{j'i'}$ must be satisfied, where Δ_i and Γ_{ji} are some functions of ξ and η to be determined later.

To determine the non-fixed functions $\{\alpha_i, \beta_i, P_i, a_i, b_i, Q_i, \xi, \eta, \Delta_i, \Gamma_{ji}\}$, we can use some remarks to simplify the calculations

Remark 1 If $\alpha_i(x, y, z)$ (or $a_i(x, y, z)$) has the form $\alpha_i = \beta_i(x, y, z)\Xi_i(\xi, \eta) + \alpha_{0i}(x, y, z)$ (or $a_i = b_i(x, y, z)\Xi_i(\xi, \eta) + a_{0i}(x, y, z)$), then we can take $\Xi_i \equiv 0$.

Remark 2 If $\beta_i(x, y, z)$ (or $b_i(x, y, z)$) has the form $\beta_i = \beta_{0i}(x, y, z)\Xi_i(\xi, \eta)$ (or $b_i = b_{0i}(x, y, z)\ \Xi_i(\xi, \eta)$), then we can take $\Xi_i \equiv C_i =$ constant.

Remark 3 If $\xi = \xi(\xi_0(x, y, z), \eta)$ (or $\eta = \eta(\xi, \eta_0(x, y, z))$), then we can take $\xi \equiv \xi_0$ (or $\eta \equiv \eta_0$).

Remark 4 If $\xi(x, y, z)$ (or $\eta(x, y, z)$) has the form $\Xi(\xi) = \xi_0(x, y, z)$ (or $H(\eta) = \eta_0(x, y, z)$), where $\Xi(\xi)$ (or $H(\eta)$) is any invertible function, then we can take $\Xi(\xi) = \xi$ (or $H(\eta) = \eta$).

To discuss further, three cases should be considered: (i) $\xi_x \neq 0$, (ii) $\xi_x = \eta_x = 0$, $\xi_y \neq 0$, and (iii) $\xi_x = \eta_x = 0$, $\xi_y = \eta_y = 0$. Only the first two cases will be discussed in detail because the last one is quite trivial.

(i) $\xi_x \neq 0$.

In this case, we can choose $\delta_j = \delta_{11} = 2\beta_j b_j\xi_x$ as common factors. As far as the choice of $\gamma_{j'i'}$, we will consider the following three cases. Using Remarks 1 $\sim$ 4 to fix the freedoms in the determination of $\{\alpha_i, \beta_i, P_i, a_i, b_i, Q_i, \xi, \eta, \Delta_i, \Gamma_{ji}\}$, and analyzing Eqs. (13) and (14) with Eqs. (11) and (12) carefully, one can obtain three possible solutions of Eqs. (1) and (3).

(a) $\eta_z \neq 0$.

Substituting Eqs. (11) into Eq. (9), we obtain

$$
\begin{aligned}
&\Delta_0 = \Delta_1 = \Delta_2 = \Delta_4 = \Delta_5 = 0\,,\\
&\Delta_6 = \Delta_{11} = \Delta_{13} = 1\,,
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
&\Delta_3 = \frac{a_j}{b_j}\,, \quad \Delta_7 = \frac{\xi_x}{2\beta_j}\,, \quad \Delta_8 = \frac{\eta_x}{\beta_j}\,,\\
&\Delta_9 = \frac{\eta_x^2}{2\beta_j\xi_x}\,, \quad \Delta_{10} = \frac{\beta_j b_{jx} + \beta_{jx}b_j}{\beta_j b_j\xi_x}\,,\\
&\Delta_{12} = \Delta_{14} = \frac{\eta_x}{\xi_x}\,,
\end{aligned} \tag{16}
$$

$$\alpha_j = \frac{\xi_{xx} - \xi_y}{2\xi_x}, \eta_y = \xi_x^2\,, \tag{17}$$

$$b_{jy} + b_{jxx} + 2\alpha_j b_{jx} + 2\alpha_{jx}b_j = 0\,. \tag{18}$$

From

$$\Delta_3 = \frac{a_j}{b_j}\,, \quad \Delta_7 = \frac{\xi_x}{2\beta_j}\,, \tag{19}$$

we can have

$$a_j = \Delta_3 b_j\,, \quad \beta_j = \frac{\xi_x}{2\Delta_7}\,. \tag{20}$$

Using Remarks 1 and 2 with Eq. (20) yields

$$a_j = 0\,, \quad \beta_j = \frac{\xi_x}{2}\,. \tag{21}$$

So one can arrive at

$$\Delta_8 = \frac{\eta_x}{\beta_j}\,. \tag{22}$$

From Eq. (22), we can have

$$\eta_x = \Delta_8\frac{\xi_x}{2}\,, \quad \eta = \frac{1}{2}\int^{\xi}\Delta_8(\xi_1)\,\mathrm{d}\xi_1 + C_1(y, z)\,. \tag{23}$$

Using Remark 3 with Eq. (23) yields

$$\eta = C_1(y, z)\,. \tag{24}$$

In the meantime, we also get $\Delta_9 = \Delta_{12} = \Delta_{14} = 0$.

From
$$\Delta_{10} = \frac{\beta_j b_{jx} + \beta_{jx} b_j}{\beta_j b_j \xi_x}\,, \tag{25}$$
we can have
$$b_j = \frac{C_{2j}(y,z)}{\beta_j}\exp\Big(\int^{\xi}\Delta_{10}(\xi_1)\mathrm{d}\xi_1\Big)\,. \tag{26}$$
Using Remark 2 with Eq. (26) yields
$$b_j = \frac{C_{2j}(y,z)}{\beta_j} = \frac{2C_{2j}(y,z)}{\xi_x}\,. \tag{27}$$
Here, we choose $\gamma_{j'i'} = \gamma_{80} = -\beta_j \xi_x^2 \eta_z$. Substituting Eqs. (12), (17), (21), (24), and (27) into Eq. (10), we arrive at
$$\Gamma_0 = \Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4 = \Gamma_5 = \Gamma_9 = \Gamma_{10} = \Gamma_{13} = \Gamma_{16} = \Gamma_{19} = \Gamma_{22} = \Gamma_{24} = \Gamma_{25} = (0,0,\ldots,0)\,, \tag{28}$$
$$\Gamma_6 = \Gamma_{15} = \Gamma_{18} = (-1,0,0,\ldots,0)\,, \tag{29}$$
$$\Gamma_7 = \Big(\frac{\xi_z}{\eta_z},0,0,\ldots,0\Big)\,, \quad \Gamma_8 = (1,0,0,\ldots,0)\,,$$
$$\Gamma_{11} = \Big(-\frac{\xi_{xx}\xi_{xz} + \xi_x\xi_{xxz}}{\xi_x^3 C_{1z}},0,0,\ldots,0\Big), \tag{30}$$
$$\Gamma_{12} = \Big(-\frac{2(2\xi_{xx}\xi_z + 3\xi_x\xi_{xz})}{\xi_x^2\eta_z},0,0,\ldots,0\Big)\,,$$
$$\Gamma_{14} = \Gamma_{17} = \Big(-\frac{\xi_z}{\eta_z},0,0,\ldots,0\Big), \tag{31}$$
$$\Gamma_{15} = \Gamma_{18} = (-1,0,0,\ldots,0)\,,$$
$$\Gamma_{20} = \Big(0,-\frac{4b_{1xx}}{\xi_x^3 C_{1z}},-\frac{4b_{2xx}}{\xi_x^3 C_{1z}},\ldots,-\frac{4b_{Nxx}}{\xi_x^3 C_{1z}}\Big)\,, \tag{32}$$
$$\Gamma_{21} = \Big(0,-\frac{4(2b_{1x}\xi_x + b_1\xi_{xx})}{\xi_x^3 C_{1z}},-\frac{4(2b_{2x}\xi_x + b_2\xi_{xx})}{\xi_x^3 C_{1z}},\ldots, -\frac{4(2b_{Nx}\xi_x + b_N\xi_{xx})}{\xi_x^3 C_{1z}}\Big), \tag{33}$$
$$\Gamma_{23} = -2(0,1,1,\ldots,1)\,, \tag{34}$$
$$\xi_{xx} = 0, C_{1z} = \frac{4C_{2i}(y,z)}{\xi_x^2}\,, \tag{35}$$
$$\beta_{jy} + 2\alpha_{jx}\beta_j = 0\,, \tag{36}$$
$$\xi_y + 2\alpha_j\xi_x = 0\,. \tag{37}$$
From Eq. (30), we can have
$$\xi_z = \Gamma_{70}\eta_z\,, \quad \xi = \int^{\eta}\Gamma_{70}(\eta_1)\mathrm{d}\eta_1 + C_3(x,y)\,. \tag{38}$$
Using Remark 3 with Eq. (38) yields
$$\xi = C_3(x,y)\,. \tag{39}$$
So we can obtain
$$\xi = C_4(y)x + C_5(y)\,, \quad \alpha_j = -\frac{C_4'(y)}{2C_4(y)}x - \frac{C_5'(y)}{2C_4(y)}\,,$$
$$\beta_j = \frac{C_4(y)}{2}\,, \quad b_j = \frac{1}{2}C_{1z}C_4(y)\,, \tag{40}$$
$$C_{1y} = C_4^2(y)\,, \quad C_{1z} = \frac{4C_{2i}(y,z)}{C_4^2(y)}\,, \tag{41}$$
$$\Gamma_{11} = \Gamma_{12} = \Gamma_{14} = \Gamma_{17} = \Gamma_{20} = \Gamma_{21} = (0,0,0,\ldots,0)\,. \tag{42}$$
Solving Eqs. (18), (36) and (37) with Eqs. (40), (41) and (42), one can arrive at
$$\alpha_j = -\frac{C_4'(y)}{2C_4(y)}x - \frac{C_5'(y)}{2C_4(y)}\,, \quad \beta_j = \frac{C_4(y)}{2}\,,$$
$$a_j = 0\,, \quad b_j = 2C_4(y)C_6(z)\,, \tag{43}$$
$$\xi = C_4(y)x + C_5(y)\,,$$
$$\eta = \int C_4^2(y)\mathrm{d}y + 4\int C_6(z)\mathrm{d}z + c_0\,, \tag{44}$$
where c_0 is an arbitrary constant, $C_4(y)$, $C_5(y)$, and $C_6(z)$ are arbitrary functions of the corresponding variables.

From the above analysis, the first type of similarity reduction of Eqs. (1) and (3) is simplified to
$$Q_{j\eta} + Q_{j\xi\xi} + P_{j\xi}Q_j + P_jQ_{j\xi} = 0\,, \tag{45}$$
$$-P_{j\eta\eta} + P_{j\xi\xi\eta} - P_{j\xi}P_{j\eta} - P_jP_{j\xi\eta} - 2\sum_{i=1}^{N}Q_{i\xi\xi} = 0\,, \tag{46}$$
with ξ and η satisfying Eq. (44).

By solving Eqs. (45) and (46), one can obtain
$$P_j = \frac{-a_3 + a_2\sqrt{2a_2a_6}\tan\big[\sqrt{2a_2a_6}(a_1 + a_2\xi + a_3\eta)/2a_2\big]}{a_2}\,,$$
$$Q_j = c_j\frac{a_4\big\{\sqrt{2}\tan\big[\sqrt{2a_2a_6}(a_1 + a_2\xi + a_3\eta)/2a_2\big]/\sqrt{a_2a_6} - a_1/a_2\big\} + a_5}{1 + \tan^2\big[\sqrt{2a_2a_6}(a_1 + a_2\xi + a_3\eta)/2a_2\big]}\,,$$
where c_j $(j = 1,2,\ldots,N)$, a_i $(i = 1,2,\ldots,6)$ are arbitrary constants and $\sum_{j=1}^{N}c_j = 0$. So the corresponding solution of Eqs. (1) and (3) reads
$$H_j = -\frac{C_4'(y)}{2C_4(y)}x - \frac{C_5'(y)}{2C_4(y)} - \frac{a_3C_4(y)}{2a_2} + \frac{a_2\sqrt{2a_2a_6}C_4(y)\tan\big[\sqrt{2a_2a_6}(a_1 + a_2\xi + a_3\eta)/2a_2\big]}{2a_2}\,, \tag{47}$$
$$G_j = 2c_jC_4(y)C_6(z)\frac{a_4\big\{\sqrt{2}\tan[\sqrt{2a_2a_6}(a_1 + a_2\xi + a_3\eta)/2a_2]/\sqrt{a_2a_6} - a_1/a_2\big\} + a_5}{1 + \tan^2\big[\sqrt{2a_2a_6}(a_1 + a_2\xi + a_3\eta)/2a_2\big]}\,, \tag{48}$$

with ξ and η satisfying Eq. (44).

If $\eta_z = 0$ and $\xi_z \neq 0$, we can consider the following case.

(b) $\eta_z = 0$, $\xi_z \neq 0$.

In this case, the corresponding solution of Eqs. (1) and (3) has the form

$$H_j = \frac{C_1'(y)}{4C_1(y)}x + \frac{1}{\sqrt{C_1(y)}}P_j(\xi,\eta)\,, \tag{49}$$

$$G_j = \frac{C_2(z)}{\sqrt{C_1(y)}}Q_j(\xi,\eta)\,, \tag{50}$$

where $C_1(y)$ and $C_2(z)$ are arbitrary functions of the corresponding variables. The reduction equations then read

$$Q_{j\eta} + Q_{j\xi\xi} + P_{j\xi}Q_j + P_jQ_{j\xi} = 0\,, \tag{51}$$

$$-P_{j\xi\eta} + P_{j\xi\xi\xi} - P_{j\xi}^2 - P_jP_{j\xi\xi} - 2\sum_{i=1}^{N} Q_{i\xi\xi} = 0\,, \tag{52}$$

with

$$\xi = \frac{2}{\sqrt{C_1(y)}}x + \int C_2(z)\,\mathrm{d}z + c_0\,,$$
$$\eta = \int \frac{4}{C_1(y)}\,\mathrm{d}y + c_1\,, \tag{53}$$

where $c_i (i = 0, 1)$ are arbitrary constants.

Integrating Eq. (49) with respect to ξ, we can obtain

$$-P_{j\eta} + P_{j\xi\xi} - P_jP_{j\xi} - 2\sum_{i=1}^{N} Q_{i\xi} = 0\,. \tag{54}$$

By taking $P_j = 2H_j$ and $Q_j = 2G_j$, one can arrive at the same equation as Eqs. (25) and (26) in Ref. [8] as follows:

$$G_{j\eta} + G_{j\xi\xi} + 2H_{j\xi}G_j + 2H_jG_{j\xi} = 0\,, \tag{55}$$

$$-H_{j\eta} + H_{j\xi\xi} - 2H_jH_{j\xi} - 2\sum_{i=1}^{N} G_{i\xi} = 0\,. \tag{56}$$

It is easy to see that equations (55) and (56) have the following solution

$$H_j = -\frac{a_1}{4} - 1 + \frac{2}{1+\exp(-2\xi - a_1\eta - a_2)}\,, \tag{57}$$

$$G_j = \frac{4c_j\exp(-2\xi - a_1\eta - a_2)}{k[1+\exp(-2\xi - a_1\eta - a_2)]^2}\,, \tag{58}$$

where a_1, a_2, and c_j $(j = 1, 2, \ldots, N)$ are arbitrary constants and $k = \sum_{i=1}^{N} c_i$. So equations (1) and (3) have the solution

$$H_j = \frac{C_1'(y)}{4C_1(y)}x + \frac{2}{\sqrt{C_1(y)}} \times\Big[-\frac{a_1}{4} - 1 + \frac{2}{1+\exp(-2\xi - a_1\eta - a_2)}\Big], \tag{59}$$

$$G_j = \frac{8c_jC_2(z)\exp(-2\xi - a_1\eta - a_2)}{k\sqrt{C_1(y)}[1+\exp(-2\xi - a_1\eta - a_2)]^2}\,, \tag{60}$$

with ξ and η satisfying Eq. (53).

(c) $\xi_z = \eta_z = 0$.

In this simple case, the corresponding solution of Eqs. (1) and (3) can be written as

$$H_j = -\frac{1}{4}x - \frac{C_1'(y)}{2c_0}\exp\Big(-\frac{1}{2}y\Big) + \frac{c_0}{2}\exp\Big(\frac{1}{2}y\Big)P_j(\xi,\eta)\,, \tag{61}$$

$$G_j = c_0\exp\Big(\frac{1}{2}y\Big)C_{2j}(z)Q_j(\xi,\eta)\,, \tag{62}$$

where $C_1(y)$ and $C_{2j}(z)$ are arbitrary functions of the corresponding variables and c_0 is an arbitrary constant. The reduction equations read

$$Q_{j\eta} + Q_{j\xi\xi} + P_{j\xi}Q_j + P_jQ_{j\xi} = 0\,, \tag{63}$$

$$\sum_{i=1}^{N} Q_{i\xi\xi} = 0\,, \tag{64}$$

with

$$\xi = c_0\exp(\frac{1}{2}y)x + C_1(y)\,, \quad \eta = c_0^2\exp(y)\,. \tag{65}$$

By solving Eqs. (63) and (64), one can arrive at

$$P_j = \frac{-(F_{1j}'(\eta)/2)\xi^2 - F_{2j}'(\eta)\xi + F_{3j}(\eta)}{F_{1j}(\eta)\xi + F_{2j}(\eta)}\,, \tag{66}$$

$$Q_j = F_{1j}(\eta)\xi + F_{2j}(\eta)\,, \tag{67}$$

where $F_{ij}(\eta)\,(i = 1, 2, 3)$ are arbitrary functions. So the solution of Eqs. (1) and (3) can be written as

$$H_j = -4x - \frac{C_1'(y)}{2c_0}\exp\Big(-\frac{1}{2}y\Big) + \frac{c_0\exp(\frac{y}{2})[-(F_{1j}'(\eta)/2)\xi^2 - F_{2j}'(\eta)\xi + F_{3j}(\eta)]}{2[F_{1j}(\eta)\xi + F_{2j}(\eta)]}, \tag{68}$$

$$G_j = c_0\exp\Big(\frac{1}{2}y\Big)C_{2j}(z)[F_{1j}(\eta)\xi + F_{2j}(\eta)]\,, \tag{69}$$

with ξ and η satisfying Eq. (65).

(ii) $\xi_x = \eta_x = 0$, $\xi_y \neq 0$.

In this case, we can choose $\delta_j = \delta_5 = b_j\xi_y$ and $\gamma_{j'i'} = \gamma_{50} = \beta_j(\xi_y\eta_z + \xi_z\eta_y)$. Using Remarks 1 ∼ 4, we obtain two possible independent solutions of Eqs. (1) and (3).

(a) $\beta_{jx} \neq 0$.

The reduction equations in this case have the forms

$$P_j + Q_{j\xi} + P_jQ_j = 0\,, \tag{70}$$

$$P_{j\xi\eta} + P_jP_{j\eta} = 0\,, \tag{71}$$

with

$$\xi = 4ay + b\,, \quad \eta = C_2(z)\,, \tag{72}$$

while a and b are arbitrary constants.

The corresponding special solution of Eqs. (1) and (3) reads

$$H_j = C_3(y) - \frac{C_4(y)}{x + C_3(y)} + a\Big[x - 2\int C_3(y)\,\mathrm{d}y\Big]P_j(\xi,\eta), \tag{73}$$

$$G_j = C_1(z)\Big[x - 2\int C_3(y)\,\mathrm{d}y\Big][1 + Q_j(\xi,\eta)], \tag{74}$$

where $C_1(z)$ and $C_i(y)$ $(i = 2, 3, 4)$ are arbitrary functions of the corresponding variables. $P_j(\xi, \eta)$ and $Q_j(\xi, \eta)$ satisfy Eqs. (70) and (71).

It is easy to know that equations (70) and (71) have the solution

$$P_j = \frac{2}{\xi + 2F_{1j}(\eta)}\,,$$
$$Q_j = \frac{-\xi^2 - 4F_{1j}(\eta)\xi + F_{2j}(\eta)}{[\xi + 2F_{1j}(\eta)]^2}\,,$$

where $F_{ij}(\eta)\,(i = 1, 2)$ are arbitrary functions. So equations (73) and (74) can be rewritten as

$$H_j = C_3(y) - \frac{C_4(y)}{x + C_3(y)} + a\Big[x - 2\int C_3(y)\,\mathrm{d}y\Big]\frac{2}{\xi + 2F_{1j}(\eta)}\,,$$
$$G_j = C_1(z)\Big[x - 2\int C_3(y)\,\mathrm{d}y\Big] \times \Big[1 + \frac{-\xi^2 - 4F_{1j}(\eta)\xi + F_{2j}(\eta)}{[\xi + 2F_{1j}(\eta)]^2}\Big],$$

with ξ and η satisfying Eq. (72).

(b) $\beta_{jx} = 0$

The reduction equations in this case can be written as

$$P_j + Q_{j\xi} = 0\,, \tag{75}$$
$$P_{j\xi\eta} = 0\,, \tag{76}$$

with $\xi = 2c_0c_1y + c_2, \eta = C_5(z)$. The reduction Eqs. (75) and (76) can be easily solved. The results read

$$P_j = F'_{1j}(\xi) + F_{2j}(\eta)\,,$$
$$Q_j = -F_{1j}(\xi) - F_{2j}(\eta)\xi + F_{3j}(\eta)\,,$$

where $F_{1j}(\xi)$, $F_{2j}(\eta)$, and $F_{3j}(\eta)$ are arbitrary functions of the corresponding variables.

The corresponding special solution of Eqs. (1) and (3) is as follows:

$$H_j = C'_{1j}(y) + C_{2j}(z) + c_0[F'_{1j}(2c_0c_1y + c_2) + F_{2j}(C_5(z))]\,,$$
$$G_j = c_1C_{4j}(z)x - 2c_1C_{4j}(z)C_{1j}(y) - 2c_1C_{2j}(z)C_{4j}(z)y + C_{3j}(z) + C_{4j}(z)[-F_{1j}(2c_0c_1y + c_2) - F_{2j}(C_5(z)) \times (2c_0c_1y + c_2) + F_{3j}(C_5(z))]\,,$$

where c_i $(i = 0, 1, 2)$ are arbitrary constants, $C_{1j}(y)$, $C_{2j}(z)$, $C_{3j}(z)$, $C_{4j}(z)$, and $C_5(z)$ are arbitrary functions of the corresponding variables.

4 Conclusions

It is proved that the (2+1)-dimensional multi-component BK system possesses Painlevé properties. The CK direct method is a useful tool of solving partial differential equations. But the application to the multi-component equations is seldom reported. In this paper, we apply the CK direct method to the (2+1)-dimensional multi-component BK system, and obtain some explicit solutions. All the solutions obtained in this paper have not been reported in other journals.

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