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Kac–Moody–Virasoro Symmetry Algebra of (2+1)-Dimensional Dispersive Long-Wave Equation with Arbitrary Order Invariant*

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Abstract *By Lie symmetry method, the Lie point symmetries and its Kac–Moody–Virasoro (KMV) symmetry algebra of (2+1)-dimensional dispersive long-wave equation (DLWE) are obtained, and the finite transformation of DLWE is given by symmetry group direct method, which can recover Lie point symmetries. Then KMV symmetry algebra of DLWE with arbitrary order invariant is also obtained. On basis of this algebra the group invariant solutions and similarity reductions are also derived.*

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1 Introduction

In recent years, the study of symmetries, symmetry groups, symmetry reductions, group invariant solutions, and soliton solutions of nonlinear partial differential equations (PDEs) have become one of the most exciting and extremely active areas of research.[1−11] The investigations of the symmetries of (2+1)-dimensional integrable differential equations show that they possess in common an isomorphic centerless Virasoro symmetry algebra:[12−14]

$$[\sigma(f_1),\sigma(f_2)]=\sigma(f_2\dot{f}_1-f_1\dot{f}_2)\,, \tag{1}$$

where f_1 and f_2 are arbitrary functions of a single independent variable. The dot means the derivative of the functions with respect to their argument. According to the centerless Virasoro symmetry algebra, similarity reduction of equation can be given out. Most recently, Lou *et al.* have developed a symmetry group direct method in a series of papers.[15−18] By the method, both the Lie point symmetry groups and the non-Lie symmetry groups can be obtained for some PDEs.

And it is well-known that the integrable dispersive long wave equations (DLWE) is an interesting topics in physics and mathematics,

$$u_{yt}+v_{xx}+u_xu_y+uu_{xy}=0\,, \tag{2}$$

$$v_t+u_x+vu_x+uv_x+u_{xxy}=0\,, \tag{3}$$

where $u\equiv u(x,y,t)$, $v\equiv v(x,y,t)$. And (2+1)-dimensions DLWE with arbitrary order can be read as

$$u_{yt}+v_{xx}+u_xu_y+uu_{xy}+F(u,v)=0\,, \tag{4}$$

$$v_t+u_x+vu_x+uv_x+u_{xxy}+G(u,v)=0\,, \tag{5}$$

where $u\equiv u(x,y,t)$, $v\equiv v(x,y,t)$, $F(u,v)=F(u,u_x,u_{xx},\ldots,u_{x^ny^m},\ldots,v,v_x,v_{xx},\ldots,v_{x^{n_1}y^{m_1}},\ldots)\equiv F$, $G(u,v)=G(u,u_x,u_{xx},\ldots,u_{x^ny^m},\ldots,v,v_x,v_{xx},\ldots,v_{x^{n_1}y^{m_1}},\ldots)\equiv G$, $n,m,n_1,m_1=0,1,2,\ldots$ Since the 1960's, many one-dimensional versions of DLWE have been proposed to model the water wave propagation in certain infinitely-long channels of finite constant depth and narrow width. Recently, to cover wide channels or open seas, one-dimensional DLWE has been extended to the coupled integrable DLWE (2) and (3). This equation has been researched by many authors. Wang *et al.*[19] obtained some solutions of DLWE equation by an extended Jacobi elliptic function rational expansion method. Emmanuel Yomba[20] got new and more general solutions by the modified extended Fan's sub-equation method.[21−23] By means of the variable separation approach,[24−26] Tang *et al.*[27] gave out the abundant localized coherent structures of DLWE.

This article is organized as follows. In Sec. 2, we give out the Lie point symmetries, Kac–Moody–Virasoro (KMV) symmetry algebra and finite transformations of the DLWE equation. In Sec. 3, we obtain a KMV symmetry algebra of DLWE with arbitrary order invariant. In Sec. 4, similarly reductions and group invariant solutions of DLWE with arbitrary order invariant are obtained. Finally, we give out the conclusion of the article.

2 Lie Point Symmetries and Finite Transformations of (2+1)-Dimensional DLWE

A symmetry of arbitrary equation can be defined as a

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solution of

$$\frac{d}{d\epsilon}H(u+\epsilon\sigma)=0\,, \tag{6}$$

where $H(u)$ needs to satisfy $H(u)=0$, σ is a symmetry of equation.

According to the definition of symmetry, we obtain the symmetry equations of the (2+1)-dimension DLWE:

$$\sigma_{1yt}+\sigma_{2xx}+u_x\sigma_{1y}+\sigma_{1x}u_y+u\sigma_{1xy}+\sigma_1 u_{xy}=0\,, \tag{7}$$

$$\sigma_{2t}+\sigma_{1x}+v\sigma_{1x}+\sigma_2 u_x+u\sigma_{2x}+\sigma_1 v_x+\sigma_{1xxy}=0\,. \tag{8}$$

Then let

$$\sigma_1=\xi u_x+\eta u_y+\tau u_t-\phi_1\,, \tag{9}$$

$$\sigma_2=\xi v_x+\eta v_y+\tau v_t-\phi_2\,, \tag{10}$$

where $\xi\equiv\xi(x,y,t,u,v)$, $\eta\equiv\eta(x,y,t,u,v)$, $\tau\equiv\tau(x,y,t,u,v)$, $\phi_1\equiv\phi_1(x,y,t,u,v)$, $\phi_2\equiv\phi_2(x,y,t,u,v)$. Substituting Eqs. (9) and (10) into Eqs. (7) and (8), ξ, η, τ, ϕ_1, ϕ_2 can be obtained by vanishing the coefficients of the polynomials of u,v and their derivatives:

$$\tau=f(t),\quad \eta=l(y),\quad \xi=\frac{1}{2}f_t(t)x+h(t)\,, \tag{11}$$

$$\phi_1=-\frac{1}{2}uf_t(t)+\frac{1}{2}f_{tt}(t)x+h_t(t)\,,$$

$$\phi_2=-\frac{1}{2}(1+v)(2l_y(y)+f_t(t))\,. \tag{12}$$

According to Eqs. (11) and (12), the Lie point symmetries of the DLWE are the linear combinations of the following generators:

$$\sigma_1(f)=\begin{pmatrix}\frac{1}{2}u_xf_t(t)x+f(t)u_t+\frac{1}{2}uf_t(t)-\frac{1}{2}f_{tt}(t)x\\ \frac{1}{2}v_xf_t(t)x+f(t)v_t+\frac{1}{2}f_t(t)+\frac{1}{2}vf_t(t)\end{pmatrix}, \tag{13}$$

$$\sigma_2(h)=\begin{pmatrix}u_xh(t)-h_t(t)\\ v_xh(t)\end{pmatrix}, \tag{14}$$

$$\sigma_3(l)=\begin{pmatrix}l(y)u_y\\ l(y)v_y+l_y(y)+vl_y(y)\end{pmatrix}. \tag{15}$$

The commutators among $\sigma_1(f)$, $\sigma_2(h)$, and $\sigma_3(l)$ are

$$[\sigma_1(f_1),\sigma_1(f_2)]=\sigma_1(f_1f_{2t}-f_2f_{1t})\,, \tag{16}$$

$$[\sigma_1(f),\sigma_2(h)]=\sigma_2\Big(fh_t-\frac{1}{2}hf_t\Big)\,,$$

$$[\sigma_2(h_1),\sigma_2(h_2)]=0\,. \tag{17}$$

According to Eqs. (16) and (17), we know that the subalgebra constituted by $\sigma_1(f)$ and $\sigma_2(h)$ is just the Virasoro algebra.

In general, we can obtain symmetry groups by Lie point symmetries, but there are many difficulties. Fortunately, Lou *et al.* have developed a symmetry group direct method, by the method, both the Lie point symmetry groups and the non-Lie symmetry groups can be obtained. In the following, we would like to use symmetry group direct method to search for finite symmetry transformation of the DLWE (2) and (3), which can recover generators $\sigma_1(f)$, $\sigma_2(h)$, $\sigma_3(l)$.

Firstly, let

$$u=\alpha_1+\beta_1U(\xi_1,\eta_1,\tau_1)+\gamma_1V(\xi_1,\eta_1,\tau_1)\,, \tag{18}$$

$$v=\alpha_2+\beta_2U(\xi_1,\eta_1,\tau_1)+\gamma_2V(\xi_1,\eta_1,\tau_1)\,, \tag{19}$$

where α_1, α_2, β_1, β_2, γ_1, γ_2, ξ_1, η_1, and τ_1 are functions of $\{x,\ y,\ t\}$. Restricting $U(\xi_1,\eta_1,\tau_1)\equiv U$, $V(\xi_1,\eta_1,\tau_1)\equiv V$, and satisfy the same form as the DLWE (2) and (3) but with new independent variables, i.e.

$$V_{\xi_1\xi_1}+U_{\eta_1\tau_1}+U_{\xi_1}U_{\eta_1}+U(\xi_1,\eta_1,\tau_1)U_{\xi_1\eta_1}=0\,, \tag{20}$$

$$U_{\xi_1\xi_1\eta_1}+V_{\tau_1}+U_{\xi_1}+V(\xi_1,\eta_1,\tau_1)U_{\xi_1}+U(\xi_1,\eta_1,\tau_1)V_{\xi_1}=0\,. \tag{21}$$

Substituting Eqs. (18) and (19) into DLWE (2) and (3), then eliminating $V_{\xi_1\xi_1}$ and $U_{\xi_1\xi_1\eta_1}$ by using Eqs. (20) and (21), from that, the remained determining equations of the functions ξ_1, η_1, τ_1, α_1, α_2, β_1, β_2, γ_1, γ_2 can be obtained by vanishing the coefficients of the polynomials of U, V and its derivatives, then it is straightforward to find out the general solution of the determining equations. The results read:

$$\xi_1=\delta\tau_{0t}^{1/2}x+\xi_0,\quad \eta_1=\eta_0,\quad \tau_1=\tau_0\,, \tag{22}$$

$$\beta_1=\delta\tau_{0t}^{1/2},\quad \beta_2=\gamma_1=0,\quad \gamma_2=\delta\tau_{0t}^{1/2}\eta_{0y}\,, \tag{23}$$

$$\alpha_1=\frac{1}{2}\frac{\tau_{0tt}x\delta}{\tau_{0t}}+\frac{2}{\xi_{0t}}\delta\tau_{0t}^{1/2}\,,$$

$$\alpha_2=-1+\delta\tau_{0t}^{1/2}\eta_{0y}\,, \tag{24}$$

where $\xi_0\equiv\xi_0(t)$, $\eta_0\equiv\eta_0(y)$, $\tau_0\equiv\tau_0(t)$, $\delta=\pm1$.

In summary, the following theorem holds:

Theorem If $U\equiv U(x,y,t)$, $V\equiv V(x,y,t)$ are a solution of the DLWE (2) and (3), then so are

$$u=-\frac{1}{2}\frac{(x\tau_{0tt}\delta+2\xi_{0t}\tau_{0t}^{1/2}-2\tau_{0t}^{3/2}U(\xi_1,\eta_1,\tau_1))}{\tau_{0t}\delta}\,,$$

$$v=-1+(\tau_{0t}^{1/2}\eta_{0y}+\tau_{0t}^{1/2}\eta_{0y}V(\xi_1,\eta_1,\tau_1))\delta$$

with Eq. (22), where $\xi_0\equiv\xi_0(t)$, $\eta_0\equiv\eta_0(y)$, $\tau_0\equiv\tau_0(t)$ and discrete value of the $\delta=\pm1$.

From Theorem, by restricting ($f\equiv f(t)$, $h\equiv h(t)$, $l\equiv l(y)$)

$$\tau=t+\epsilon f,\quad \xi_0=\epsilon h,\quad \eta_0=y+\epsilon l\,,$$

we can obtain the general Lie point symmetries of DLWE: Eqs. (13)–(15).

3 Kac–Moody–Virasoro Symmetry Algebra of DLWE with Arbitrary Order Invariant

In this section, we would discuss Kac–Moody–Virasoro symmetry algebra of DLWE with arbitrary order invariant.

Firstly, let $u\to u+\epsilon\sigma_1$, $v\to v+\epsilon\sigma_2$, according to Eq. (6) the symmetry equation of Eqs. (4) and (5) is obtained

$$\sigma_{1yt}+\sigma_{2xx}+u_x\sigma_{1y}+\sigma_{1x}u_y+u\sigma_{1xy}+\sigma_1u_{xy}+F_1'\sigma_1+F_2'\sigma_2=0\,, \tag{25}$$

$$\sigma_{2t}+\sigma_{1x}+v\sigma_{1x}+\sigma_2u_x+u\sigma_{2x}+\sigma_1v_x+\sigma_{1xxy}+G_1'\sigma_1+G_2'\sigma_2=0\,, \tag{26}$$

where F_i' and G_i' are the linearized operator, which respectively defined by

$$F_1'H = \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}F_1(u+\epsilon H), \qquad F_2'H = \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}F_2(v+\epsilon H)\,, \tag{27}$$

$$G_1'H = \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}G_1(u+\epsilon H), \qquad G_2'H = \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}G_2(v+\epsilon H)\,. \tag{28}$$

Because DLWE has Lie point symmetries Eqs. (13)–(15), we would divide two cases to consider Eqs. (13) and (14) symmetries of the DLWE.

Case 1 Substituting Eq. (13) into Eqs. (25) and (26) with the help of Eqs. (4) and (5), we get

$$-\frac{1}{2}f_txF_1'u_x - \frac{1}{2}f_txF_2'v_x - \frac{3}{2}f_tF - fF_1'u_t - fF_2v_t + F_1'\Big(\frac{1}{2}u_xf_tx + fu_t - \frac{1}{2}f_{tt}x + \frac{1}{2}uf_t\Big) + F_2'\Big(\frac{1}{2}v_xf_tx + fv_t + \frac{1}{2}f_t + \frac{1}{2}vf_t\Big) = 0\,, \tag{29}$$

$$-\frac{1}{2}f_txG_1'u_x - \frac{1}{2}f_txG_2'v_x - \frac{3}{2}f_tG - fG_1'u_t - fG_2'v_t + G_1'\Big(\frac{1}{2}u_xf_tx + fu_t - \frac{1}{2}f_{tt}x + \frac{1}{2}uf_t\Big) + G_2'\Big(\frac{1}{2}v_xf_tx + fv_t + \frac{1}{2}f_t + \frac{1}{2}vf_t\Big) = 0\,, \tag{30}$$

where

$$H_1' = \sum_{m,n}\frac{\partial H}{\partial u_{x^ny^m}}\frac{\partial^{m+n}}{\partial x^n\partial y^m}, \quad H_2' = \sum_{m,n}\frac{\partial H}{\partial v_{x^ny^m}}\frac{\partial^{m+n}}{\partial x^n\partial y^m}, \quad m,n\geq 0, \quad H = F,G\,. \tag{31}$$

Then substitute Eq. (31) into Eqs. (29) and (30), we obtain

$$\frac{1}{2}f_t\Big(\sum_{m,n}(n+1)\frac{\partial F}{\partial u_{x^ny^m}}u_{x^ny^m} + \sum_{m,n}(n+1)\frac{\partial F}{\partial v_{x^ny^m}}v_{x^ny^m}\Big) - \frac{1}{2}f_{tt}\Big(x\frac{\partial F}{\partial u} + \frac{\partial F}{\partial u_x}\Big) = \frac{3}{2}f_tF\,, \tag{32}$$

$$\frac{1}{2}f_t\Big(\sum_{m,n}(n+1)\frac{\partial G}{\partial u_{x^ny^m}}u_{x^ny^m} + \sum_{m,n}(n+1)\frac{\partial G}{\partial v_{x^ny^m}}v_{x^ny^m}\Big) - \frac{1}{2}f_{tt}\Big(x\frac{\partial G}{\partial u} + \frac{\partial G}{\partial u_x}\Big) = \frac{3}{2}f_tG\,. \tag{33}$$

Under the f-independent requirement and the autonomous condition of F, G, according to Eqs. (32) and (33), we have

$$\frac{\partial F}{\partial u} = 0, \quad \frac{\partial F}{\partial u_x} = 0, \quad \frac{\partial G}{\partial u} = 0, \quad \frac{\partial G}{\partial u_x} = 0\,, \tag{34}$$

$$\sum_{n,m}(n+1)\frac{\partial F}{\partial u_{x^ny^m}}u_{x^ny^m} + \sum_{n,m}(n+1)\frac{\partial F}{\partial v_{x^ny^m}}v_{x^ny^m} = 3F\,, \tag{35}$$

$$\sum_{n,m}(n+1)\frac{\partial G}{\partial u_{x^ny^m}}u_{x^ny^m} + \sum_{n,m}(n+1)\frac{\partial G}{\partial v_{x^ny^m}}v_{x^ny^m} = 3G\,. \tag{36}$$

The solution of Eqs. (34), (35), and (36) can be written as:

$$F = u_y^3M_1(1-\delta_{n0}\delta_{m0}-\delta_{n1}\delta_{m0})u_{x^ny^m}u_y^{-(n+1)}, v_{x^ny^m}u_y^{-(n+1)}, n,m\geq 0) \equiv u_y^3M_1\,, \tag{37}$$

$$G = u_y^3N_1((1-\delta_{n0}\delta_{m0}-\delta_{n1}\delta_{m0})u_{x^ny^m}u_y^{-(n+1)}, v_{x^ny^m}u_y^{-(n+1)}, n,m\geq 0) \equiv u_y^3N_1\,, \tag{38}$$

where

$$\delta_{nm} = \begin{cases} 0, & n\neq m\,, \\ 1, & n=m\,. \end{cases}$$

In all, if equation has the form

$$u_{yt} + v_{xx} + u_xu_y + uu_{xy} + u_y^3M_1 = 0\,, \tag{39}$$

$$v_t + u_x + vu_x + uv_x + u_{xxy} + u_y^3N_1 = 0\,, \tag{40}$$

it possesses a Virasoro symmetry algebra Eq. (13).

Case 2 In a similar way, substituting Eq. (14) into Eqs. (25) and (26) along with Eqs. (4) and (5), we obtain

$$-hF_1'u_x - hF_2'v_x + F_1'(u_xh - h_t) + F_2'(v_xh) = 0\,, \tag{41}$$

$$-hG_1'u_x - hG_2'v_x + G_1'(u_xh - h_t) + G_2'(v_xh) = 0\,, \tag{42}$$

where F_i', G_i', $i = 1,2$ satisfy Eq. (31).

Then substitute Eq. (31) into Eqs. (41) and (42) with the help of Eqs. (4) and (5), we get

$$h_t\frac{\partial F}{\partial u} = 0, \quad h_t\frac{\partial G}{\partial u} = 0\,. \tag{43}$$

Thus the general autonomous solution of Eq. (43) read

$$F = M_2(u_{x^{n_1}y^{m_1}}, v_{x^{n_2}y^{m_2}}, n_i, m_i = 0, 1, 2, \ldots, (n_1, m_1) \neq (0, 0)) \equiv M_2 \,, \tag{44}$$

$$G = N_2(u_{x^{n_1}y^{m_1}}, v_{x^{n_2}y^{m_2}}, n_i, m_i = 0, 1, 2, \ldots, (n_1, m_1) \neq (0, 0)) \equiv N_2 \,. \tag{45}$$

We can find that Eqs. (37) and (38) are special cases of Eqs. (44) and (45).

Thus, from Case 1 and Case 2, if equation has the form

$$u_{yt} + v_{xx} + u_x u_y + u u_{xy} + M_2 = 0, \quad v_t + u_x + v u_x + u v_x + u_{xxy} + N_2 = 0 \,,$$

it possesses not only a Virasoro symmetry algebra, but also a Kac–Moody–Virasoro symmetry algebra constructed by $\sigma_1(f)$ and $\sigma_2(h)$.

4 Group Invariant Solutions of DLWE with Arbitrary Order Invariant

When a symmetry σ is known, one can find the corresponding group invariant solutions of the model by solving the symmetry constraint condition $\sigma = 0$ and the original equation at the same time.

Using the results of the above section, we find that the general Lie point symmetries of DLWE with arbitrary order invariant have the form

$$\sigma_1(f) + \sigma_2(h) = 0 \,, \tag{46}$$

where $\sigma_1(f)$, $\sigma_2(h)$ should satisfy Eqs. (13) and (14). After solving Eq. (46) with the help of Eqs. (4) and (5), we get

Case 1 $f(t) \neq 0$

The general solution of Eq. (46) reads:

$$u = \frac{1}{2}\frac{f_t x f^{1/2} + 2f\int h_t f^{-1/2}dt - f\int h f_t f^{-3/2}dt + 2fU(\xi,\eta)}{f^{3/2}}, \quad v = -1 + \frac{V(\xi,\eta)}{f^{1/2}} \,, \tag{47}$$

where

$$\xi = \frac{x - f^{1/2}\int h f^{-3/2}dt}{f^{1/2}}, \quad \eta = y \,.$$

Then the group invariant solutions $U(\xi,\eta) \equiv U$, $V(\xi,\eta) \equiv V$ should satisfy

$$U_\xi V + V_\xi U + U_{\xi\xi\eta} + U_\eta^3 M_1(U_{\xi^n\eta^m} U_\eta^{-(n+1)}, V_{\xi^n\eta^m} U_\eta^{-(n+1)}) = 0 \,,$$

$$V_{\xi\xi} + U_\eta U_\xi + U_{\xi\eta} U + U_\eta^3 N_1(U_{\xi^n\eta^m} U_\eta^{-(n+1)}, V_{\xi^n\eta^m} U_\eta^{-(n+1)}) = 0 \,.$$

Case 2 $f(t) = 0$

The general solution of Eq. (46) reads:

$$u = \frac{h_t x + U(\xi,\eta)}{h}, \quad v = \frac{V(\xi,\eta)}{h} - 1 \,, \tag{48}$$

where

$$\xi = y, \qquad \eta = t \,.$$

Then the group invariant solutions $U(\xi,\eta) \equiv U$, $V(\xi,\eta) \equiv V$ should satisfy

$$U_{\xi\eta} = 0, \quad V_\eta = 0 \,.$$

5 Conclusions

In this article, the Lie point symmetries and Kac–Moody–Virasoro (KMV) symmetry algebra of (2+1)-dimensions dispersive long-wave equation (DLWE) are obtained by Lie symmetry method. Then finite transformation of DLWE is derived by symmetry group direct method, which can recover Lie point symmetries. Finally, KMV symmetry algebra of DLWE with arbitrary order invariant is obtained, on basis of this algebra the group invariant solutions and similarity reductions of the DLWE with arbitrary order invariant are also derived. Similar to the DLWE, there may exist group invariant solutions and similarity reduction for other (2+1)-dimensional integrable models with arbitrary order invariant.

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