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Homotopy perturbation method for a type of nonlinear coupled equations with parameters derivative

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ABSTRACT

In this paper, homotopy perturbation method is directly extended to investigate nonlinear coupled equations with parameters derivative and to derive their numerical solutions. These nonlinear coupled equations with parameters derivative contain many important mathematical physics equations and reaction–diffusion equations. By choosing different values of the parameters in general formal numerical solutions, as a result, a very rapidly convergent series solution is obtained. The efficiency and accuracy of the method are verified by using two famous examples: coupled Burgers and mKdV equations. Numerical solutions show that good results have been achieved.

1. Introduction

In the recent years, with the rapid development of nonlinear science, the development of numerical techniques for solving nonlinear equations is a subject of considerable interest. Because many scientists and engineers have done the excellent work, the applications of homotopy theory have become a powerful mathematical tool in the nonlinear problems (references cited therein). For example, since 1986, Watson presented a series of probability-one homotopy algorithms for solving the nonlinear systems of equations that are globally convergent for a wide range of problems in science and engineering. With the development of computer simulation, the mathematical software packages HOMPACK90 and POLSYSPLP were given [1–3]. A more extensive list of references as well as a survey on progress made on this class of problems may be found in [4,5]. In 1992, based on the homotopy in topology, Liao [6] proposed a method, named homotopy analysis method (HAM), which transforms a nonlinear problem into an infinite number of linear problems without using the perturbation techniques. The homotopy perturbation method (HPM) [7,8] has been widely used by scientists and engineers to study the linear and nonlinear problems. As we all know, there exists a number of effective methods [9–17] that are applied to investigate the explicit and numerical solutions of various equations. Compared with other methods, the HPM always deforms the difficult problem into a simple and easily solvable one, which is a coupling of the traditional perturbation method and homotopy in topology. With this method, a series solution can be obtained that is usually rapidly convergent and with easily computable components.

Recently, many researchers have applied HPM method to various linear and nonlinear problems including reaction–diffusion equations [18], the fifth-order KdV equation [19], the fractional KdV equation [20], etc. [21–26]. In the recent decades, because of the extensive applications of the fractional differential equations, it has attracted great attention and interest in the areas of physics and engineering [27]. With fractional derivative equations, many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science can be well described [28–31]. However,

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as we know, for the nonlinear coupled equations with parameters derivative, especially fractional parameter derivative, not much work has been done.

In this paper, we extend the application of HPM to discuss the explicit numerical solutions of a type of nonlinear-coupled equations with parameters derivative in this form:

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= L_1(u,v) + N_1(u,v),\ t>0, \\ \frac{\partial^\beta v}{\partial t^\beta} &= L_2(u,v) + N_2(u,v),\ t>0, \end{aligned} \tag{1}$$

where L_i and N_i $(i=1,2)$ are the linear and nonlinear functions of u and v, respectively, α and β are the parameters that describe the order of the derivative. Different nonlinear coupled systems can be obtained when one of the parameters α or β varies. The study to Eq. (1) is very necessary and significant that is because when α and β are integers, it contains many important mathematical physics equations such as the coupled Burgers equations [32], mKdV equations [33] and many coupled reaction–diffusion equations [34,35]. The aim of this paper is to investigate the numerical solutions of Eq. (1) by introducing the Caputo derivative [31] and directly extending the application of the HPM in details.

The paper is organized as follows: in Section 2, some necessary description on the fractional calculus as well as the HPM for the nonlinear equations is given. In Section 3, two famous coupled examples: Burgers and mKdV equations are given to verify the effectiveness and accuracy of the proposed method. Finally, conclusions are followed.

2. Preliminaries and notations

2.1. Description on the fractional calculus

For the concept of fractional derivative, there exist many mathematical definitions [27–31]. In this paper, the two most commonly used definitions: the Caputo derivative and its reverse operator Riemann–Liouville integral are adopted. That is because Caputo fractional derivative [31] allows the traditional assumption of initial and boundary conditions. The Caputo fractional derivative is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\mathrm{d}\tau \quad (n-1 < Re(\alpha) \leqslant n, n \in N), \tag{2}$$

and the Riemann–Liouville fractional integral is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad x>0. \tag{3}$$

Here, we also need two basic properties about them:

$$\begin{aligned} &D^\alpha J^\alpha f(x) = f(x), \\ &J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{\infty} f^k(0^+)\frac{x^k}{k!}, \quad x>0. \end{aligned} \tag{4}$$

For more details on the fractional derivative and integral, one can consult Ref. [30].

Remark 1. In this paper, we need to discuss the fractional derivative for the type of nonlinear-coupled equations. When $\alpha \in R^+$ we just need to copy (2); when $\alpha = n \in N$, the fractional derivative reduces to the commonly used integer derivative. That is to say

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,\tau)}{\partial \tau^n}\mathrm{d}\tau, & n-1<\alpha<n, \\ \frac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n \in N. \end{cases} \tag{5}$$

2.2. Analysis on the homotopy perturbation method

To illustrate the basic ideas of the HPM, consider the operator form of Eq. (1):

$$\begin{aligned} D_t^\alpha u &= L_1(u,v) + N_1(u,v), \quad t>0, \\ D_t^\beta v &= L_2(u,v) + N_2(u,v), \quad t>0, \end{aligned} \tag{6}$$

where the operators D_t^α and D_t^β stand for the fractional derivative and are defined as in Eq. (5). The initial conditions are assumed as

$$u(x,0) = f(x), \quad v(x,0) = g(x). \tag{7}$$

According to the homotopy perturbation method, we construct the following simple homotopy:

$$\begin{aligned}&(1-p)D_t^{\alpha}u + p(D_t^{\alpha}u - L_1(u,v) - N_1(u,v)) = 0,\\&(1-p)D_t^{\beta}u + p(D_t^{\beta}v - L_2(u,v) - N_2(u,v)) = 0,\end{aligned} \tag{8}$$

or

$$\begin{aligned}&D_t^{\alpha}u - p(L_1(u,v) + N_1(u,v)) = 0,\\&D_t^{\beta}u - p(L_2(u,v) + N_2(u,v)) = 0,\end{aligned} \tag{9}$$

where p is an embedding parameter and it monotonously increases from zero to unit. When $p=0$, Eq. (9) becomes a set of linear equations $D_t^{\alpha}u = 0$ and $D_t^{\beta}v = 0$, which are easy to be solved; when $p=1$, Eq. (9) becomes the original one Eq. (6). In topology, this is called deformation, and $D_t^{\alpha}u,\ D_t^{\alpha}u - L_1(u,v) - N_1(u,v)$ and $D_t^{\beta}v,\ D_t^{\beta}v - L_2(u,v) - N_2(u,v)$ are called homotopic.

The basic assumption is that the solutions of Eq. (6) can be expressed as a power series of p:

$$\begin{aligned}u &= \sum_{i=0}^{+\infty} p^i u_i = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots,\\v &= \sum_{i=0}^{+\infty} p^i v_i = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots.\end{aligned} \tag{10}$$

The approximate solutions of Eq. (6) can be obtained when $p \to 1$:

$$\begin{aligned}u &= \lim_{p\to1} \sum p^i u_i = u_0 + u_1 + u_2 + u_3 + \cdots,\\v &= \lim_{p\to1} \sum p^i v_i = v_0 + v_1 + v_2 + v_3 + \cdots.\end{aligned} \tag{11}$$

The convergence of the method has been proved in Ref. [38].

Substituting Eq. (10) into Eq. (9), we can obtain a set of algebraic equations of $p^i\ (i = 0,1,\ldots)$. Equating the terms with the identical powers of p^i and setting them to zero yields a series of over-determined differential equations with respect to $u_n,\ v_n\ (n = 0,1,\ldots)$. Solving these over-determined differential equations, we can get the corresponding u_n and v_n. That is to say, we can obtain the numerical solutions of Eq. (6).

Remark 2. The parameters α and β can be arbitrarily chosen as, integer or fraction, bigger or smaller than 1. When the parameters are bigger than 1, collecting the coefficient of p^0 yields $D_t^{\alpha}u_0 = 0$. Applying the inverse operator J^{α} to it, we get $J^{\alpha}D_t^{\alpha}u_0 = u_0(x,t) - u_0(x,0) - tu_{0t}(x,0) - \cdots - \frac{t^n}{n!}u_{0t}^{(n)},\ n = [\alpha]$. That is to say, we will need more initial and boundary conditions such as $u_{0t}(x,0), u''_{0t}(x,0), \ldots$ and the calculations will become more complicated correspondingly. In order to illustrate the problem and make it convenient for the readers, we only confine the parameters to $[0,1]$ to discuss.

In the following, we will give two famous nonlinear-coupled examples: Burgers and mKdV equation to illustrate the effectiveness of the method in details.

3. Applications of the HPM with two famous nonlinear-coupled examples

3.1. The nonlinear coupled Burgers equations with parameters derivative

To construct the explicit and numerical solutions for the coupled Burgers equations, we take the system written in an operator form:

$$\begin{aligned}D_t^{\alpha}u &= L_{xx}u + 2uL_xu - L_xuv, (0 < \alpha \leqslant 1),\\D_t^{\beta}v &= L_{xx}v + 2vL_xv - L_xuv, (0 < \beta \leqslant 1),\end{aligned} \tag{12}$$

where $t > 0$, $L_x = \partial/\partial x$ and the fractional operators D_t^{α} and D_t^{β} are defined as in Eq. (5).

As for the choice of the initial conditions, we generally take them based on its exact solutions that are known to compare an approximate error. Assuming the initial value as

$$\begin{aligned}u(x,0) &= \sin x,\\v(x,0) &= \sin x.\end{aligned} \tag{13}$$

The exact solutions of Eq. (12) for the special case: $\alpha = \beta = 1$ are

$$\begin{aligned}u(x,t) &= \mathrm{e}^{-t}\sin x,\\v(x,t) &= \mathrm{e}^{-t}\sin x.\end{aligned} \tag{14}$$

Extending the HPM to Eq. (12), according to Eq. (9), we get

$$\begin{aligned}&D_t^{\alpha}u = p(L_{xx}u + 2uL_xu - L_xuv),\\&D_t^{\beta}v = p(L_{xx}v + 2vL_xv - L_xuv).\end{aligned} \tag{15}$$

With the aid of *Maple*, substituting Eq. (10) into Eq. (15) yields a set of algebraic equations of p^i $(i = 0, 1, \cdots)$. Setting the coefficients of these terms p^i to zero, we can obtain a set of equations of $u_n,\ v_n$ $(n = 0, 1, \ldots)$. For convenience, we just list the first few terms:

$$\begin{aligned}&D_t^{\alpha}u_0 = 0,\\&D_t^{\beta}v_0 = 0,\\&D_t^{\alpha}u_1 = L_{xx}u_0 + 2u_0L_xu_0 - L_xu_0v_0,\\&D_t^{\beta}v_1 = L_{xx}v_0 + 2v_0L_xv_0 - L_xu_0v_0,\\&D_t^{\alpha}u_2 = L_{xx}u_1 + 2L_xu_0u_1 - L_xu_0v_1 - L_xu_1v_0,\\&D_t^{\beta}v_2 = L_{xx}v_1 + 2L_xv_0v_1 - L_xu_0v_1 - L_xu_1v_0, \ldots\end{aligned} \tag{16}$$

Applying the operators J^{α}, J^{β}, the inverse operators of D^{α}, D^{β} on both sides of the corresponding Eq. (16), yields

$$\begin{aligned}&u_0 = f_0(x), u_1 = f_1(x) + f^1\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\&v_0 = g_0(x), v_1 = g_1(x) + g^1\frac{t^{\beta}}{\Gamma(\beta+1)},\\&u_2 = f_2(x) + f^2\frac{t^{\alpha}}{\Gamma(\alpha+1)} + f^3\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + f^4\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},\\&v_2 = g_2(x) + g^2\frac{t^{\beta}}{\Gamma(\beta+1)} + g^3\frac{t^{2\beta}}{\Gamma(2\beta+1)} + g^4\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}, \ldots,\end{aligned}$$

where

$$\begin{aligned}&f_i(x) = u_i(x,0), \quad \sum_{i=0}^{\infty} f_i(x) = f(x), \quad f^1 = f_{0xx} + 2f_0f_{0x} - (f_0g_0)_x,\\&g_i(x) = v_i(x,0), \quad \sum_{i=0}^{\infty} g_i(x) = g(x), \quad g^1 = g_{0xx} + 2g_0g_{0x} - (f_0g_0)_x,\\&f^2 = f_{1xx} + 2(f_0f_1)_x - (f_0g_1 + g_0f_1)_x, \quad f^3 = f^1_{xx} + 2(f_0f^1)_x - (g_0f^1)_x,\\&g^2 = g_{1xx} + 2(g_0g_1)_x - (f_0g_1 + g_0f_1)_x, \quad g^3 = g^1_{xx} + 2(g_0g^1)_x - (f_0g^1)_x,\\&f^4 = -f_{0x}g^1 - f_0g^1_x, \quad g^4 = -g_{0x}f^1 - g_0f^1_x.\end{aligned}$$

So we get the approximate solutions of nonlinear coupled Burgers equations with parameters derivative Eq. (14) in a finite series as

$$\begin{aligned}&u = \sum_{i=0}^{\infty} u_i = f(x) + f^1\frac{t^{\alpha}}{\Gamma(\alpha+1)} + f^2\frac{t^{\alpha}}{\Gamma(\alpha+1)} + f^3\frac{2\alpha}{\Gamma(2\alpha+1)} + f^4\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \cdots,\\&v = \sum_{i=0}^{\infty} v_i = g(x) + g^1\frac{t^{\beta}}{\Gamma(\beta+1)} + g^2\frac{t^{\beta}}{\Gamma(\beta+1)} + g^3\frac{2\beta}{\Gamma(2\beta+1)} + g^4\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \cdots\end{aligned} \tag{17}$$

Remark 3. In fact, as we all know, for various perturbation methods, low-order approximate solution can produce high accuracy, usually 2–4 terms are enough only if we choose proper $f_i(x)$ and $g_i(x)$. In order to illustrate the convergent speed that depends on the initial conditions and the derivative operator, not to lose the generality, we take the parameters as $\alpha = \beta = 1,\ f_0(x) = g_0(x) = \sin x$ and $f_i(x) = g_i(x) \equiv 0$; substituting these into the above expressions, we can easily get $u_i = v_i = \frac{(-t)^i}{i!}\sin x$. So the series solutions of Eq. (14) are $u = v = \sum_{i=0}^{\infty}\frac{(-t)^i}{i!}\sin x = e^{-t}\sin x$, which are rapidly convergent to the exact solutions.

Remark 4. Up to now, there is no valid method to obtain the exact solutions for the differential equation of fractional order according to the existing literature. So it is nearly impossible to analyze the actual error between the exact solutions and the numerical solutions of the fractional differential equation. Here, we only limit to discuss the fractional differential equation in which the fractional order equals to integer. However, in this paper, we just employ a graphical simulation to roughly evaluate the accuracy of the numerical solutions obtained. As to the higher accurate analysis, it will be further discussed and investigated (see Table 1).

Table 1
The approximate solution of $\phi_n = \sum_{i=0}^{n} u_n$, exact solution $u(x,t)$ and relative error when $\alpha = \beta = 1$

x	t	ϕ_n	$u(x,t)$	$\lvert u(x,t) - \phi_n \rvert$
−2	0.01	−0.9002497658	−0.9002497662	0.4546487134e−10
−2	0.02	−0.8912921314	−0.8912921314	0.9092974268e−11
−2	0.03	−0.8824236263	−0.8824236265	0.4546487134e−10
5	0.01	−0.9493828183	−0.9493828187	0.4794621374e−10
5	0.02	−0.9399363019	−0.9399363019	0.9589242747e−11
5	0.03	−0.9305837792	−0.9305837793	0.4794621374e−10
10	0.01	−0.5386080102	−0.5386080104	0.2720105554e−10
10	0.02	−0.5332487712	−0.5332487712	0.5440211109e−11
10	0.03	−0.5279428571	−0.5279428572	0.2720105554e−10

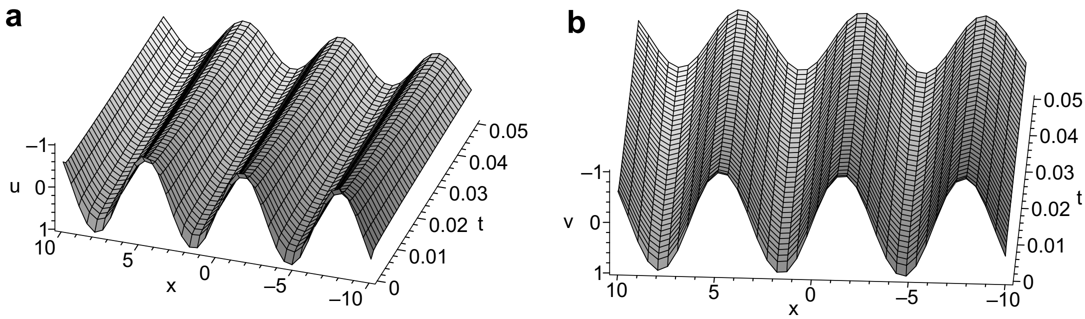

Fig. 1. Explicit numerical solutions for Burgers Eq. (12): (a) $u(x,t)$, (b) $v(x,t)$ with $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{3}$.

From the table, we can obviously see the approximate numerical solutions ϕ_5 in Eq. (17) when $\alpha = \beta = 1$, the exact solution (14) as well as the relative error between them. Fig. 1 shows the numerical solutions of the nonlinear coupled Burgers equations with parameters derivative with $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{3}$. Fig. 2 depicts the exact solutions of the classical Burger equations with $\alpha = \beta = 1$. Fig. 3 shows the residual graph between the two different kind of solutions with $\alpha = \beta = 1$ at $t = 0.02$. The table and the residual figures fully show that the numerical solutions obtained by us can rapidly converge to the exact solutions derived by Liu et al. So we conclude that HPM is an effective method for the nonlinear coupled Burgers equations with parameters derivative.

3.2. The nonlinear coupled mKdV equations with parameters derivative

In this section, we will take the nonlinear coupled mKdV equations with parameters derivative as an example to illustrate the feasibleness and accuracy of the HPM.

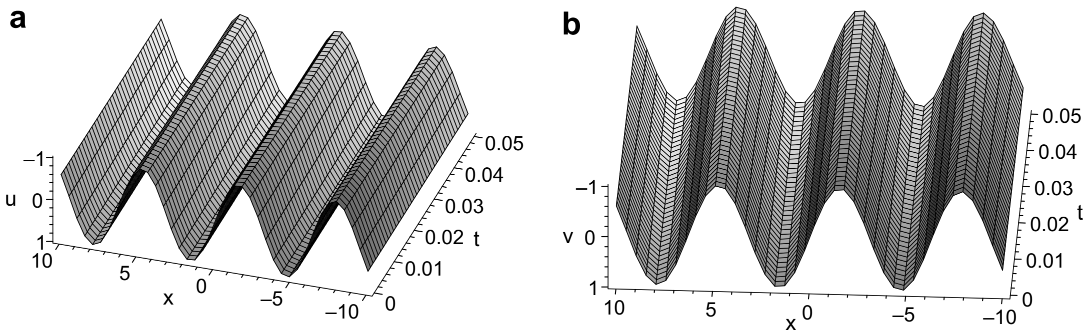

Fig. 2. Exact solutions (14) for Eq. (12): (a) $u(x,t)$, (b) $v(x,t)$ with $\alpha = 1$ and $\beta = 1$.

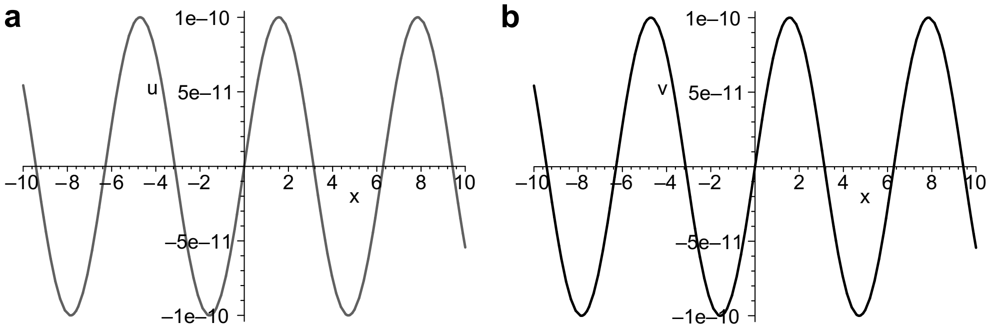

Fig. 3. Residual graph between the numerical solutions (17) and the exact ones (14) at $t = 0.02$: (a) $u(x,t)$, (b) $v(x,t)$ with $\alpha = 1$ and $\beta = 1$.

The coupled mKdV equations are given in the operator form:

$$\begin{aligned} D_t^\alpha u &= \frac{1}{2}u_{xxx} - 3u^2u_x + \frac{3}{2}v_{xx} + 3(uv)_x - 3\lambda u_x, \\ D_t^\beta v &= -v_{xxx} - 3vv_x - 3u_xv_x + 3u^2v_x + 3\lambda v_x, \end{aligned} \tag{18}$$

with the initial conditions

$$u(x,0) = \frac{b}{2k} + k\tanh(kx), \quad v(x,0) = \frac{\lambda}{2}\left(1 + \frac{k}{b}\right) + b\tanh(kx).$$

Implementing the HPM into Eq. (18), and according to the homotopic expressions (9), yields

$$\begin{aligned} D_t^\alpha u &= p\left(\frac{1}{2}u_{xxx} - 3u^2u_x + \frac{3}{2}v_{xx} + 3(uv)_x - 3\lambda u_x\right), \\ D_t^\beta v &= p(-v_{xxx} - 3vv_x - 3u_xv_x + 3u^2v_x + 3\lambda v_x). \end{aligned} \tag{19}$$

The steps are the same as above: substituting the series $u = \sum p^i u_i$ and $v = \sum p^i v_i$, balancing the same powers of p^i and setting them into them as zero, then we can obtain a series of over-determined equations, and by solving them we can get the results. Here, we just give the first three terms of p^i $(i = 0, 1, 2)$ and the corresponding numerical solutions:

$$\begin{aligned} D_t^\alpha u_0 &= 0, \\ D_t^\beta v_0 &= 0, \\ D_t^\alpha u_1 &= \frac{1}{2}u_{0xxx} - 3u_0^2u_{0x} + \frac{3}{2}v_{0xx} + 3(u_0v_0)_x - 3\lambda u_{0x}, \\ D_t^\beta v_1 &= -v_{0xxx} - 3v_0v_{0x} - 3u_{0x}v_{0x} + 3u_0^2v_{0x} + 3\lambda v_{0x}, \\ D_t^\alpha u_2 &= \frac{1}{2}u_{1xxx} - 3(u_0^2u_{1x} + 2u_0u_{0x}u_1) + \frac{3}{2}v_{1xx} + 3(u_0v_1 + u_1v_0)_x - 3\lambda u_{1x}, \\ D_t^\beta v_2 &= -v_{1xxx} - 3(v_0v_1)_x - 3(u_{0x}v_{1x} + u_{1x}v_{0x}) + 3(u_0^2v_{1x} + 2u_0u_1v_{0x}) + 3\lambda v_{1x}. \end{aligned}$$

Solve them and get $u_i,\ v_i$ $(i = 0, 1, 2)$ as

$$\begin{aligned} u_0 &= f_0(x), \quad u_1 = f_1(x) + f^1\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ v_0 &= g_0(x), \quad v_1 = g_1(x) + g^1\frac{t^\beta}{\Gamma(\beta+1)}, \\ u_2 &= f_2(x) + f^2\frac{t^\alpha}{\Gamma(\alpha+1)} + f^3\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + f^4\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}, \\ v_2 &= g_2(x) + g^2\frac{t^\beta}{\Gamma(\beta+1)} + g^3\frac{t^{2\beta}}{\Gamma(2\beta+1)} + g^4\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}, \cdots, \end{aligned}$$

where

$$\begin{aligned} &f_i(x) = u_i(x,0), \quad \sum_{i=0}^{\infty} f_i(x) = f(x), \quad f^1 = \frac{1}{2}f_{0xxx} - 3f_0^2f_{0x} + \frac{3}{2}g_{0xx} + 3(f_0g_0)_x - 3\lambda f_{0x}, \\ &g_i(x) = v_i(x,0), \quad \sum_{i=0}^{\infty} g_i(x) = g(x), \quad g^1 = -g_{0xxx} - 3g_0g_{0x} - 3f_{0x}g_{0x} + 3f_0^2g_{0x} + 3\lambda g_{0x}, \\ &f^2 = \frac{1}{2}f_{1xxx} - 3(f_0^2f_1)_x + \frac{3}{2}g_{1xx} + 3(f_0g_1 + f_1g_0)_x - 3\lambda f_{1x}, \\ &g^2 = -g_{1xxx} - 3(g_0g_1)_x - 3(f_{0x}g_{1x} + f_{1x}g_{0x}) + 3(f_0^2g_{1x} + 2f_0f_1g_{0x}) + 3\lambda g_{1x}, \end{aligned}$$

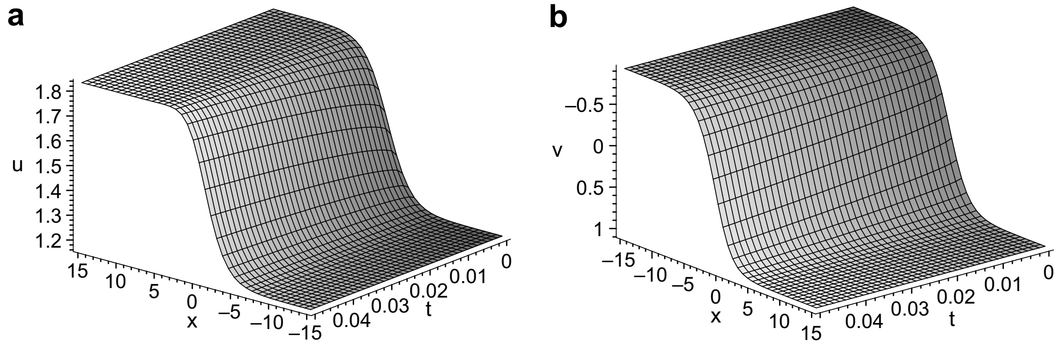

Fig. 4. Explicit numerical solutions for Eq. (20): (a) $u(x,t)$, (b) $v(x,t)$ with $\alpha=\frac{1}{2}$, $\beta=\frac{2}{3}$, $\lambda=0.1$, $b=1$ and $k=\frac{1}{3}$.

$$
\begin{aligned}
f^3 &= \frac{1}{2}f^1_{xxx}+3(g_0f^1-f_0^2f^1)_x-3\lambda f^1_x,\\
g^3 &= -3g^1_{xxx}+3f_0^2g^1_x-3(g_0g^1)_x-3f_{0x}g^1_x)+3\lambda g^1_x,\\
f^4 &= \frac{3}{2}g^1_{xx}+3(f_0g^1)_x, \quad g^4=6f_0f^1g_{0x}-3g_0f^1_x.
\end{aligned}
$$

So the final numerical results are

$$
\begin{aligned}
u &= \sum_{i=0}^{\infty}u_i=f(x)+f^1\frac{t^{\alpha}}{\Gamma(\alpha+1)}+f^2\frac{t^{\alpha}}{\Gamma(\alpha+1)}+f^3\frac{2\alpha}{\Gamma(2\alpha+1)}+f^4\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\cdots,\\
v &= \sum_{i=0}^{\infty}v_i=g(x)+g^1\frac{t^{\beta}}{\Gamma(\beta+1)}+g^2\frac{t^{\beta}}{\Gamma(\beta+1)}+g^3\frac{2\beta}{\Gamma(2\beta+1)}+g^4\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\cdots
\end{aligned}
\tag{20}
$$

As we know, when $\alpha=\beta=1$, Eq. (18) has the kink-type soliton solutions

$$
\begin{aligned}
u(x,t) &= \frac{b}{2k}+k\tanh(k\xi),\\
v(x,t) &= \frac{\lambda}{2}\left(1+\frac{k}{b}\right)+b\tanh(k\xi),
\end{aligned}
\tag{21}
$$

constructed by Fan [39], where $\xi=x+\frac{1}{4}(-4k^2-6\lambda+\frac{6k\lambda}{b}+\frac{3b^2}{k^2})t$, $k\neq0$ and $b\neq0$.

The effectiveness and accuracy of the numerical results can be seen by making a comparison of the figures. Figs. 3 and 4 show the numerical solutions (20) with $\alpha=\frac{1}{2}$, $\beta=\frac{2}{3}$ and the exact ones (21) with $\alpha=\beta=1$ when $\lambda=0.1$, $b=1$, $k=\frac{1}{3}$, respectively. Fig. 6 is the error graph between the two solutions with $\alpha=\beta=1$ at $t=0.002$. From Fig. 6, we can know that the series solutions converge rapidly by HPM, so we say that a good approximation has been achieved (Fig. 5).

Remark 5. In fact, the operators $L_i(u,v)$ and $N_i(u,v)$ $(i=1,2)$ can also contain fractional derivative. In this case, the system will become another type of coupled system with time- and space-parameters derivative. The HPM is also effective to them. As the main steps are similar to the above, we omit these complex calculations here.

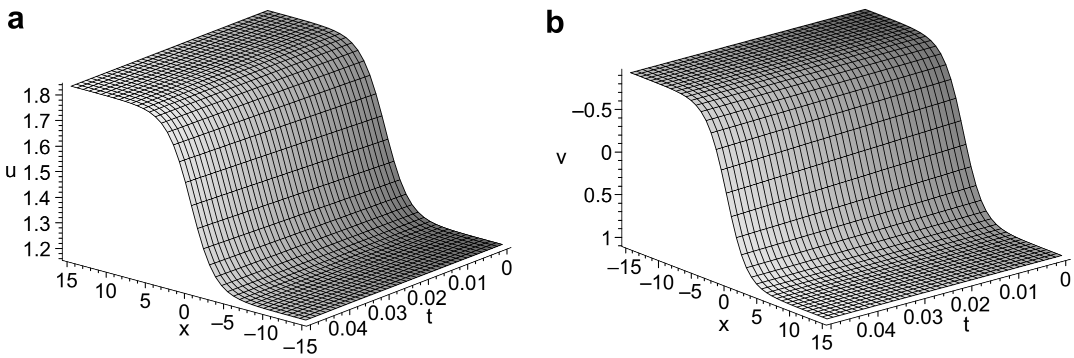

Fig. 5. Exact solutions (21) for (18): (a) $u(x,t)$, (b) $v(x,t)$ with $\alpha=1$, $\beta=1$, $\lambda=0.1$, $b=1$ and $k=\frac{1}{3}$.

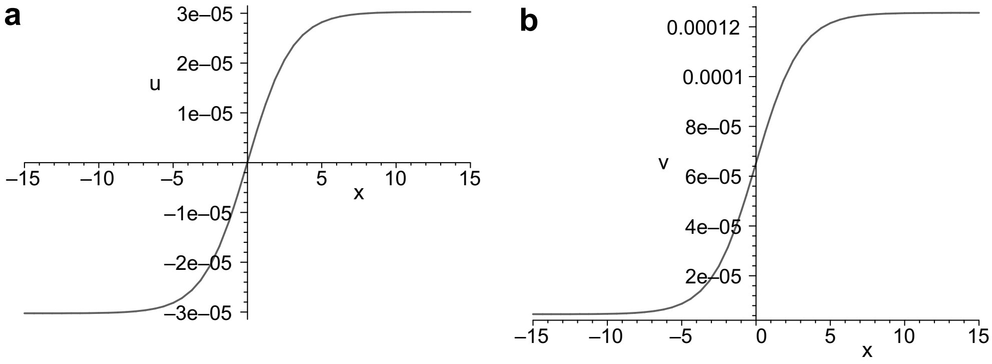

Fig. 6. Residual graph between the numerical solutions and the exact ones at $t = 0.001$: (a) $u(x,t)$, (b) $v(x,t)$ with $\alpha = 1$ and $\beta = 1$.

4. Conclusion

In this paper, we have investigated a type of nonlinear coupled equations with parameters derivative. When the parameters derivative is a fractional derivative, we assume them to satisfy the Caputo derivative. The HPM has been successfully extended to derive its explicit numerical solutions. Actually, the study on the type of coupled equations with parameters derivative is very interesting and significant because when the parameters are integers, the system contains many important mathematical physics models as well as many coupled reaction–diffusion equations. Two different test systems: coupled Burgers and mKdV equations are used to illustrate the validity of the proposed method. A comparison of the results between the numerical and the exact solutions implies that we have achieved a good result. So the HPM is a very powerful tool to solve a wide class of linear and nonlinear problems, especially the weak nonlinearity. Hayat and Sajid [36,37] proved that the HPM [7,8] is a special case when $\hbar = -1$ of the HAM. The HAM has more advantages, such as providing a simple way to control and adjust the convergence region both for weak and for strong nonlinearity. However, in this paper, we employ the HPM to investigate the problems of weak nonlinearity in order to avoid the heavy calculation by auxiliary parameter $\hbar$. In addition, the technique can also be extended to a generalized type of coupled equations with time- and space-parameters derivative. The question of whether we can introduce other new feasible derivative operators or algorithms to solve the systems and whether we can adopt other techniques to accelerate the convergent speed of the solutions will be further studied. At the same time, how to find an effective method to obtain the exact solution for the differential equation of fractional order will be deeply explored later.

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