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Kac–Moody–Virasoro Symmetry Algebra of (2+1)-Dimensional Dispersive Long-Wave Equation with Arbitrary Order Invariant∗

ZHANG Huan-Ping (张焕萍),1 LI Biao (李彪),1,3,† and CHEN Yong (陈勇)2

1Nonlinear Science Center, Ningbo University, Ningbo 315211, China
2Institute of Theoretical Computing, East China Normal University, Shanghai 200062, China
3MM Key Lab, Chinese Academy of Sciences, Beijing 100080, China

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Abstract By Lie symmetry method, the Lie point symmetries and its Kac–Moody–Virasoro (KMV) symmetry algebra of (2+1)-dimensional dispersive long-wave equation (DLWE) are obtained, and the finite transformation of DLWE is given by symmetry group direct method, which can recover Lie point symmetries. Then KMV symmetry algebra of DLWE with arbitrary order invariant is also obtained. On basis of this algebra the group invariant solutions and similarity reductions are also derived.

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Key words: Kac–Moody–Virasoro symmetry algebra, dispersive long-wave equation, symmetry reduction, group invariant solutions

1 Introduction

In recent years, the study of symmetries, symmetry groups, symmetry reductions, group invariant solutions, and soliton solutions of nonlinear partial differential equations (PDEs) have become one of the most exciting and extremely active areas of research.11–13 The investigations of the symmetries of (2+1)-dimensional integrable differential equations show that they possess in common an isomorphic centerless Virasoro symmetry algebra:12–14

\[ [\sigma(f_1), \sigma(f_2)] = \sigma(f_2 \dot{f}_1 - f_1 \dot{f}_2), \]

where \( f_1 \) and \( f_2 \) are arbitrary functions of a single independent variable. The dot means the derivative of the functions with respect to their argument. According to the centerless Virasoro symmetry algebra, similarity reduction of equation can be given out. Most recently, Lou et al. have developed a symmetry group direct method in a series of papers.15–18 By the method, both the Lie point symmetry groups and the non-Lie symmetry groups can be obtained for some PDEs.

And it is well-known that the integrable dispersive long wave equations (DLWE) is an interesting topics in physics and mathematics,

\[ u_{yt} + v_{xx} + u_x u_y + uu_{xy} = 0, \]

\[ v_t + u_x + uu_x + uv_y + u u_{xy} = 0, \]

where \( u \equiv u(x, y, t), \ v \equiv v(x, y, t). \) And (2+1)-dimensions DLWE with arbitrary order can be read as

\[ u_{yt} + v_{xx} + u_x u_y + uu_{xy} + F(u, v) = 0, \]

\[ v_t + u_x + uu_x + uv_y + G(u, v) = 0, \]

where \( u \equiv u(x, y, t), \ v \equiv v(x, y, t). \) F and G are arbitrary functions with respect to their argument. According to the centerless Virasoro symmetry algebra, similarity reduction of equation can be given out. Most recently, Lou et al. have developed a symmetry group direct method in a series of papers.15–18 By the method, both the Lie point symmetry groups and the non-Lie symmetry groups can be obtained for some PDEs.

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2 Lie Point Symmetries and Finite Transformations of (2+1)-Dimensional DLWE

A symmetry of arbitrary equation can be defined as a

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†E-mail: biaolee2000@yahoo.com.cn
solution of
\[ \frac{d}{de} H(u + e\sigma) = 0, \]  
where \( H(u) \) needs to satisfy \( H(u) = 0, \sigma \) is a symmetry of equation.

According to the definition of symmetry, we obtain the symmetry equations of the (2+1)-dimensional DLWE:
\begin{align*}
\sigma_{1yt} + \sigma_{2xx} + u_x \sigma_{1y} + \sigma_{1x} u_y + u \sigma_{1xy} + \sigma_{1uxy} &= 0, \\
\sigma_{2t} + \sigma_{1x} + \nu \sigma_{1x} + \sigma_{2ux} + \sigma_{2x} + \sigma_{1v_x} + \sigma_{1xxxy} &= 0.
\end{align*}

Then let
\begin{align*}
\sigma_1 &= \xi u_x + \eta u_y + \tau u_x - \phi_1, \\
\sigma_2 &= \xi v_x + \eta v_y + \tau v_t - \phi_2,
\end{align*}
where \( \xi \equiv \xi(x, y, t, u, v), \eta \equiv \eta(x, y, t, u, v), \tau \equiv \tau(x, y, t, u, v), \phi_1 \equiv \phi_1(x, y, t, u, v), \phi_2 \equiv \phi_2(x, y, t, u, v). \)

Substituting Eqs. (9) and (10) into Eqs. (7) and (8), then eliminating \( \nu \), \( \xi, \eta, \tau, \phi_1, \phi_2 \) can be obtained by vanishing the coefficients of the polynomials of \( u, v \) and their derivatives:
\begin{align*}
\tau &= f(t), \quad \eta = l(y), \quad \xi = \frac{1}{2} f_1(t) x + h(t), \\
\phi_1 &= -\frac{1}{2} u f_1(t) + \frac{1}{2} f_1(t) x + h(t), \\
\phi_2 &= -\frac{1}{2(1 + v)} (2 l(y) + f_1(t)).
\end{align*}

According to Eqs. (11) and (12), the Lie point symmetries of the DLWE are the linear combinations of the following generators:
\begin{align*}
\sigma_1(f) &= \left( u_x f_1(t) x + f(t) u_x + \frac{1}{2} u f_1(t) - \frac{1}{2} f_1(t) x \right), \\
\sigma_2(h) &= \left( u_x h(t) - h(t) \right), \\
\sigma_3(l) &= \left( l(y) u_y + l_y(y) + v l_y(y) \right).
\end{align*}

The commutators among \( \sigma_1(f), \sigma_2(h), \) and \( \sigma_3(l) \) are
\begin{align*}
[\sigma_1(f_1), \sigma_1(f_2)] &= \sigma_1(f_1 f_2 - f_2 f_1), \\
[\sigma_1(f), \sigma_2(h)] &= \sigma_2(f h_t - \frac{1}{2} h f_t), \\
[\sigma_2(h_1), \sigma_2(h_2)] &= 0.
\end{align*}

According to Eqs. (16) and (17), we know that the subalgebra constituted by \( \sigma_1(f) \) and \( \sigma_2(h) \) is just the Virasoro algebra.

In general, we can obtain symmetry groups by Lie point symmetries, but there are many difficulties. Fortunately, Lou et al. have developed a symmetry group direct method, by the method, both the Lie point symmetry groups and the non-Lie symmetry groups can be obtained. In the following, we would like to use symmetry group direct method to search for finite symmetry transformation of the DLWE (2) and (3), which can recover generators \( \sigma_1(f), \sigma_2(h), \sigma_3(l) \).

Firstly, let
\begin{align*}
u &= \alpha_1 + \beta_1 U(\xi_1, \eta_1, \tau_1) + \gamma_1 V(\xi_1, \eta_1, \tau_1), \\
v &= \alpha_2 + \beta_2 U(\xi_1, \eta_1, \tau_1) + \gamma_2 V(\xi_1, \eta_1, \tau_1),
\end{align*}
where \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \xi_1, \eta_1, \tau_1 \) and \( \tau_1 \) are functions of \( x, y, t \). Restricting \( U(\xi_1, \eta_1, \tau_1) \equiv U, V(\xi_1, \eta_1, \tau_1) \equiv V, \) and satisfy the same form as the DLWE (2) and (3) but with new independent variables, i.e.
\begin{align*}
V_{\xi_1 \xi_1} + U_{\eta_1 \tau_1} + U_{\xi_1} U_{\eta_1} + U(\xi_1, \eta_1, \tau_1) U_{\xi_1 \eta_1} &= 0, \\
U_{\xi_1 \xi_1 \eta_1} + V_{\tau_1} + U_{\xi_1} + V(\xi_1, \eta_1, \tau_1) U_{\xi_1} + U(\xi_1, \eta_1, \tau_1) V_{\xi_1} &= 0.
\end{align*}

Substituting Eqs. (18) and (19) into DLWE (2) and (3), then eliminating \( V_{\xi_1 \xi_1} \) and \( U_{\xi_1 \xi_1 \eta_1} \) by using Eqs. (20) and (21), from that, the remained determining equations of the functions \( \xi_1, \eta_1, \tau_1, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \) can be obtained by vanishing the coefficients of the polynomials of \( U, V \) and its derivatives, then it is straightforward to find out the general solution of the determining equations.

The results read:
\begin{align*}
\xi_1 &= \delta_{0t}^{\frac{1}{2}} x + \xi_0, \quad \eta_1 = \eta_0, \quad \tau_1 = \tau_0, \\
\beta_1 &= \delta_{0t}^{\frac{1}{2}}, \quad \beta_2 = \gamma_1 = 0, \quad \gamma_2 = \delta_{0t}^{\frac{3}{2}} \eta_0 y, \\
\alpha_1 &= \frac{1}{2} \delta_{0t}^{\frac{3}{2}} \eta_0 y + \frac{2}{\delta_{0t}^{\frac{1}{2}}}, \\
\alpha_2 &= -1 + \delta_{0t}^{\frac{1}{2}} \eta_0 y,
\end{align*}
where \( \xi_0 \equiv \xi_0(t), \eta_0 \equiv \eta_0(y), \tau_0 \equiv \tau_0(t), \delta = \pm 1. \)

In summary, the following theorem holds:

**Theorem** If \( U \equiv U(x, y, t), V \equiv V(x, y, t) \) are a solution of the DLWE (2) and (3), then so are
\begin{align*}
u &= -\frac{1}{2} \left( x \delta_{0t}^{\frac{3}{2}} \delta - 2 \delta_{0t}^{\frac{1}{2}} \eta_0 y \right), \\
v &= -1 + \left( \delta_{0t}^{\frac{1}{2}} \eta_0 y + \delta_{0t}^{\frac{1}{2}} \eta_0 y V(\xi_1, \eta_1, \tau_1) \right),
\end{align*}
with Eq. (22), where \( \xi_0 \equiv \xi_0(t), \eta_0 \equiv \eta_0(y), \tau_0 \equiv \tau_0(t) \) and discrete value of the \( \delta = \pm 1. \)

From Theorem, by restricting \( f \equiv f(t), h \equiv h(t), l \equiv l(y) \)
\begin{align*}
\tau = t + \epsilon f, \quad \xi_0 = \epsilon h, \quad \eta_0 = y + \epsilon l,
\end{align*}
we can obtain the general Lie point symmetries of DLWE: Eqs. (13)–(15).

### 3 Kac–Moody–Virasoro Symmetry Algebra of DLWE with Arbitrary Order Invariant

In this section, we would discuss Kac–Moody–Virasoro symmetry algebra of DLWE with arbitrary order invariant.

Firstly, let \( u \rightarrow u + \epsilon \sigma_1, v \rightarrow v + \epsilon \sigma_2, \) according to Eq. (6) the symmetry equation of Eqs. (4) and (5) is obtained
\begin{align*}
\sigma_{1yt} + \sigma_{2xx} + u_x \sigma_{1y} + \sigma_{1x} u_y + u \sigma_{1xy} + \sigma_{1uxy} + F'_1 \sigma_1 + F'_2 \sigma_2 &= 0, \\
\sigma_{2t} + \sigma_{1x} + v \sigma_{1x} + \sigma_{2ux} + u \sigma_{2x} + \sigma_{1v_x} + \sigma_{1xxxy} + G'_1 \sigma_1 + G'_2 \sigma_2 &= 0,
\end{align*}
where $F'_i$ and $G'_i$ are the linearized operator, which respectively defined by

$$F'_1 H = \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} F_1 (u + \epsilon H), \quad F'_2 H = \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} F_2 (v + \epsilon H),$$

$$G'_1 H = \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} G_1 (u + \epsilon H), \quad G'_2 H = \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} G_2 (v + \epsilon H).$$

(27)

(28)

Because DLWE has Lie point symmetries Eqs. (13)–(15), we would divide two cases to consider Eqs. (13) and (14) symmetries of the DLWE.

**Case 1** Substituting Eq. (13) into Eqs. (25) and (26) with the help of Eqs. (4) and (5), we get

$$- \frac{1}{2} f_{x x} F'_1 u_x - \frac{1}{2} f_{x x} F'_2 v_x - \frac{3}{2} f_{x} F'_{1} u_t - F F'_1 u_t - f F'_2 v_t + F_1 (\frac{1}{2} f_x f_x + f u_t - \frac{1}{2} f_{t t} x + \frac{1}{2} f_t)$$

$$+ F'_2 (\frac{1}{2} f_{x x} f_x + f v_t + \frac{1}{2} f_{t t} + \frac{1}{2} v f_t) = 0,$$

$$- \frac{1}{2} f_{x x} G'_1 u_x - \frac{1}{2} f_{x x} G'_2 v_x - \frac{3}{2} f_{x} G'_{1} u_t - G G'_1 u_t - f G'_2 v_t + G_1 (\frac{1}{2} f_x f_x + f u_t - \frac{1}{2} f_{t t} x + \frac{1}{2} u f_t)$$

$$+ G'_2 (\frac{1}{2} f_{x x} f_x + f v_t + \frac{1}{2} f_{t t} + \frac{1}{2} v f_t) = 0,$$

(29)

(30)

where

$$H'_1 = \sum_{n,m} \frac{\partial H}{\partial u_{x^n y^m}} \frac{\partial^{m+n}}{\partial x^n \partial y^m}, \quad H'_2 = \sum_{n,m} \frac{\partial H}{\partial v_{x^n y^m}} \frac{\partial^{m+n}}{\partial x^n \partial y^m}, \quad m, n \geq 0, \quad H = F, G.$$

Then substitute Eq. (31) into Eqs. (29) and (30), we obtain

$$\frac{1}{2} f_t (\sum_{n,m} (n + 1) \frac{\partial F}{\partial u_{x^n y^m}} u_x + \sum_{n,m} (n + 1) \frac{\partial F}{\partial v_{x^n y^m}} v_{x^n y^m}) - \frac{1}{2} f_u \left( x \frac{\partial F}{\partial u} + \frac{\partial F}{\partial u_x} \right) = \frac{3}{2} f_t F,$$

$$\frac{1}{2} f_t (\sum_{n,m} (n + 1) \frac{\partial G}{\partial u_{x^n y^m}} u_x + \sum_{n,m} (n + 1) \frac{\partial G}{\partial v_{x^n y^m}} v_{x^n y^m}) - \frac{1}{2} f_u \left( x \frac{\partial G}{\partial u} + \frac{\partial G}{\partial u_x} \right) = \frac{3}{2} f_t G.$$

(32)

(33)

Under the $f$-independent requirement and the autonomous condition of $F, G$, according to Eqs. (32) and (33), we have

$$\frac{\partial F}{\partial u} = 0, \quad \frac{\partial F}{\partial u_x} = 0, \quad \frac{\partial G}{\partial u} = 0, \quad \frac{\partial G}{\partial u_x} = 0,$$

$$\sum_{n,m} (n + 1) \frac{\partial F}{\partial u_{x^n y^m}} u_x + \sum_{n,m} (n + 1) \frac{\partial F}{\partial v_{x^n y^m}} v_{x^n y^m} = 3F,$$

$$\sum_{n,m} (n + 1) \frac{\partial G}{\partial u_{x^n y^m}} u_x + \sum_{n,m} (n + 1) \frac{\partial G}{\partial v_{x^n y^m}} v_{x^n y^m} = 3G.$$

(34)

(35)

(36)

The solution of Eqs. (34), (35), and (36) can be written as:

$$F = u_y^3 M_1 (1 - \delta_{00} \delta_{m0} - \delta_{n1} \delta_{m0}) u_{x^n y^m} u_{y^n}^{(n+1)} u_{x^n y^m} u_{y^n}^{(n+1)}, n, m \geq 0 \equiv u^3_y M_1,$$

$$G = u_y^3 N_1 (1 - \delta_{00} \delta_{m0} - \delta_{n1} \delta_{m0}) u_{x^n y^m} u_{y^n}^{(n+1)} u_{x^n y^m} u_{y^n}^{(n+1)}, n, m \geq 0 \equiv u^3_y N_1,$$

(37)

(38)

where

$$\delta_{nm} = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases}$$

In all, if equation has the form

$$u_{yt} + v_{xx} + u_x u_y + uu_{xy} + u^3_y M_1 = 0,$$

$$v_t + u_x + uu + u_{xy} + u^3_y N_1 = 0,$$

(39)

(40)

it possesses a Virasoro symmetry algebra Eq. (13).

**Case 2** In a similar way, substituting Eq. (14) into Eqs. (25) and (26) along with Eqs. (4) and (5), we obtain

$$- h F'_1 u_x - h F'_2 v_x + F'_1 (u_x h - h_t) + F'_2 (v_x h) = 0,$$

$$- h G'_1 u_x - h G'_2 v_x + G'_1 (u_x h - h_t) + G'_2 (v_x h) = 0,$$

(41)

(42)

where $F'_i, G'_i, i = 1, 2$ satisfy Eq. (31).

Then substitute Eq. (31) into Eqs. (41) and (42) with the help of Eqs. (4) and (5), we get

$$h_t \frac{\partial F}{\partial u} = 0, \quad h_t \frac{\partial G}{\partial u} = 0.$$

(43)
Thus the general autonomous solution of Eq. (43) read
\[ F = M_2(u_{x_1 y_{m_1}} + u_{x_2 y_{m_2}}, n_i, m_i = 0, 1, 2, \ldots, (n_1, m_1) \neq (0, 0)) = M_2, \quad (44) \]
\[ G = N_2(u_{x_1 y_{m_1}} + u_{x_2 y_{m_2}}, n_i, m_i = 0, 1, 2, \ldots, (n_1, m_1) \neq (0, 0)) = N_2. \quad (45) \]
We can find that Eqs. (37) and (38) are special cases of Eqs. (44) and (45).

Thus, from Case 1 and Case 2, if equation has the form
\[ u_{yt} + v_{xx} + u_x u_y + u u_{xy} + M_2 = 0, \quad v_t + u_x + v u_x + w x + u u_{xy} + N_2 = 0, \]
it possesses not only a Virasoro symmetry algebra, but also a Kac–Moody–Virasoro symmetry algebra constructed by \( \sigma_1(f) \) and \( \sigma_2(h) \).

### 4 Group Invariant Solutions of DLWE with Arbitrary Order Invariant

When a symmetry \( \sigma \) is known, one can find the corresponding group invariant solutions of the model by solving the symmetry constraint condition \( \sigma = 0 \) and the original equation at the same time.

Using the results of the above section, we find that the general Lie point symmetries of DLWE with arbitrary order invariant have the form
\[ \sigma_1(f) + \sigma_2(h) = 0, \quad (46) \]
where \( \sigma_1(f), \sigma_2(h) \) should satisfy Eqs. (13) and (14). After solving Eq. (46) with the help of Eqs. (4) and (5), we get

**Case 1** \( f(t) \neq 0 \)

The general solution of Eq. (46) reads:
\[ u = \frac{1}{2} f_t e f^{1/2} + 2 f \int h_t f^{-1/2} dt - f \int h f^{1/2} dt + 2 f U(\xi, \eta), \quad v = -1 + \frac{V(\xi, \eta)}{f^{1/2}}, \quad (47) \]
where
\[ \xi = \frac{x - f^{1/2} \int h f^{-3/2} dt}{f^{1/2}}, \quad \eta = y. \]

Then the group invariant solutions \( U(\xi, \eta) \equiv U, \quad V(\xi, \eta) \equiv V \) should satisfy
\[ U_\xi V + U_\xi U + U_\xi \eta + U_\xi^{3} M_1(U^{\xi_1 \eta_1} U^{-(n+1)}_\eta), \quad (48) \]
\[ V_\xi^{n\eta} U^{-1} = 0, \quad (49) \]
\[ V_\xi + U_\xi U + U_\xi \eta + U_\xi^{3} N_1(U^{\xi_1 \eta_1} U^{-(n+1)}_\eta), \quad (50) \]
\[ V_\xi^{n\eta} U^{-1} = 0. \quad (51) \]

**Case 2** \( f(t) = 0 \)

The general solution of Eq. (46) reads:
\[ u = \frac{h_t x + U(\xi, \eta)}{h}, \quad v = \frac{V(\xi, \eta)}{h} - 1, \quad (52) \]
where
\[ \xi = y, \quad \eta = t. \]

Then the group invariant solutions \( U(\xi, \eta) \equiv U, \quad V(\xi, \eta) \equiv V \) should satisfy
\[ U_\xi = 0, \quad V_\eta = 0. \quad (53) \]

### 5 Conclusions

In this article, the Lie point symmetries and Kac–Moody–Virasoro (KMV) symmetry algebra of (2+1)-dimensional dispersive long-wave equation (DLWE) are obtained by Lie symmetry method. Then finite transformation of DLWE is derived by symmetry group direct method, which can recover Lie point symmetries. Finally, KMV symmetry algebra of DLWE with arbitrary order invariant is obtained, on basis of this algebra the group invariant solutions and similarity reductions of the DLWE with arbitrary order invariant are also derived. Similar to the DLWE, there may exist group invariant solutions and similarity reduction for other (2+1)-dimensional integrable models with arbitrary order invariant.

### References